Negative-energy perturbations in general and in arbitrary one-dimensional Vlasov-Maxwell equilibria

Darío Correa-Restrepo and Dieter Pfirsch
Max-Planck-Institut für Plasmaphysik, EURATOM Association, Postfach 1533, D-85740 Garching bei München, Federal Republic of Germany
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The expression for the free energy of arbitrary perturbations of general Vlasov-Maxwell equilibria derived by Morrison and Pfirsch [Phys. Rev. A **40**, 3898 (1989); Phys. Fluids B **2**, 1105 (1990)] is transformed and put in a concise form, which is subsequently evaluated for *arbitrary, double-symmetric equilibria* in the case of internal perturbations, i.e., perturbations which vanish outside the plasma, and on its boundary. With the single exception of the configurations in which the equilibrium distribution functions are everywhere isotropic and monotonically decreasing functions of the particle energy, these equilibria always allow negative-energy perturbations, *without requiring a large spatial variation of the perturbation across the equilibrium magnetic field.*

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I. INTRODUCTION

Considering arbitrary perturbations of general Vlasov-Maxwell equilibria, Morrison and Pfirsch [1,2] derived expressions for the second variation of the free energy and concluded that negative-energy modes (which are potentially dangerous because they may become nonlinearly unstable and cause anomalous transport [3–5]) exist in any Vlasov-Maxwell equilibrium whenever the unperturbed distribution function $f_\nu^{(0)}$ of any particle species ν deviates from monotonicity and/or isotropy in the vicinity of a single point, i.e., whenever the condition

$$(\mathbf{v}\cdot\mathbf{k})\left[\mathbf{k}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right]>0 \tag{1}$$

holds (in the frame of reference of minimum equilibrium energy) for any particle species ν for some position vector $\mathbf{x}$ and velocity $\mathbf{v}$ and for some local wave vector $\mathbf{k}$. The proof of this result obtained by Morrison and Pfirsch was based on infinitely strongly localized perturbations, which correspond to $|\mathbf{k}|\rightarrow\infty$. This raises the question of the degree of localization actually required for negative-energy modes to exist in a certain equilibrium. Studying a homogeneous Vlasov-Maxwell plasma with constant magnetic field, Correa-Restrepo and Pfirsh [6] showed that negative-energy modes exist for any deviation of the equilibrium distribution function of any of the species from monotonicity and/or isotropy, without having to impose any restricting conditions on the perpendicular wave number $k_\perp$, i.e., without requiring large $k_\perp$. These results were later extended to the more interesting case of an inhomogeneous, force-free equilibrium with a sheared magnetic field [7]. In the present paper, the investigations are carried out for a whole class of equilibria (which includes the previous configurations as particular cases), in that the general expression for the perturbation energy is evaluated for arbitrary double-symmetric, i.e., one-dimensional, equilibria. In generalized coordinates q_1,q_2,q_3, such equilibria depend only on q_1, the equilibrium magnetic field $\mathbf{B}^{(0)}$ is perpendicular to ∇q_1, $\mathbf{B}^{(0)}\cdot\nabla q_1=0$, and the equilibrium distribution function of each particle species ν has the general form $f_\nu^{(0)}=f_\nu^{(0)}(\mathcal{H}_\nu,p_{\nu2},p_{\nu3})$, where $\mathcal{H}_\nu$ is the (conserved) particle energy and $p_{\nu2},p_{\nu3}$ are the (conserved) canonical momenta corresponding to the two ignorable coordinates q_2,q_3, respectively. This class of configurations investigated here includes, for instance, all cylindrical axisymmetric (dependent only on the radius r) and all plane symmetric (dependent only on one Cartesian coordinate x) equilibria.

For double-symmetric equilibria, one obtains a *sufficient* (but not necessary) condition for the existence of negative-energy perturbations which is somewhat similar to inequality (1), namely,

$$\langle \mathbf{k}_{23}\cdot\mathbf{v}\rangle\left\langle \mathbf{k}_{23}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right\rangle>0\ , \tag{2}$$

where the angles are mean values along the unperturbed particle orbits and the wave vector $\mathbf{k}_{23}$ is given by $\mathbf{k}_{23}=k_2\nabla q_2+k_3\nabla q_3$. Unlike the case of inequality (1), $k_{23}=|\mathbf{k}_{23}|$ does not have to be large, the only condition imposed on $\mathbf{k}_{23}$ being $\mathbf{k}_{23}\neq 0$.

Negative-energy waves are also possible even if inequality (2) is not satisfied, namely, if the equilibrium distribution function of any of the particle species is nonmonotonic ($\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu>0$) and/or locally anisotropic in phase space. ($\partial f_\nu^{(0)}/\partial p_{\nu2}$ and $\partial f_\nu^{(0)}/\partial p_{\nu3}$ are not both identically zero. This does not exclude isotropic pressure tensors.) Large spatial variation of the perturbations across the equilibrium magnetic field is not required in these cases either. If there is only anisotropy, however, $\mathbf{k}_{23}$ is not completely arbitrary because, at given k_2/k_3, the quotient n_ν/k_3, where n_ν is an arbitrary integer (positive or negative), can assume values only in a certain range. The result then is that it is only configurations for which the equilibrium distribution functions of all species are everywhere isotropic and monotonically decreasing functions of the particle energy that do not allow the kind of negative-energy perturbations studied here.

In Sec. II, the expression for the free energy $\delta^2 H$ available upon arbitrary perturbations of general Vlasov-Maxwell equilibria derived by Morrison and Pfirsch [1] is transformed and put in a clear and concise form, which is then evaluated in Sec. III for arbitrary, double-symmetric equilibria. For these equilibria, a convenient representation of the generating function of the perturbations further simplifies the expression for $\delta^2 H$. Considering internal perturbations, i.e., those which vanish outside the plasma, and on its boundary, the minimizing perturbations are obtained in Sec. IV, where the expression for the minimized energy is also obtained. In deriving this expression, the difference between particles with periodic motion (PPM) and particles with nonperiodic motion (PNPM) plays a major role. Section V is devoted to an extensive discussion of the energy expression. This discussion leads to the main results, which are then summarized in Sec. VI.

A considerable part of the calculations is done in the appendixes. The relations that are necessary to transform the general expression for the perturbation energy are derived in Appendix A. A convenient representation of derivatives in **x**-**v** space is introduced in Appendix B. The motion of the charged particles is treated in Appendix C, and the two different groups of particles, namely, the particles with periodic motion and the particles with nonperiodic motion, are introduced.

In Appendix D, the constant of the motion C_ν, which plays a crucial role in the expression for the minimized perturbation energy, is determined. Appendix E introduces coordinates in **x**-**v** space which are particularly suited to treating the expression for the perturbation energy. Finally, in Appendix F, an expression is derived for the perturbed electric charge density, and it is shown that this can be made to vanish by an appropriate nontrivial choice of the perturbations.

II. PERTURBATION ENERGY FOR GENERAL EQUILIBRIA

The expressions for the free energy $\delta^2 H$ available upon arbitrary perturbations of general Vlasov-Maxwell equilibria derived by Morrison and Pfirsch [1,2] can be written as [6]

$$\begin{aligned}\delta^2 H=\sum_\nu \int \frac{d^3x\, d^3v}{2m_\nu} \Bigg\{ & \frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\Bigg[-\left[\mathbf{v}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]\frac{\partial G_\nu}{\partial \mathbf{x}}-\left[\mathbf{a}_\nu^{(0)}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}\right]\left[\frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial G_\nu}{\partial \mathbf{v}}+2\frac{\partial G_\nu}{\partial \mathbf{x}}\right] \\ & +\frac{e_\nu}{m_\nu c}G_\nu \mathbf{v}\times\frac{\partial}{\partial \mathbf{v}}\left[\mathbf{B}^{(0)}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]-\frac{e_\nu}{m_\nu}G_\nu\frac{\partial}{\partial \mathbf{v}}\times\left[\frac{\partial G_\nu}{\partial \mathbf{v}}\times\mathbf{E}^{(0)}\right]\Bigg] \\ & +\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\cdot\left[-\left[\frac{\partial G_\nu}{\partial \mathbf{x}}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}\right]\mathbf{v}+(d_\nu G_\nu)\frac{\partial G_\nu}{\partial \mathbf{v}}\right] \\ & +f_\nu^{(0)}\left[\frac{e_\nu}{c}\delta\mathbf{A}\right]^2-2\frac{e_\nu}{c}\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\left[d_\nu(G_\nu\delta\mathbf{A})-G_\nu\frac{\partial}{\partial \mathbf{x}}(\mathbf{v}\cdot\delta\mathbf{A})\right]\Bigg\}+\frac{1}{8\pi}\int d^3x(\delta E^2+\delta B^2)\,. \end{aligned} \tag{3}$$

Here, the species ν with equilibrium distribution function $f_\nu^{(0)}(\mathbf{x},\mathbf{v})$ consists of particles of electric charge e_ν and mass m_ν (c is the velocity of light). $\mathbf{E}^{(0)}$ and $\mathbf{B}^{(0)}$ are the equilibrium electric and magnetic fields, respectively. $G_\nu(\mathbf{x},\mathbf{v})$ is a generating function for the perturbations $\delta\mathbf{x}$ and $\delta\mathbf{v}$ of the particle position and velocity, as given explicitly by Eqs. (F8) and (F9). $\delta\mathbf{A}$ is the perturbation of the vector potential and $\delta E^2/8\pi$ and $\delta B^2/8\pi$ are the perturbations in the electric- and magnetic-field energy densities. The operator d_ν is the equilibrium Vlasov operator, i.e.,

$$d_\nu=\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}+\mathbf{a}_\nu^{(0)}\cdot\frac{\partial}{\partial \mathbf{v}}\,, \tag{4}$$

where

$$\mathbf{a}_\nu^{(0)}=\frac{e_\nu}{m_\nu}\left[\mathbf{E}^{(0)}+\frac{\mathbf{v}\times\mathbf{B}^{(0)}}{c}\right]. \tag{5}$$

Taking into account the relations derived in Appendix A, Eqs. (A1)–(A4), Eq. (3) can easily be transformed to yield

$$\begin{aligned}\delta^2 H=\sum_\nu \int \frac{d^3x\, d^3v}{2m_\nu} \Bigg\{ & (d_\nu G_\nu)\left[\mathbf{F}_\nu^{(0)}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}-\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right] \\ & +f_\nu^{(0)}\left[\frac{e_\nu}{c}\delta\mathbf{A}\right]^2-2\frac{e_\nu}{c}\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\left[d_\nu(G_\nu\delta\mathbf{A})-G_\nu\frac{\partial}{\partial \mathbf{x}}(\mathbf{v}\cdot\delta\mathbf{A})\right]\Bigg\}+\frac{1}{8\pi}\int d^3x(\delta E^2+\delta B^2)\,, \end{aligned} \tag{6}$$

where

$$\mathbf{F}_\nu^{(0)}=\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}+\frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\,, \tag{7}$$

so that the equilibrium Vlasov's equation is

$$d_\nu f_\nu^{(0)} = \mathbf{F}_\nu^{(0)} \cdot \mathbf{v} + \frac{e_\nu}{m_\nu} \mathbf{E}^{(0)} \cdot \frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} = 0 \ . \tag{8}$$

Equation (6) becomes particularly simple if one evaluates $\delta^2 H$ in terms of *electrostatic initial perturbations,* which have $\delta \mathbf{A} = \mathbf{0}$, $\delta \mathbf{B} = \nabla \times \delta \mathbf{A} = \mathbf{0}$. This yields

$$\delta^2 H = \sum_\nu \int \frac{d^3x\, d^3v}{2m_\nu} \left\{ (d_\nu G_\nu) \left[\mathbf{F}_\nu^{(0)} \cdot \frac{\partial G_\nu}{\partial \mathbf{v}} - \frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} \cdot \frac{\partial G_\nu}{\partial \mathbf{x}} \right] \right\} + \frac{1}{8\pi} \int d^3x\, \delta E^2 \ . \tag{9}$$

For time-independent equilibrium fields $\mathbf{E}^{(0)} = -\nabla \Phi^{(0)}$ and $\mathbf{B}^{(0)} = \nabla \times \mathbf{A}^{(0)}$, the particle energy $\mathcal{H}_\nu$,

$$\mathcal{H}_\nu = \frac{m_\nu}{2} \mathbf{v}^2 + e_\nu \Phi^{(0)} \ , \tag{10}$$

is a constant of the motion. The equilibrium distribution functions $f_\nu^{(0)}$ can be written as

$$f_\nu^{(0)}(\mathbf{x},\mathbf{v}) = f_\nu^{(0)}(\mathcal{H}_\nu(\mathbf{x},\mathbf{v}), \mathcal{K}_{\nu\kappa}(\mathbf{x},\mathbf{v})) \ , \tag{11}$$

where κ runs over as many indices as there are other constants of the motion $\mathcal{K}_{\nu\kappa}$ in the problem under consideration.

If one introduces generalized coordinates $q_i(\mathbf{x})$, $i = 1, \ldots 3$, with the corresponding covariant velocity components $v_i(\mathbf{x},\mathbf{v})$ and takes into account the relations derived in Appendix B, in particular Eq. (B10), the perturbation energy $\delta^2 H$, Eq. (9), can be expressed as

$$\delta^2 H = \sum_{\nu,\kappa} \int \frac{d^3x\, d^3v}{2m_\nu} \left\{ -m_\nu (d_\nu G_\nu)^2 \left.\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\right|_{\mathcal{K}_{\nu\kappa}} - (d_\nu G_\nu) \left.\frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu\kappa}}\right|_{\mathcal{H}_\nu} \left.\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{v}}\right|_{\mathbf{x}} \cdot \left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{v_i} \right.$$
$$\left. + (d_\nu G_\nu) \left.\frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu\kappa}}\right|_{\mathcal{H}_\nu} \left.\frac{\partial G_\nu}{\partial \mathbf{v}}\right|_{\mathbf{x}} \cdot \left[\left.\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{x}}\right|_{v_i} + \frac{e_\nu}{m_\nu c} \mathbf{B}^{(0)} \times \left.\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{v}}\right|_{\mathbf{x}} \right] \right\} + \frac{1}{8\pi} \int d^3x\, \delta E^2 \ . \tag{12}$$

Note that, in Eq. (12), the derivatives with respect to $\mathbf{x}$ are now performed at constant $v_i(\mathbf{x},\mathbf{v}) = \mathbf{v} \cdot (\partial \mathbf{x} / \partial q_i)$, and *not at constant* $\mathbf{v}$. It is evident from Eq. (12) that there cannot be negative-energy perturbations if all $f_\nu^{(0)} = f_\nu^{(0)}(\mathcal{H}_\nu)$ and if $\partial f_\nu^{(0)} / \partial \mathcal{H}_\nu < 0$, a result already proved in Ref. [8].

III. PERTURBATION ENERGY FOR ARBITRARY, ONE-DIMENSIONAL EQUILIBRIA

Double-symmetric equilibria are now considered, i.e., configurations in which the equilibrium scalar quantities depend only on *one* of the three spatial coordinates q_1, q_2, q_3. Let q_1 be the generalized coordinate on which the equilibrium depends. Since q_2 and q_3 are ignorable, the corresponding canonical momenta are constants of the motion.

The equilibrium magnetic field $\mathbf{B}^{(0)}$ has the following general form:

$$\begin{aligned} \mathbf{B}^{(0)} &= \nabla \times \mathbf{A}^{(0)} = \nabla \times [A_i^{(0)}(q_1) \nabla q_i] \\ &= -\frac{dA_3^{(0)}}{dq_1} \nabla q_3 \times \nabla q_1 + \frac{dA_2^{(0)}}{dq_1} \nabla q_1 \times \nabla q_2 \\ &= \frac{1}{J(q_1)} \left[-\frac{dA_3^{(0)}}{dq_1} \frac{\partial \mathbf{x}}{\partial q_2} + \frac{dA_2^{(0)}}{dq_1} \frac{\partial \mathbf{x}}{\partial q_3} \right] , \end{aligned} \tag{13}$$

where

$$J(q_1) = \frac{\partial \mathbf{x}}{\partial q_1} \cdot \left[\frac{\partial \mathbf{x}}{\partial q_2} \times \frac{\partial \mathbf{x}}{\partial q_3} \right] \tag{14}$$

($A_1^{(0)} \equiv 0$ without loss of generality).

The Lagrangian L_ν of a particle of species ν is

$$L_\nu = \frac{m_\nu}{2} \mathbf{v}^2 + \frac{e_\nu}{c} \mathbf{A}^{(0)}(\mathbf{x}) \cdot \mathbf{v} - e_\nu \Phi^{(0)}(\mathbf{x}) \ , \tag{15}$$

from which the momentum canonically conjugated to $\mathbf{x}$ follows:

$$\mathbf{p}_\nu = \frac{\partial L_\nu}{\partial \mathbf{v}} = m_\nu \mathbf{v} + \frac{e_\nu}{c} \mathbf{A}^{(0)} \ , \tag{16}$$

with covariant components

$$\begin{aligned} p_{\nu i} &= m_\nu v_i(\mathbf{x},\mathbf{v}) + \frac{e_\nu}{c} A_i^{(0)}(q_1) \\ &= m_\nu \mathbf{v} \cdot \frac{\partial \mathbf{x}}{\partial q_i} + \frac{e_\nu}{c} A_i^{(0)}(q_1) \ , \end{aligned} \tag{17}$$

which are the momenta canonically conjugated to the q_i's.

Besides the particle energy $\mathcal{H}_\nu$, also the canonical momenta $p_{\nu 2}$ and $p_{\nu 3}$ are constants of the motion. In the *four-dimensional* space (v_i, q_1), $i = 1, \ldots, 3$, the general equilibrium solution of Vlasov's equation is

$$f_\nu^{(0)} = f_\nu^{(0)}(\mathcal{H}_\nu, p_{\nu 2}, p_{\nu 3}) \ . \tag{18}$$

Then, in Eq. (12), $\mathcal{H}_{\nu\kappa}=p_{\nu\kappa}$, $\kappa=2,3$. From Eq. (17) one has

$$\left.\frac{\partial p_{\nu\kappa}}{\partial \mathbf{x}}\right|_{v_i}=\frac{e_\nu}{c}\frac{dA_\kappa^{(0)}}{dq_1}\nabla q_1=\frac{e_\nu}{c}\nabla A_\kappa^{(0)} \tag{19}$$

and

$$\left.\frac{\partial p_{\nu\kappa}}{\partial \mathbf{v}}\right|_{\mathbf{x}}=m_\nu\frac{\partial \mathbf{x}}{\partial q_\kappa}, \tag{20}$$

and therefore

$$\begin{aligned}\left.\frac{\partial \mathcal{H}_{\nu\kappa}}{\partial \mathbf{x}}+\frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial \mathcal{H}_{\nu\kappa}}{\partial \mathbf{v}}\right|_{\mathbf{x}}&=\left.\frac{\partial p_{\nu\kappa}}{\partial \mathbf{x}}\right|_{v_i}+\left.\frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial p_{\nu\kappa}}{\partial \mathbf{v}}\right|_{\mathbf{x}}\\&=\left.\frac{\partial p_{\nu\kappa}}{\partial \mathbf{x}}\right|_{v_i}+\left.\frac{e_\nu}{m_\nu c}[\nabla A_i^{(0)}\times\nabla q_i]\times\frac{\partial p_{\nu\kappa}}{\partial \mathbf{v}}\right|_{\mathbf{x}}\\&=\frac{e_\nu}{c}\left[\nabla A_\kappa^{(0)}-\delta_{i\kappa}\nabla A_i^{(0)}+\frac{\partial A_i^{(0)}}{\partial q_\kappa}\nabla q_i\right]=0,\end{aligned} \tag{21}$$

because $A_i^{(0)}$ does not depend on q_κ for $\kappa=2,3$.

Hence, for equilibria which depend only on one spatial coordinate q_1, Eq. (12) reduces to

$$\begin{aligned}\delta H=\sum_{\substack{\nu\\ \kappa=2,3}}\int\frac{d^3x\,d^3v}{2}\Bigg\{&-(d_\nu G_\nu)^2\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\\&-(d_\nu G_\nu)\left[\left.\frac{\partial \mathbf{x}}{\partial q_\kappa}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{v_i}\right]\frac{\partial f_\nu^{(0)}}{\partial p_{\nu\kappa}}\Bigg\}\\&+\frac{1}{8\pi}\int d^3x\,\delta E^2\,.\end{aligned} \tag{22}$$

Note that, in this representation, derivatives of G_ν in $\mathbf{v}$ space appear only in $d_\nu G_\nu$, the derivative of G_ν along the unperturbed particle orbits.

Since the equilibrium is independent of q_2 and q_3, an approximate ansatz for the generating function G_ν is

$$\begin{aligned}G_\nu(\mathbf{x},\mathbf{v})&=G_\nu(\mathbf{x},v_i(\mathbf{x},\mathbf{v}))\\&=\tfrac{1}{2}[g_\nu(q_1,v_i)e^{i[k_2q_2+k_3q_3]}\\&\quad+g_\nu^*(q_1,v_i)e^{-i[k_2q_2+k_3q_3]}]\,.\end{aligned} \tag{23}$$

G_ν is obviously a real function since g_ν^* is the complex conjugate of g_ν.

The constants k_2 and k_3 are the covariant components of a wave vector $\mathbf{k}_{23}$, given by

$$\mathbf{k}_{23}=k_2\nabla q_2+k_3\nabla q_3\,. \tag{24}$$

Derivation of the expressions in the exponents of Eq. (23) along the unperturbed orbits yields

$$\begin{aligned}d_\nu(k_2q_2+k_3q_3)&=k_2\dot q_2+k_3\dot q_3\\&=k_2v^2+k_3v^3\\&=\mathbf{k}_{23}\cdot\mathbf{v}\end{aligned} \tag{25}$$

(v^i, $i=1,\dots,3$ are the contravariant components of the velocity).

Inserting Eq. (23) in Eq. (22), integrating with respect to q_2 between q_{20} and $q_{20}+2\pi/k_2$ and with respect to q_3 between q_{30} and $q_{30}+2\pi/k_3$, taking into account that

$$d^3x=J(q_1)dq_1dq_2dq_3 \tag{26}$$

and defining $s(q_1)$ by the relation

$$s(q_1)=J(q_1)\int_{q_{20}}^{q_{20}+2\pi/k_2}\int_{q_{30}}^{q_{30}+2\pi/k_3}dq_2dq_3\,, \tag{27}$$

yields

$$\begin{aligned}\delta^2H=\sum_{\substack{\nu\\ \kappa=2,3}}\int\frac{s(q_1)}{4}dq_1d^3v\Bigg\{&-\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}|[d_\nu g_\nu+i(\mathbf{v}\cdot\mathbf{k}_{23})g_\nu]|^2-\frac{i}{2}\frac{\partial f_\nu^{(0)}}{\partial p_{\nu\kappa}}k_\kappa[g_\nu d_\nu g_\nu^*-g_\nu^*d_\nu g_\nu-2i(\mathbf{v}\cdot\mathbf{k}_{23})g_\nu g_\nu^*]\Bigg\}\\&+\frac{1}{2\pi}\int d^3x\,\delta E^2\,.\end{aligned} \tag{28}$$

The complex functions g_ν are conveniently represented as

$$g_\nu(q_1,v_i)=\Psi_\nu(q_1,v_i)e^{i\Gamma_\nu(q_1,v_i)}\,, \tag{29}$$

where Ψ_ν and Γ_ν are real functions and are such that the g_ν's are single-valued functions of q_1 and v_i.

Inserting Eq. (29) in Eq. (28) yields

$$\delta^2 H = \sum_{\substack{\nu \\ \kappa=2,3}} \int \frac{s(q_1)}{4} dq_1 d^3v \left\{ -\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} \{ (d_\nu \Psi_\nu)^2 + \Psi_\nu^2 [d_\nu \Gamma_\nu + (\mathbf{v}\cdot\mathbf{k}_{23})]^2 \} - \frac{\partial f_\nu^{(0)}}{\partial p_{\nu\kappa}} k_\kappa \Psi_\nu^2 [d_\nu \Gamma_\nu + (\mathbf{v}\cdot\mathbf{k}_{23})] \right\} . \tag{30}$$

Here, the electrostatic energy term $(1/8\pi)\int d^3x\, \delta E^2$ has been dropped since the perturbed charge density can be made zero by an appropriate choice of the *signs* of Ψ_ν, which do not influence Eq. (30). This is explicitly shown in Appendix F. Note that $\delta^2 H$ is a functional of Ψ_ν, which appears as Ψ_ν and $d_\nu \Psi_\nu$, and of Γ_ν, *which appears only through its derivative $d_\nu \Gamma_\nu$ along the unperturbed orbits.*

IV. EXTREMIZATION OF THE SECOND-ORDER PERTURBATION ENERGY

Complete minimization of the expression for the perturbation energy, Eq. (30), with respect to Γ_ν is now possible. In order to do this, we first consider the variation of $\delta^2 H$ brought about by a variation $\delta\Gamma_\nu$ of Γ_ν. This quantity can easily be calculated and is

$$\begin{aligned} \delta_{\Gamma_\nu}(\delta^2 H) &= \delta^2 H(\Gamma_\nu + \delta\Gamma_\nu) - \delta^2 H(\Gamma_\nu) \\ &= \sum_{\substack{\nu \\ \kappa=2,3}} \int \tfrac{1}{4} s(q_1) dq_1 d^3v [d_\nu \delta\Gamma_\nu] \left\{ -2\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} \Psi_\nu^2 [d_\nu \Gamma_\nu + \mathbf{v}\cdot\mathbf{k}_{23}] - \frac{\partial f_\nu^{(0)}}{\partial p_{\kappa\nu}} k_\kappa \Psi_\nu^2 \right\} \\ &= \sum_{\substack{\nu \\ \kappa=2,3}} \int \tfrac{1}{4} s(q_1) dq_1 d^3v \left\{ d_\nu \left[\delta\Gamma_\nu \left[-2\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} \Psi_\nu^2 [d_\nu \Gamma_\nu + \mathbf{v}\cdot\mathbf{k}_{23}] - \frac{\partial f_\nu^{(0)}}{\partial p_{\nu\kappa}} k_\kappa \Psi_\nu^2 \right] \right] \right. \\ &\qquad \left. + [\delta\Gamma_\nu] d_\nu \left[2\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} \Psi_\nu^2 [d_\nu \Gamma_\nu + \mathbf{v}\cdot\mathbf{k}_{23}] + \frac{\partial f_\nu^{(0)}}{\partial p_{\nu\kappa}} k_\kappa \Psi_\nu^2 \right] \right\} . \end{aligned} \tag{31}$$

Taking $\delta\Gamma_\nu$ to vanish outside the plasma, and on its boundary, as is appropriate for the internal perturbations considered here, Eq. (31) reduces to

$$\delta_{\Gamma_\nu}(\delta^2 H) = \sum_{\substack{\nu \\ \kappa=2,3}} \int \tfrac{1}{4} s(q_1) dq_1 d^3v [\delta\Gamma_\nu] d_\nu \left[2\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} \Psi_\nu^2 [d_\nu \Gamma_\nu + \mathbf{v}\cdot\mathbf{k}_{23}] + \frac{\partial f_\nu^{(0)}}{\partial p_{\nu\kappa}} k_\kappa \Psi_\nu^2 \right] , \tag{32}$$

and, since $\delta\Gamma_\nu$ is arbitrary in the internal region, the condition for the vanishing of $\delta_{\Gamma_\nu}(\delta^2 H)$ is

$$d_\nu \left[2\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} \Psi_\nu^2 [d_\nu \Gamma_\nu + \mathbf{v}\cdot\mathbf{k}_{23}] + \frac{\partial f_\nu^{(0)}}{\partial p_{\nu\kappa}} k_\kappa \Psi_\nu^2 \right] = 0 . \tag{33}$$

Since d_ν is the rate of change seen by the moving particles along the unperturbed orbits, the general solution of Eq. (33), in the four-dimensional space q_1, v_i, $i=1,\ldots,3$, is given by

$$2\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} \Psi_\nu^2 [d_\nu \Gamma_\nu + \mathbf{v}\cdot\mathbf{k}_{23}] + \left[k_2 \frac{\partial f_\nu^{(0)}}{\partial p_{\nu 2}} + k_3 \frac{\partial f_\nu^{(0)}}{\partial p_{\nu 3}} \right] \Psi_\nu^2 = C_\nu(\mathcal{H}_\nu, p_{\nu 2}, p_{\nu 3}) , \tag{34}$$

where C_ν is a single-valued function of the constants of the motion. $C_\nu(\mathcal{H}_\nu, p_{\nu 2}, p_{\nu 3})$ is explicitly determined in Appendix D by using the fact that Γ_ν must be such that the generating function g_ν for the perturbations, Eq. (29), must be single valued. Inserting Eq. (34) in Eq. (30) yields the minimized perturbation energy in the form

$$\delta^2 H = \sum_\nu \int \tfrac{1}{4} s(q_1) dq_1 d^3v \left[\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} \right] \left\{ -[d_\nu \Psi_\nu]^2 + \tfrac{1}{4}\Psi_\nu^2 \frac{\left[k_2 \frac{\partial f_\nu^{(0)}}{\partial p_{\nu 2}} + k_3 \frac{\partial f_\nu^{(0)}}{\partial p_{\nu 3}} \right]^2}{\left[\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} \right]^2} - \frac{1}{4\left[\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} \right]^2} \frac{C_\nu^2}{\Psi_\nu^2} \right\} . \tag{35}$$

According to Appendixes C and D, the particles of each species ν can be divided into two classes, namely, the particles with periodic motion (PPM), for which $q_1(t)$ is a periodic function of time, and the particles with nonperiodic motion (PNPM). This is utilized to split the perturbation energy into two parts:

$$\delta^2 H = (\delta^2 H)_{\mathrm{PPM}} + (\delta^2 H)_{\mathrm{PNPM}} , \tag{36}$$

where $(\delta^2 H)_{\rm PPM}$ is the contribution of the particles with periodic motion, and $(\delta^2 H)_{\rm PNPM}$ that of the particles with nonperiodic motion. According to Appendixes C and D, these contributions are, explicitly,

$$(\delta^2 H)_{\rm PNPM}=\sum_\nu \int \tfrac{1}{4}s(q_1)dq_1 d^3v \left[\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\right]\left\{-[d_\nu \Psi_\nu]^2+\tfrac{1}{4}\Psi_\nu^2 \frac{\left[k_2\frac{\partial f_\nu^{(0)}}{\partial p_{\nu 2}}+k_3\frac{\partial f_\nu^{(0)}}{\partial p_{\nu 3}}\right]^2}{\left[\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\right]^2}-\frac{1}{4\left[\frac{f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\right]^2}\frac{C_\nu^2}{\Psi_\nu^2}\right\}, \tag{37}$$

where $C_\nu(\mathcal{H}_\nu, p_{\nu 2}, p_{\nu 3})$ is a *completely arbitrary function,* and

$$(\delta^2 H)_{\rm PPM}=\sum_\nu \int \tfrac{1}{4}s(q_1)dq_1 d^3v \left[\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\right]\left\{-[d_\nu \Psi_\nu]^2+\tfrac{1}{4}\Psi_\nu^2 \frac{\left[k_2\frac{\partial f_\nu^{(0)}}{\partial p_{\nu 2}}+k_3\frac{\partial f_\nu^{(0)}}{\partial p_{\nu 3}}\right]^2}{\left[\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\right]^2}\right.$$

$$\left.-\frac{1}{\Psi^2}\frac{1}{\left\langle\frac{1}{\Psi_\nu^2}\right\rangle^2}\left[\frac{2\pi}{\tau}\right]^2 \times\left[n_\nu+\frac{\tau}{4\pi}\frac{1}{\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}}\left[2\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\langle \mathbf{k}_{23}\cdot \mathbf{v}\rangle+k_2\frac{\partial f_\nu^{(0)}}{\partial p_{\nu 2}}+k_3\frac{\partial f_\nu^{(0)}}{\partial p_{\nu 3}}\right]\right]^2\right\}, \tag{38}$$

where n_ν is *any integer number,* i.e., $n_\nu=0,\pm 1,\ldots$. This integer n_ν appears upon integration of $d_\nu \Gamma_\nu$ along unperturbed orbits as a consequence of the fact that the perturbations are single valued in **x**-**v** space, and that, for PPM, the coordinates q_1, v_i, $i=1,\ldots,3$, are periodic along the unperturbed orbits. This is discussed in more detail in Appendix D.

By employing the coordinate system $t, \mathcal{H}_\nu, p_{\nu 2}, p_{\nu 3}$ introduced in Appendix E the contribution of the particles with periodic motion to the perturbation energy can be expressed as

$$(\delta^2 H)_{\rm PPM}=\sum_\nu \int \frac{s_0}{4m_\nu^3}d\mathcal{H}_\nu dp_{\nu 2}dp_{\nu 3}dt\left[\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\right]$$

$$\times\left\{-\left[\frac{\partial \Psi_\nu}{\partial t}\right]^2+\tfrac{1}{4}\Psi_\nu^2\frac{\left[k_2\frac{\partial f_\nu^{(0)}}{\partial p_{\nu 2}}+k_3\frac{\partial f_\nu^{(0)}}{\partial p_{\nu 3}}\right]^2}{\left[\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\right]^2}\right.$$

$$\left.-\frac{1}{\Psi_\nu^2}\frac{1}{\left\langle\frac{1}{\Psi_\nu^2}\right\rangle^2}\left[\frac{2\pi}{\tau}\right]^2\left[n_\nu+\frac{\tau}{4\pi}\frac{1}{\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}}\left[2\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\langle \mathbf{k}_{23}\cdot \mathbf{v}\rangle+k_2\frac{\partial f_\nu^{(0)}}{\partial p_{\nu 2}}+k_3\frac{\partial f_\nu^{(0)}}{\partial p_{\nu 3}}\right]\right]^2\right\}, \tag{39}$$

where

$$s_0=\frac{s}{J(q_1)}=\frac{4\pi^2}{k_2 k_3}\ . \tag{40}$$

Performing the integration over t in Eq. (39) yields

$$(\delta^2 H)_{\rm PPM} = \sum_\nu \int \frac{s_0}{4m_\nu^3} d\mathcal{H}_\nu dp_{\nu 2} dp_{\nu 3} \tau \left| \frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} \right| \times \left\{ -\left\langle \left[\frac{\partial \Psi_\nu}{\partial t} \right]^2 \right\rangle + \frac{1}{4} \langle \Psi_\nu^2 \rangle \frac{\left[k_2 \frac{\partial f_\nu^{(0)}}{\partial p_{\nu 2}} + k_3 \frac{\partial f_\nu^{(0)}}{\partial p_{\nu 3}} \right]^2}{\left[\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} \right]^2} - \frac{1}{\left\langle \frac{1}{\Psi_\nu^2} \right\rangle} \left[\frac{2\pi}{\tau} \right]^2 \left[n_\nu + \frac{\tau}{4\pi} \frac{1}{\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}} \left[2 \frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} \langle \mathbf{k}_{23} \cdot \mathbf{v} \rangle + k_2 \frac{\partial f_\nu^{(0)}}{\partial p_{\nu 2}} + k_3 \frac{\partial f_\nu^{(0)}}{\partial p_{\nu 3}} \right] \right]^2 \right\}, \quad (41)$$

where the term $(\partial f_\nu^{(0)}/\partial \mathcal{H}_\nu)\langle \mathbf{k}_{23}\cdot\mathbf{v}\rangle$ could also be expressed in a different way according to the relations

$$\frac{1}{m_\nu} \frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} \bigg|_{\mathbf{x}} = \mathbf{v} \frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} + \frac{\partial \mathbf{x}}{\partial q_2} \frac{\partial f_\nu^{(0)}}{\partial p_{\nu 2}} + \frac{\partial \mathbf{x}}{\partial q_3} \frac{\partial f_\nu^{(0)}}{\partial p_{\nu 3}}, \quad (42)$$

$$\frac{1}{m_\nu} \mathbf{k}_{23} \cdot \frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} \bigg|_{\mathbf{x}} = \mathbf{k}_{23} \cdot \mathbf{v} \frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} + k_2 \frac{\partial f_\nu^{(0)}}{\partial p_{\nu 2}} + k_3 \frac{\partial f_\nu^{(0)}}{\partial p_{\nu 3}}. \quad (43)$$

Note that the only derivative of Ψ_ν which appears in the expressions for $\delta^2 H$, Eqs. (37)–(41), is

$$d_\nu \Psi_\nu = \frac{d\Psi_\nu}{dt} \bigg|_{\text{along orbits}},$$

the rate of change of Ψ_ν along the unperturbed orbits. In particular, *there are no explicit spatial derivatives.*

V. DISCUSSION OF THE EXPRESSION FOR THE SECOND-ORDER PERTURBATION ENERGY

A. The perturbation energy $(\delta^2 H)_{\rm PPM}$ for particles with periodic motion

To study the sign of $(\delta^2 H)_{\rm PPM}$, Eq. (41), one has to distinguish the following two cases.

1. $k_2 = k_3 = 0 \rightarrow \mathbf{k}_{23} = 0$, *perpendicular wave propagation*

In this case, the wave propagation is perpendicular to $\mathbf{B}^{(0)}$ since $\mathbf{k}_{23}\cdot\mathbf{B}^{(0)} = 0$ for all q_1. Let us consider, in phase space, the subspace defined by $\mathcal{H}_\nu = \mathcal{H}_{\nu 0}$, $p_{\nu 2} = p_{\nu 20}$, $p_{\nu 3} = p_{\nu 30}$. It follows immediately from Eq. (41) that $\delta^2 H < 0$ if $\partial f_\nu^{(0)}/\partial \mathcal{H}_\nu > 0$ for some $\mathcal{H}_\nu, p_{\nu 2}, p_{\nu 3}$ around $\mathcal{H}_{\nu 0}, p_{\nu 20}, p_{\nu 30}$ corresponding to PPM, and for any particle species ν. This means that the presence of a local minimum with respect to $\mathcal{H}_\nu$ in $f_\nu^{(0)}(\mathcal{H}_\nu, p_{\nu 2}, p_{\nu 3})$ guarantees $\delta^2 H < 0$, *without any restrictions on the spatial variation of the perturbations perpendicular to* $B^{(0)}$: it suffices to localize Ψ_ν ($d_\nu \Psi_\nu$ is then also localized) to the region around $\mathcal{H}_{\nu 0}, p_{\nu 20}, p_{\nu 30}$ where $\partial f_\nu^{(0)}/\partial \mathcal{H}_\nu > 0$. Outside this region Ψ_ν vanishes. All other Ψ_μ are set equal to zero. The Ψ_ν corresponding to the PNPM are likewise all set equal to zero, so that $(\delta^2 H)_{\rm PNPM} = 0$. The sign of $\delta^2 H = (\delta^2 H)_{\rm PPM}$ is then determined only by the sign of the integrand in the region of localization, which is then negative.

The kind of localization introduced here means that, for every q_1 in configuration space, only the particles whose constants of the motion have values near $\mathcal{H}_{\nu 0}, p_{\nu 20}, p_{\nu 30}$ are perturbed; this localization is thus quite different from a localization in configuration space.

2. k_2, k_3 *are not both zero,* $\mathbf{k}_{23}\cdot\mathbf{B}^{(0)} \neq 0$

In this case, the wave vector $\mathbf{k}_{23}$ has a component in the direction of $\mathbf{B}^{(0)}$ (with the possible exception of some isolated points q_1) since

$$\mathbf{k}_{23}\cdot\mathbf{B}^{(0)} = k_2 B^{(0)2}(q_1) + k_3 B^{(0)3}(q_1) = \frac{1}{J(q_1)} \left[-k_2 \frac{dA_3^{(0)}}{dq_1} + k_3 \frac{dA_2^{(0)}}{dq_1} \right]. \quad (44)$$

If $\partial f_\nu^{(0)}/\partial \mathcal{H}_\nu > 0$ for some $\mathcal{H}_\nu, p_{\nu 2}, p_{\nu 3}$ around $\mathcal{H}_{\nu 0}, p_{\nu 20}, p_{\nu 30}$ corresponding to PPM, and for any particle species ν, one again localizes the perturbations Ψ_ν around these values, as in the preceding cases. All Ψ_ν corresponding to PNPM are set equal to zero; therefore $(\delta^2 H)_{\rm PNPM} = 0$.

If $\partial f_\nu^{(0)}/\partial p_{\nu 2} = \partial f_\nu^{(0)}/\partial p_{\nu 3} = 0$ (local isotropy), all terms in Eq. (41) are negative. If $\partial f_\nu^{(0)}/\partial p_{\nu 2}$ and/or $\partial f_\nu^{(0)}/\partial p_{\nu 3} \neq 0$, one can use the arbitrary n_ν to make the integrand in Eq. (41) negative. This is most easily shown if one chooses Ψ_ν independently of t, i.e., $\Psi_\nu = \Psi_\nu(\mathcal{H}_\nu, p_{\nu 2}, p_{\nu 3})$. In this case, the integrand in Eq. (41) is given by

$$-\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\left[n_\nu+\frac{\tau}{2\pi}\langle \mathbf{k}_{23}\cdot\mathbf{v}\rangle\right]\times\left[n_\nu+\frac{\tau}{2\pi}\frac{1}{m_\nu}\frac{1}{\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}}\left\langle \mathbf{k}_{23}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\bigg|_x\right\rangle\right]. \quad (45)$$

If

$$\langle \mathbf{v}\cdot\mathbf{k}_{23}\rangle\left\langle \mathbf{k}_{23}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\bigg|_x\right\rangle>0 ,$$

it suffices to take $n_\nu=0$ to make the expression (45) (and thus $\delta^2 H$) negative. For any $\langle \mathbf{v}\cdot\mathbf{k}_{23}\rangle\langle \mathbf{k}_{23}\cdot(\partial f_\nu^{(0)}/\partial\mathbf{v})|_x\rangle$, it is negative if the factors in the square brackets are either both positive or both negative. Both factors are positive if

$$n_\nu>-\frac{\tau}{2\pi}\langle \mathbf{k}_{23}\cdot\mathbf{v}\rangle$$

and (46)

$$n_\nu>-\frac{\tau}{2\pi}\frac{1}{m_\nu}\frac{1}{\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}}\left\langle \mathbf{k}_{23}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\bigg|_x\right\rangle ,$$

which can easily be satisfied by the appropriate choice of n_ν, i.e., by choosing n_ν large enough to satisfy both inequalities.

The expression (45) is also negative if both factors are negative, i.e., if

$$n_\nu<-\frac{\tau}{2\pi}\langle \mathbf{k}_{23}\cdot\mathbf{v}\rangle$$

and (47)

$$n_\nu<-\frac{\tau}{2\pi}\frac{1}{m_\nu}\frac{1}{\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}}\left\langle \mathbf{k}_{23}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\bigg|_x\right\rangle ,$$

which can be satisfied by choosing the arbitrary n_ν appropriately small.

Note that when $\partial f_\nu^{(0)}(\mathcal{H}_{\nu 0},p_{\nu 20},p_{\nu 30})/\partial\mathcal{H}_\nu>0$, $\delta^2 H<0$ is possible without imposing any conditions on $\mathbf{k}_{23}$. *It is not necessary either to assume large derivatives of the perturbations across the magnetic field.*

If $\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu<0$ for some $\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3}$ around $\mathcal{H}_{\nu 0},p_{\nu 20},p_{\nu 30}$ corresponding to PPM, and for any particle species ν, one again localizes Ψ_ν around $\mathcal{H}_{\nu 0},p_{\nu 20},p_{\nu 30}$. All other Ψ_μ, and all Ψ_ν for the PNPM are set equal to zero. The positive contribution of $[d_\nu\Psi_\nu]^2=[\partial\Psi_\nu/\partial t]^2$ to the integral in Eq. (41) can be eliminated by choosing $\Psi_\nu=\Psi_\nu(\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3})$. In this case, Eq. (41) reduces to

$$\delta^2 H=(\delta^2 H)_{\rm PPM}=\sum_\nu\int\frac{s_0}{4m_\nu^3}d\mathcal{H}_\nu dp_{\nu 2}dp_{\nu 3}\frac{4\pi^2}{\tau}\Psi_\nu^2\left[-\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\right]\left[n_\nu+\frac{\tau}{2\pi}\langle \mathbf{k}_{23}\cdot\mathbf{v}\rangle\right]\left[n_\nu+\frac{\tau}{2\pi}\frac{1}{m_\nu}\frac{1}{\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}}\left\langle \mathbf{k}_{23}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\bigg|_x\right\rangle\right]. \quad (48)$$

Since Ψ_ν is localized around $\mathcal{H}_{\nu 0},p_{\nu 20},p_{\nu 30}$, the condition for $\delta^2 H<0$ is

$$\left[-\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\right]\left[n_\nu+\frac{\tau}{2\pi}\langle \mathbf{k}_{23}\cdot\mathbf{v}\rangle\right]\left[n_\nu+\frac{\tau}{2\pi}\frac{1}{m_\nu}\frac{1}{\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}}\left\langle \mathbf{k}_{23}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\bigg|_x\right\rangle\right]<0 , \quad (49)$$

or, equivalently, when Eq. (42) is taken into account,

$$\left[-\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\right]\left[n_\nu+\frac{\tau}{2\pi}\langle \mathbf{k}_{23}\cdot\mathbf{v}\rangle\right]\left[n_\nu+\frac{\tau}{2\pi}\langle \mathbf{k}_{23}\cdot\mathbf{v}\rangle+\frac{\tau}{2\pi}\frac{1}{\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}}\left[k_2\frac{\partial f_\nu^{(0)}}{\partial p_{\nu 2}}+k_3\frac{\partial f_\nu^{(0)}}{\partial p_{\nu 3}}\right]\right]<0 . \quad (50)$$

If $\langle \mathbf{k}_{23}\cdot\mathbf{v}\rangle\langle \mathbf{k}_{23}\cdot(\partial f_\nu^{(0)}/\partial\mathbf{v})|_x\rangle>0$, it is clear that choosing $n_\nu=0$ satisfies inequality (49) *without any conditions being imposed on* $\mathbf{k}_{23}$, *except* $\mathbf{k}_{23}\neq 0$. For a homogeneous plasma with constant $\mathbf{B}^{(0)}$, and choosing $\mathbf{k}_{23}=k_{\parallel 0}\mathbf{e}_B$, one obtains the result of Morrison and Pfirsch, Eq. (144.b) of Ref. [2], which was obtained in the context of drift-kinetic theory. For a y-dependent, force-free plasma slab, and choosing $\mathbf{k}_{23}=k_{\parallel 0}\mathbf{e}_B(y=y_0)$, one obtains the result of Ref. [7], which is also valid for a guiding center plasma [9].

If $\langle \mathbf{k}_{23}\cdot\mathbf{v}\rangle\langle \mathbf{k}_{23}\cdot(\partial f_\nu^{(0)}/\partial\mathbf{v})|_x\rangle<0$, inequality (50) can also be satisfied. With the arguments of the mean values given explicitly for the sake of clarity, this inequality can be written as

$$\left[-\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\right]k_3^2\left[\frac{n_\nu}{k_3}+b_\nu\left[\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3};\frac{k_2}{k_3}\right]\right]\left[\frac{n_\nu}{k_3}+b_\nu\left[\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3};\frac{k_2}{k_3}\right]+h_\nu\left[\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3};\frac{k_2}{k_3}\right]\right]<0 , \quad (51)$$

where

$$b_\nu\left[\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3};\frac{k_2}{k_3}\right]=\frac{\tau}{2\pi}\left\langle\frac{\mathbf{k}_{23}}{k_3}\cdot\mathbf{v}\right\rangle , \tag{52}$$

$$h_\nu\left[\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3};\frac{k_2}{k_3}\right]=\frac{\tau}{2\pi}\frac{1}{\dfrac{\partial f_\nu^{(0)}}{\partial\mathcal{H}_\nu}}\times\left[\frac{k_2}{k_3}\frac{\partial f_\nu^{(0)}}{\partial p_{\nu 2}}+\frac{\partial f_\nu^{(0)}}{\partial p_{\nu 3}}\right] . \tag{53}$$

Instead of prescribing the two arbitrary components k_2,k_3, one can consider k_3 and the quotient k_2/k_3 as independent of each other.

Inequality (51) is satisfied if one factor is positive and the other is negative. This is the case (since $-\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu>0$ here) if either

$$0<\frac{n_\nu}{k_3}+b_\nu<-h_\nu \quad \text{for } h_\nu<0 \tag{54}$$

or

$$-h_\nu<\frac{n_\nu}{k_3}+b_\nu<0 \quad \text{for } h_\nu>0 , \tag{55}$$

which can always be satisfied by choosing the arbitrary n_ν/k_3 correspondingly.

Inequalities (54) and (55) extend to the general one-dimensional case the results obtained for a homogeneous plasma in Ref. [6], and for a force-free plasma slab with shear in Ref. [7]. The quantity

$$1+\frac{h_\nu}{b_\nu} \tag{56}$$

can be interpreted as the local anisotropy of the distribution function in phase space, and coincides with the previous definition of the anisotropy in the homogeneous and force-free cases.

It has just been shown that, when $\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu>0$, it is always possible to have $\delta^2H<0$ without any restriction on $\mathbf{k}_{23}$ or the spatial variation of the perturbation across $\mathbf{B}^{(0)}$. When $\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu<0$ and $\langle\mathbf{k}_{23}\cdot\mathbf{v}\rangle\langle\mathbf{k}_{23}\cdot(\partial f_\nu^{(0)}/\partial\mathbf{v})|_\mathbf{x}\rangle>0$, it is also possible to have $\delta^2H<0$ without any restriction on $\mathbf{k}_{23}$, except $\mathbf{k}_{23}\neq 0$, and without any restrictions on the spatial variation of the perturbation across $\mathbf{B}^{(0)}$. In the case where $\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu<0$ and $\langle\mathbf{k}_{23}\cdot(\partial f_\nu^{(0)}/\partial\mathbf{v})|_\mathbf{x}\rangle<0$, $\delta^2H<0$ is also possible. In this case, however, n_ν/k_3 is restricted by inequalities (54) or (55), which reflect the explicit dependence of the equilibrium distribution function on the canonical momenta, i.e., the (local) anisotropy of $f_\nu^{(0)}$.

If $\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu<0$ and $\partial f_\nu^{(0)}/\partial p_{\nu 2}=0$, $\partial f_\nu^{(0)}/\partial p_{\nu 3}=0$, then $h_\nu=0$ for $\mathcal{H}_\nu=\mathcal{H}_{\nu 0}$, $p_{\nu 2}=p_{\nu 20}$, $p_{\nu 3}=p_{\nu 30}$, the equilibrium distribution function is *locally* monotonically decreasing and isotropic, and inequality (51) cannot be satisfied for these $\mathcal{H}_{\nu 0},p_{\nu 20},p_{\nu 30}$. If $\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu<0$ and $h_\nu=0$ for all $\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3}$, then $f_\nu^{(0)}=f_\nu^{(0)}(\mathcal{H}_\nu)$, the equilibrium is everywhere isotropic and homogeneous, and there is no electric current since $\nabla\times\mathbf{B}^{(0)}=0$ in this case. The equilibrium distribution functions are monotonically decreasing functions of the particle energy, and no negative-energy modes are possible, in accordance with the general results obtained in Sec. II and Ref. [8].

B. Perturbation energy $(\delta^2H)_{\mathrm{PNPM}}$ for particles with nonperiodic motion

It should be noted that the particles with nonperiodic motion usually do not have the same importance as those with periodic motion. For instance, in a homogeneous equilibrium, there are only particles with periodic motion (Appendix C and Ref. [7]); also, in the case of a force-free plane slab configuration, the overwhelming majority of particles perform a periodic motion, as shown in Ref. [7]. The particles with nonperiodic motion, however, must be taken into account when the equilibrium distribution functions allow arbitrarily large velocities and energies, e.g., when one considers Maxwell-like distributions, which could then lead to the particles being untrapped and having a nonperiodic motion, as described in Appendix C.

To study the sign of $(\delta^2H)_{\mathrm{PNPM}}$, Eq. (37), one again has to distinguish the following two cases.

1. $k_2=k_3=0$, *perpendicular wave propagation*

It follows immediately from Eq. (37) that $\delta^2H<0$ if $\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu>0$ for some $\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3}$ corresponding to PNPM, and for any particle species ν. This means that the presence of a local minimum with respect to $\mathcal{H}_\nu$ in $f_\nu^{(0)}(\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3})$ guarantees $\delta^2H<0$, *without any restrictions on the spatial variation of the perturbations perpendicular to* $B^{(0)}$*:* it suffices to localize Ψ_ν ($d_n\Psi_\nu$ is then also localized) to the region in $\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3}$, where $\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu>0$. Outside this region Ψ_ν vanishes. All other Ψ_μ are set equal to zero. The Ψ_ν corresponding to the PPM are likewise all set equal to zero, so that $(\delta H)_{\mathrm{PPM}}=0$. The sign of $\delta^2H=(\delta^2H)_{\mathrm{PNPM}}$ is then determined only by the sign of the integrand in the region of localization, which is then negative.

2. $k_2\neq 0$ *and/or* $k_3\neq 0$, $\mathbf{k}_{23}\cdot\mathbf{B}^{(0)}\neq 0$

If $\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu>0$ for some $\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3}$, the positive contribution of the term dependent on k_2,k_3 in Eq. (37) can be completely eliminated with the help of the arbitrary C_ν. Then the same line of reasoning as in the preceding case shows that δ^2H is negative.

If $\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu<0$ for some $\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3}$, the positive contribution of $[d_\nu\Psi_\nu]^2$ can be eliminated by choosing Ψ_ν as a function of the constants of the motion only, i.e., $\Psi_\nu=\Psi_\nu(\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3})$, $d_\nu\Psi_\nu=0$, and the contribution of C_ν^2 is eliminated by choosing $C_\nu=0$. No condition is imposed on k_2,k_3 or, alternatively, on $\mathbf{k}_{23}$, except $\mathbf{k}_{23}\neq 0$. If the equilibrium is locally monotonically decreasing and isotropic, i.e., if $\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu<0$ and $\partial f_\nu^{(0)}/\partial p_{\nu 2}=0$, $\partial f_\nu^{(0)}/\partial p_{\nu 3}=0$ for $\mathcal{H}_\nu=\mathcal{H}_{\nu 0}$, $p_{\nu 2}=p_{\nu 20}$, and $p_{\nu 3}=p_{\nu 30}$, then δ^2H cannot be made negative at these values of $\mathcal{H}_\nu$, $p_{\nu 2}$, and $p_{\nu 3}$, as was also the case for the PPM.

It has just been shown that if there is nonmonotonicity

$(\partial f_\nu^{(0)}/\partial \mathcal{H}_\nu > 0)$ for some $\mathcal{H}_\nu, p_{\nu 2}, p_{\nu 3}$ corresponding to particles with nonperiodic motion, $\delta^2 H$ can be made negative without imposing any condition on $\mathbf{k}_{23}$. If $f_\nu^{(0)}$ is locally monotonic $(\partial f_\nu^{(0)}/\partial \mathcal{H}_\nu < 0)$, but anisotropic $(\partial f_\nu^{(0)}/\partial p_{\nu 2} \neq 0$ and/or $\partial f_\nu^{(0)}/\partial p_{\nu 3} \neq 0$; this local anisotropy in phase space does not exclude isotropic pressure tensors), $\delta^2 H$ can also be made negative without imposing any condition on $\mathbf{k}_{23}$, except $\mathbf{k}_{23} \neq 0$. No restrictive assumptions have to be made concerning the behavior of the perturbations across the magnetic field.

VI. CONCLUSIONS

The general expression for the free energy $\delta^2 H$ available upon arbitrary perturbations of general Vlasov-Maxwell equilibria derived by Morrison and Pfirsch [1] was transformed to a relatively simple and compact expression [Eqs. (6) and (12)] which is very convenient for applications. From this expression, a previous result of Weitzner and Pfirsch [8] is immediately obtained, namely, that equilibria for which the equilibrium distribution functions depend only on the particle energy and are monotonically decreasing do not allow negative-energy perturbations.

The general expression for the perturbation energy is then evaluated for *arbitrary, double-symmetric,* i.e., *one-dimensional,* equilibria. In generalized coordinates q_1, q_2, q_3, such equilibria depend only on q_1, the equilibrium magnetic field $\mathbf{B}^{(0)}$ is perpendicular to ∇q_1, $\mathbf{B}^{(0)} \cdot \nabla q_1 = 0$, and the equilibrium distribution functions of each particle species ν are of the general form $f_\nu^{(0)} = f_\nu^{(0)}(\mathcal{H}_\nu, p_{\nu 2}, p_{\nu 3})$, where $\mathcal{H}_\nu$ is the (conserved) particle energy and $p_{\nu 2}, p_{\nu 3}$ are the (conserved) canonical momenta corresponding to the two ignorable coordinates q_2, q_3, respectively. For these equilibria, the following results are obtained.

Perturbations of negative energy $(\delta^2 H < 0)$ exist for any local deviation from monotonicity (i.e., if $\partial f_\nu^{(0)}/\partial \mathcal{H}_\nu > 0$ for some $\mathcal{H}_\nu, p_{\nu 2}, p_{\nu 3}$) of the distribution function of any of the particle species ν, and *for any wave vector* $\mathbf{k}_{23} = k^2 \nabla q_2 + k_3 \nabla q_3$, without restrictions on the behavior of the perturbations across the equilibrium magnetic field, i.e., large gradients of the perturbations across $\mathbf{B}^{(0)}$ *are not needed.*

If $\partial f_\nu^{(0)}/\partial \mathcal{H}_\nu < 0$, only waves with $\mathbf{k}_{23} \neq 0$ (which therefore have a component in the direction of $\mathbf{B}^{(0)}$ can possess negative energy.

For any $\partial f_\nu^{(0)}/\partial \mathcal{H}_\nu$, if $\langle \mathbf{k}_{23} \cdot \mathbf{v} \rangle \langle \mathbf{k}_{23} \cdot (\partial f_\nu^{(0)}/\partial \mathbf{v})|_{\mathbf{x}} \rangle > 0$ (the angular brackets mean averages along the unperturbed particle orbits), negative-energy perturbations also exist, *with no restriction on* $\mathbf{k}_{23}$, *except* $\mathbf{k}_{23} \neq 0$, and without requiring large gradients of the perturbations across $\mathbf{B}^{(0)}$.

If both $\partial f_\nu^{(0)}/\partial \mathcal{H}_\nu < 0$ and $\langle \mathbf{k}_{23} \cdot \mathbf{v} \rangle \langle \mathbf{k}_{23} \cdot (\partial f_\nu^{(0)}/\partial \mathbf{v})|_{\mathbf{x}} \rangle < 0$, but if $\partial f_\nu^{(0)}/\partial p_{\nu 2}$ and $\partial f_\nu^{(0)}/\partial p_{\nu 3}$ are not both identically zero for all ν, negative-energy perturbations also exist $[\partial f_\nu^{(0)}/\partial p_{\nu 2} = \partial f_\nu^{(0)}/\partial p_{\nu 3} \equiv 0$ for all ν only for equilibria which are everywhere isotropic, $f_\nu^{(0)} = f_\nu^{(0)}(\mathcal{H}_\nu)$, which therefore have no electric current; for all cases of practical interest, $\partial f_\nu^{(0)}/\partial p_{\nu 2}$ and $\partial f_\nu^{(0)}/\partial p_{\nu 3}$ do not vanish identically]. In this case, however, $\mathbf{k}_{23}$ is not completely arbitrary, since n_ν / k_3 is restricted by inequalities (54) and (55). As in the preceding situations, no large gradients across $\mathbf{B}^{(0)}$ are needed in this case either.

The results derived here include those previously obtained in the case of a homogeneous plasma [6], and in the case of a force-free plasma with shear [7]; they are, however, much more general since they apply to *all one-dimensional equilibria.*

APPENDIX A: RELATIONS FOR THE TRANSFORMATION OF THE PERTURBATION ENERGY

The second-order perturbation energy, Eq. (3), can be put in a very convenient form by means of the relations derived in this appendix. By taking the identities $(\partial/\partial\mathbf{v}) \times \mathbf{v} = \mathbf{0}$ and $(\partial/\partial\mathbf{v}) \times (\partial f_\nu^{(0)}/\partial\mathbf{v}) = \mathbf{0}$ into account the term

$$(\partial f_\nu^{(0)}/\partial\mathbf{v}) \cdot (e_\nu / m_\nu c) G_\nu \mathbf{v} \times (\partial/\partial\mathbf{v})[\mathbf{B}^{(0)} \cdot (\partial G_\nu / \partial\mathbf{x})]$$

can be expressed as the sum of a divergence in $\mathbf{v}$ space, which vanishes after integration, and another term, according to the equation

$$\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} \cdot \frac{e_\nu}{m_\nu c} G_\nu \mathbf{v} \times \frac{\partial}{\partial \mathbf{v}} \left[\mathbf{B}^{(0)} \cdot \frac{\partial G_\nu}{\partial \mathbf{x}} \right] = -\frac{e_\nu}{m_\nu c} G_\nu \frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{v}} \times \left[\left[\mathbf{B}^{(0)} \cdot \frac{\partial G_\nu}{\partial \mathbf{x}} \right] \mathbf{v} \right]$$
$$= \frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{e_\nu}{m_\nu c} G_\nu \left[\mathbf{B}^{(0)} \cdot \frac{\partial G_\nu}{\partial \mathbf{x}} \right] \frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} \times \mathbf{v} \right] + \frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} \cdot \frac{e_\nu}{m_\nu c} \left[\mathbf{B}^{(0)} \cdot \frac{G_\nu}{\partial \mathbf{x}} \right] \frac{\partial G_\nu}{\partial \mathbf{v}} \times \mathbf{v} \; . \tag{A1}$$

The term $-(\partial f_\nu^{(0)}/\partial\mathbf{v}) \cdot (e_\nu / m_\nu) G_\nu (\partial/\partial\mathbf{v}) \times [(\partial G_\nu / \partial\mathbf{x}) \times \mathbf{E}^{(0)}]$ can be similarly transformed to yield

$$-\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\frac{e_\nu}{m_\nu}G_\nu\frac{\partial}{\partial\mathbf{v}}\times\left[\frac{\partial G_\nu}{\partial\mathbf{x}}\times\mathbf{E}^{(0)}\right]=\frac{\partial}{\partial\mathbf{v}}\cdot\left[\frac{e_\nu}{m_\nu}G_\nu\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\times\left[\frac{\partial G_\nu}{\partial \mathbf{x}}\times\mathbf{E}^{(0)}\right]\right]-\frac{e_\nu}{m_\nu}\left[\frac{\partial G_\nu}{\partial\mathbf{x}}\times\mathbf{E}^{(0)}\right]\cdot\frac{\partial}{\partial\mathbf{v}}\times\left[G_\nu\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\right]$$
$$=\frac{\partial}{\partial\mathbf{v}}\cdot\left[\frac{e_\nu}{m_\nu}G_\nu\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\times\left[\frac{\partial G_\nu}{\partial \mathbf{x}}\times\mathbf{E}^{(0)}\right]\right]-\frac{e_\nu}{m_\nu}\left[\left[\frac{\partial G_\nu}{\partial\mathbf{x}}\times\mathbf{E}^{(0)}\right]\times\frac{\partial G_\nu}{\partial\mathbf{v}}\right]\cdot\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\;. \tag{A2}$$

The stationary Vlasov's equation is $d_\nu f_\nu^{(0)}=0$, where d_ν is the operator defined in Eqs. (4) and (5). With the help of the vector $\mathbf{F}_\nu^{(0)}$, defined by Eq. (7), namely,

$$\mathbf{F}_\nu^{(0)}=\frac{\partial f_\nu^{(0)}}{\partial\mathbf{x}}+\frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\;,$$

the equilibrium Vlasov's equation then takes the form given by Eq. (8):

$$d_\nu f_\nu^{(0)}=\mathbf{F}_\nu^{(0)}\cdot\mathbf{v}+\frac{e_\nu}{m_\nu}\mathbf{E}^{(0)}\cdot\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}=0,$$

and one has

$$\frac{\partial f_\nu^{(0)}}{\partial\mathbf{x}}\cdot\left[-\left[\frac{\partial G_\nu}{\partial\mathbf{x}}\cdot\frac{\partial G_\nu}{\partial\mathbf{v}}\right]\mathbf{v}+(d_\nu G_\nu)\frac{\partial G_\nu}{\partial\mathbf{v}}\right]=\left[\mathbf{F}_\nu^{(0)}-\frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\right]\cdot\left[-\left[\frac{\partial G_\nu}{\partial\mathbf{x}}\cdot\frac{\partial G_\nu}{\partial\mathbf{v}}\right]\mathbf{v}+(d_\nu G_\nu)\frac{\partial G_\nu}{\partial\mathbf{v}}\right]$$
$$=\frac{e_\nu}{m_\nu}\left[\frac{\partial G_\nu}{\partial\mathbf{x}}\cdot\frac{\partial G_\nu}{\partial\mathbf{v}}\right]\mathbf{E}^{(0)}\cdot\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}+(d_\nu G_\nu)\mathbf{F}_\nu^{(0)}\cdot\frac{\partial G_\nu}{\partial\mathbf{v}}$$
$$+\frac{e_\nu}{m_\nu c}\left[(\mathbf{v}\times\mathbf{B}^{(0)}\cdot\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\right]\left[\frac{\partial G_\nu}{\partial\mathbf{x}}\cdot\frac{\partial G_\nu}{\partial\mathbf{v}}\right]$$
$$+\frac{e_\nu}{m_\nu c}\left[\mathbf{B}^{(0)}\times\frac{\partial G_\nu}{\partial\mathbf{v}}\right]\cdot\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}(d_\nu G_\nu)$$
$$=\left[\mathbf{a}_\nu^{(0)}\cdot\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\right]\left[\frac{\partial G_\nu}{\partial\mathbf{x}}\cdot\frac{\partial G_\nu}{\partial\mathbf{v}}\right]+(d_\nu G_\nu)\mathbf{F}_\nu^{(0)}\cdot\frac{\partial G_\nu}{\partial\mathbf{v}}$$
$$+\frac{e_\nu}{m_\nu c}\left[\mathbf{B}^{(0)}\times\frac{\partial G_\nu}{\partial\mathbf{v}}\right]\cdot\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}(d_\nu G_\nu)\;. \tag{A3}$$

The last term in Eq. (A1) can also be further transformed if one takes into account the relation

$$\frac{e_\nu}{m_\nu c}\left[\mathbf{B}^{(0)}\cdot\frac{\partial G_\nu}{\partial\mathbf{x}}\right]\frac{\partial G_\nu}{\partial\mathbf{v}}\times\mathbf{v}=\frac{e_\nu}{m_\nu c}\frac{\partial G_\nu}{\partial\mathbf{v}}\times\left[\frac{\partial G_\nu}{\partial\mathbf{x}}\times(\mathbf{v}\times\mathbf{B}^{(0)})+\left[\mathbf{v}\cdot\frac{\partial G_\nu}{\partial\mathbf{x}}\right]\mathbf{B}^{(0)}\right]$$
$$=\frac{\partial G_\nu}{\partial\mathbf{v}}\times\left[\frac{\partial G_\nu}{\partial\mathbf{x}}\times\mathbf{a}_\nu^{(0)}-\frac{e_\nu}{m_\nu}\frac{\partial G_\nu}{\partial\mathbf{x}}\times\mathbf{E}^{(0)}+\frac{e_\nu}{m_\nu c}(d_\nu G_\nu)\mathbf{B}^{(0)}-\frac{e_\nu}{m_\nu c}\left[\mathbf{a}_\nu^{(0)}\cdot\frac{\partial G_\nu}{\partial\mathbf{v}}\right]\mathbf{B}^{(0)}\right]. \tag{A4}$$

APPENDIX B: CONVENIENT REPRESENTATION OF DERIVATIVES IN x-v SPACE

For time-independent equilibrium fields $\mathbf{E}^{(0)}=-\nabla\Phi^{(0)}$ and $\mathbf{B}^{(0)}=\nabla\times\mathbf{A}^{(0)}$, the particle energy $\mathcal{H}_\nu=(m_\nu/2)(\mathbf{v})^2+e_\nu\Phi^{(0)}$ is a constant of the motion. The equilibrium distribution functions $f_\nu^{(0)}$ can be written as

$$f_\nu^{(0)}(\mathbf{x},\mathbf{v})=f_\nu^{(0)}(\mathcal{H}_\nu(\mathbf{x},\mathbf{v}),\mathcal{K}_{\nu\kappa}(\mathbf{x},\mathbf{v}))\;,$$

where κ runs over as many indices as there are other constants of the motion $\mathcal{K}_{\nu\kappa}$ in the problem under consideration. The derivatives of $f_\nu^{(0)}$ are then

$$\left.\frac{\partial f_\nu^{(0)}}{\partial\mathbf{x}}\right|_{\mathbf{v}}=-e_\nu\mathbf{E}^{(0)}\left.\frac{\partial f_\nu^{(0)}}{\partial\mathcal{H}_\nu}\right|_{\mathcal{K}_{\nu\kappa}}+\left.\frac{\partial\mathcal{K}_{\nu\kappa}}{\partial\mathbf{x}}\right|_{\mathbf{v}}\left.\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu\kappa}}\right|_{\mathcal{H}_\nu}, \tag{B1}$$

$$\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right|_{\mathbf{x}} = m_\nu \mathbf{v}\left.\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\right|_{\mathcal{K}_{\nu\kappa}} + \left.\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{v}}\right|_{\mathbf{x}} \left.\frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu\kappa}}\right|_{\mathcal{H}_\nu}, \tag{B2}$$

where the summation convention has been adopted (the quantities kept constant when partial derivatives are calculated are given explicitly only when this is particularly convenient). The vector $\mathbf{F}_\nu^{(0)}$, Eq. (7), is then

$$\mathbf{F}_\nu^{(0)} = -m_\nu \mathbf{a}_\nu^{(0)} \frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} + \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu\kappa}}\left[\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{x}} + \frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)} \times \frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{v}}\right], \tag{B3}$$

and the quantity $\mathbf{F}_\nu^{(0)}\cdot(\partial G_\nu/\partial \mathbf{v}) - (\partial f_\nu^{(0)}/\partial \mathbf{v})\cdot(\partial G_\nu/\partial \mathbf{x})$, which appears in Eq. (6), takes the form

$$\begin{aligned}\mathbf{F}_\nu^{(0)}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}} - \frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}} &= -m_\nu\left[\mathbf{a}_\nu^{(0)}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}\right]\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} + \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu\kappa}}\frac{\partial G_\nu}{\partial \mathbf{v}}\cdot\left[\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{x}} + \frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{v}}\right] \\ &\quad - m_\nu\left[\mathbf{v}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} - \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu\kappa}}\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{v}}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}} \\ &= -m_\nu(d_\nu G_\nu)\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} + \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu\kappa}}\frac{\partial G_\nu}{\partial \mathbf{v}}\cdot\left[\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{x}} + \frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{v}}\right] \\ &\quad - \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu\kappa}}\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{v}}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\ .\end{aligned} \tag{B4}$$

Let $q_i(\mathbf{x})$, $i=1,\ldots,3$ now be generalized coordinates with covariant basis $\partial\mathbf{x}/\partial q_i$ and contravariant basis $\partial q_i/\partial \mathbf{x} = \nabla q_i$. The corresponding covariant and contravariant velocity components are, respectively,

$$v_i(\mathbf{x},\mathbf{v}) = \mathbf{v}\cdot\frac{\partial \mathbf{x}}{\partial q_i} \tag{B5}$$

and

$$v^i(\mathbf{x},\mathbf{v}) = \mathbf{v}\cdot\nabla q_i = \dot{q}_i\ . \tag{B6}$$

Since $\mathbf{v} = v_i \nabla q_i$, one has

$$\left.\frac{\partial \mathbf{v}}{\partial v_i}\right|_{\mathbf{x}} = \nabla q_i\ . \tag{B7}$$

With $\mathcal{K}_{\nu\kappa}(\mathbf{x},\mathbf{v})$ taken as $\mathcal{K}_{\nu\kappa}(\mathbf{x}, v_i(\mathbf{x},\mathbf{v}))$, the derivatives are

$$\left.\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{x}}\right|_{\mathbf{v}} = \left.\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{x}}\right|_{v_i} + \left.\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial v_i}\right|_{\mathbf{x}}\left.\frac{\partial v_i}{\partial \mathbf{x}}\right|_{\mathbf{v}} = \left.\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{x}}\right|_{v_i} + \left[\frac{\partial \mathbf{v}}{\partial v_i}\cdot\left.\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{v}}\right|_{\mathbf{x}}\right]\left.\frac{\partial v_i}{\partial \mathbf{x}}\right|_{\mathbf{v}} = \left.\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{x}}\right|_{v_i} + \left[\nabla q_i\cdot\left.\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{v}}\right|_{\mathbf{x}}\right]\left.\frac{\partial v_i}{\partial \mathbf{x}}\right|_{\mathbf{v}}, \tag{B8}$$

and correspondingly for G_ν,

$$\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{\mathbf{v}} = \left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{v_i} + \left[\nabla q_i\cdot\left.\frac{\partial G_\nu}{\partial \mathbf{v}}\right|_{\mathbf{x}}\right]\left.\frac{\partial v_i}{\partial \mathbf{x}}\right|_{\mathbf{v}}\ . \tag{B9}$$

Equations (B4), (B8), and (B9) then yield

$$\begin{aligned}\mathbf{F}_\nu^{(0)}\cdot\left.\frac{\partial G_\nu}{\partial \mathbf{v}}\right|_{\mathbf{x}} - \left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right|_{\mathbf{x}}\cdot\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{\mathbf{v}} &= -m_\nu(d_\nu G_\nu)\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} - \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu\kappa}}\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{v}}\cdot\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{v_i} \\ &\quad + \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu\kappa}}\frac{\partial G_\nu}{\partial \mathbf{v}}\cdot\left[\left.\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{x}}\right|_{v_i} + \left[\nabla q_i\cdot\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{v}}\right]\frac{\partial v_i}{\partial \mathbf{x}}\right. \\ &\qquad\qquad \left. + \frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{v}} - \left[\frac{\partial v_i}{\partial \mathbf{x}}\cdot\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{v}}\right]\nabla q_i\right] \\ &= -m_\nu(d_\nu G_\nu)\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} - \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu\kappa}}\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{v}}\cdot\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{v_i} \\ &\quad + \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu\kappa}}\frac{\partial G_\nu}{\partial \mathbf{v}}\cdot\left[\left.\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{x}}\right|_{v_i} + \frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial \mathcal{K}_{\nu\kappa}}{\partial \mathbf{v}}\right],\end{aligned} \tag{B10}$$

where the relation

$$\left[\nabla q_i\cdot\frac{\partial\mathcal{K}_{\nu\kappa}}{\partial\mathbf{v}}\right]\frac{\partial v_i}{\partial\mathbf{x}}-\left[\frac{\partial v_i}{\partial\mathbf{x}}\cdot\frac{\partial\mathcal{K}_{\nu\kappa}}{\partial\mathbf{v}}\right]\nabla q_i=\frac{\partial\mathcal{K}_{\nu\kappa}}{\partial\mathbf{v}}\times\left[\frac{\partial v_i}{\partial\mathbf{x}}\times\nabla q_i\right]$$

$$=\frac{\partial\mathcal{K}_{\nu\kappa}}{\partial\mathbf{v}}\times\left[\frac{\partial}{\partial\mathbf{x}}\times(v_i\nabla q_i)\right]=\frac{\partial\mathcal{K}_{\nu\kappa}}{\partial\mathbf{v}}\times\left[\frac{\partial}{\partial\mathbf{x}}\times\mathbf{v}\right]=0 \tag{B11}$$

has been taken into account. The derivatives of G_ν and $\mathcal{K}_{\nu\kappa}$ with respect to $\mathbf{x}$ on the right-hand side of Eq. (B10) are now performed at constant $v_i(\mathbf{x},\mathbf{v})=\mathbf{v}\cdot(\partial\mathbf{x}/\partial q_i)$, and *not at constant* $\mathbf{v}$.

APPENDIX C: PARTICLE ORBITS: PERIODIC MOTION AND NONPERIODIC MOTION IN ONE-DIMENSIONAL EQUILIBRIA

The extremization of the second-order perturbation energy for configurations in which the equilibrium quantities depend on only one spatial coordinate q_1 involves the determination of the constant of the motion $C_\nu(\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3})$, Eq. (34). To determine this constant, it is necessary to know whether the particles with given conserved energy $\mathcal{H}_\nu$ and conserved momenta $p_{\nu 2},p_{\nu 3}$ perform a periodic motion in (q_1,v_1,v_2,v_3) space. If $q_1(t)$ is a periodic variable, then so are $v_2(t)$ and $v_3(t)$, because, at constant $p_{\nu 2}$ and $p_{\nu 3}$, respectively, they depend only on q_1 [Eq. (C2)]. $v_1(t)$ is then also periodic since it depends only on $v^1=\dot q_1$, v_2, v_3, and q_1 [Eq. (C5)]. This problem is investigated in this and in the following appendix.

Owing to the fact that the canonical momenta $p_{\nu 2},p_{\nu 3}$ are constants of the motion, the particles moving in the magnetic field

$$\mathbf{B}^{(0)}=\nabla\times[A_i^{(0)}(q_1)\nabla q_i]$$

$$=\frac{1}{J(q_1)}\left[-\frac{dA_3^{(0)}}{dq_1}\frac{\partial\mathbf{x}}{\partial q_2}+\frac{dA_2^{(0)}}{dq_1}\frac{\partial\mathbf{x}}{\partial q_3}\right]$$

can be considered as effectively being in a one-dimensional potential $V_\nu(q_1)$, as will presently be shown. (One can choose $A_1^{(0)}\equiv 0$ without loss of generality.)

The Hamiltonian H_ν of a particle of species ν is given by

$$H_\nu=\frac{m_\nu}{2}v^i v_i+e_\nu\Phi^{(0)}(q_1),\quad i=1,\ldots,3\ , \tag{C1}$$

where the velocities v^i,v_i are related to the canonical momenta p_ν by the equations

$$v_i=v_i(p_{\nu i},q_1)=\frac{1}{m_\nu}\left[p_{\nu i}-\frac{e_\nu}{c}A_i^{(0)}(q_1)\right]\ , \tag{C2}$$

$$v^i=g^{ik}v_k,\quad i,k=1,\ldots,3\ , \tag{C3}$$

$$g^{ik}=g^{ik}(q_1)=\nabla q_i\cdot\nabla q_k\ . \tag{C4}$$

From Eq. (C3) one obtains

$$v_1=\frac{v^1}{g^{11}}-\frac{g^{1\lambda}}{g^{11}}v_\lambda,\quad \lambda=2,3\ , \tag{C5}$$

and H_ν can be written as

$$H_\nu=\frac{m_\nu}{2}[v^1v_1+v^\mu v_\mu]+e_\nu\Phi^{(0)},\quad \mu=2,3\ , \tag{C6}$$

$$H_\nu=\frac{m_\nu}{2}[v^1v_1+g^{1\mu}v_1v_\mu+g^{\mu\lambda}v_\mu v_\lambda]+e_\nu\Phi^{(0)},\quad \mu,\lambda=2,3\ . \tag{C7}$$

Inserting v_1 from Eq. (C5) and taking into account that $v^1=\dot q_1$, one can express this as

$$H_\nu=\frac{m_\nu}{2}\frac{\dot q_1^2}{g^{11}}+V_\nu(p_{\nu 2},p_{\nu 3},q_1)\ , \tag{C8}$$

with

$$V_\nu(p_{\nu 2},p_{\nu 3},q_1)=\frac{m_\nu}{2}\left[-\frac{g^{1\mu}g^{1\lambda}}{g^{11}}v_\mu v_\lambda+g^{\mu\lambda}v_\mu v_\lambda\right]+e_\nu\Phi^{(0)},\quad \mu,\lambda=2,3\ . \tag{C9}$$

[The dependence on $p_{\nu 2},p_{\nu 3}$ is, of course, obtained from Eq. (C2).]

From Eq. (C8) one has, for a particle with *conserved energy* $\mathcal{H}_\nu$ and *conserved momenta* $p_{\nu 2},p_{\nu 3}$,

$$\dot q_1^2(\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3},q_1)=\frac{2g^{11}(q_1)}{m_\nu}[\mathcal{H}_\nu-V_\nu(p_{\nu 2},p_{\nu 3},q_1)]\ . \tag{C10}$$

If the potential $V_\nu(p_{\nu 2},p_{\nu 3},q_1)$, as a function of q_1, has troughs [for instance, a trough between two maxima $V_{\nu\max}(q_{1A})$ and $V_{\nu\max}(q_{1B})$], a particle of energy $\mathcal{H}_\nu$ is trapped if $\mathcal{H}_\nu<\min(V_{\nu\max})$, where $\min(V_{\nu\max})$ is the smaller of the two maxima $V_{\nu\max}(q_{1A}),V_{\nu\max}(q_{1B})$. Otherwise, the particle is untrapped. For trapped particles, the motion is periodic; the particle oscillates between the turning points $q_{1\alpha}$ and $q_{1\beta}$, which can be determined from Eq. (C10) by setting $\dot q_1=0$ and solving for q_1. A detailed discussion of the particle orbits can be found in Ref. [7] for the special case of a force-free plane slab configuration. In that case, it was found that the overwhelming majority of particles were trapped and performed a periodic motion.

As an example, consider a *homogeneous equilibrium* with no electric field, $\Phi^{(0)}=0$, constant magnetic field $\mathbf{B}^{(0)}=B^{(0)}\mathbf{e}_z$, and vector potential $\mathbf{A}^{(0)}=B^{(0)}(x-x_0)\mathbf{e}_y$, i.e., $A_x^{(0)}=0$, $A_y^{(0)}=B^{(0)}(x-x_0)$, $A_z^{(0)}=0$. Therefore

$$q_1=x,\quad q_2=y,\quad q_3=z\ , \tag{C11}$$

and

$$p_{\nu x}=m_\nu v_x,\quad p_{\nu y}=m_\nu v_y+\frac{e_\nu}{c}B^{(0)}(x-x_0)\ ,$$
$$p_{\nu z}=m_\nu v_z\ . \tag{C12}$$

The potential, Eq. (C9), is

$$V_\nu=\frac{m_\nu}{e}[v_y^2(p_{\nu y},x)+v_z^2(p_{\nu z})]$$
$$=\frac{m_\nu}{2}\left[\left[\frac{p_{\nu y}}{m_\nu}-\omega_\nu(x-x_0)\right]^2+\frac{p_{\nu z}^2}{m_\nu^2}\right]\ , \tag{C13}$$

where $\omega_\nu=e_\nu B^{(0)}/cm_\nu$ is the gyration frequency. Since the potential is parabolic, *all particles are trapped* and have a periodic motion. The turning points follow from Eq. (C10) with $v_x=0$:

$$(x-x_0)_{\alpha,\beta}=\frac{1}{\omega_\nu}\left[\frac{p_{\nu y}}{m_\nu}\pm\left[\frac{2}{m_\nu}\mathcal{H}_\nu-\frac{p_{\nu z}^2}{m_\nu^2}\right]^{1/2}\right]$$
$$=\frac{1}{\omega_\nu}[v_y\pm v_y]$$
$$=R_g\pm R_g\ , \tag{C14}$$

where R_g is the gyroradius.

The general form of the equilibrium distribution functions, which do not depend on either y or z is, in this case, $f_\nu^{(0)}=f_\nu^{(0)}(\mathcal{H}_\nu,p_{\nu y},p_{\nu z})$. However, if it is further required that $\partial f_\nu^{(0)}/\partial \mathbf{x}|_\nu=0$, then $f_\nu^{(0)}=f_\nu^{(0)}(\mathcal{H}_\nu,p_{\nu z})$. This case, which is a special case of the theory developed here, was treated in Ref. [6].

APPENDIX D: CONSTANT OF THE MOTION $C_\nu(\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3})$

The minimization of the perturbation energy $\delta^2 H$, Eq. (30), with respect to $\Gamma_\nu(q_1,v_1,v_2,v_3)$ leads to an equation of the form

$$\left[\frac{d\Gamma_\nu}{dt}\right]_{\text{along orbit}}+\alpha_1(q_1,v_1,v_2,v_3)$$
$$=\alpha_2(q_1,v_1,v_2,v_3)C_\nu(\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3})\ . \tag{D1}$$

This follows from Eq. (34) since $d_\nu\Gamma_\nu$ is the rate of change experienced by the moving particle along the unperturbed orbit, i.e., $d_\nu\Gamma_\nu=(d\Gamma_\nu/dt)_{\text{along orbit}}$.

The question of interest in determining the constant of the motion $C_\nu(\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3})$ is whether the motion of the particle in (q_1,v_1,v_2,v_3) space is periodic or not. If it is periodic, $q_1(t)$, $v_1(t)$, $v_2(t)$, and $v_3(t)$ are periodic functions along the unperturbed particle orbits, with period τ,

$$\tau=\oint_{q_1}dt\ . \tag{D2}$$

These particles constitute the group of particles with periodic motion. All other particles are the particles with nonperiodic motion.

For particles with periodic motion, mean values along the unperturbed orbits are now defined by the expression

$$\langle\cdots\rangle=\left[\int_{t_0}^{t_0+\tau}\cdots dt\right]\Big/\tau\ . \tag{D3}$$

Integrating Eq. (D1) between t and $t+\tau$ along orbits yields

$$\Gamma_\nu(q_1(t+\tau),v_1(t+\tau),v_2(t+\tau),v_3(t+\tau))-\Gamma_\nu(q_1(t),v_1(t),v_2(t),v_3(t))+\tau\langle\alpha_1\rangle=\tau\langle\alpha_2\rangle C_\nu\ . \tag{D4}$$

Since, for the particles with periodic motion, $q_1(t+\tau)=q_1(t)$, etc., $\Gamma_\nu(t+\tau)-\Gamma_\nu(t)$ in this equation is then determined from the fact that the generating function $g_\nu(q_1,v_1,v_2,v_3)$ for the perturbations, Eq. (29), must be a single-valued function of its variables. Since

$$g_\nu(q_1(t),\ldots)=\Psi_\nu(q_1(t),\ldots)e^{i\Gamma_\nu(q_1(t),\ldots)}\ , \tag{D5}$$

this means that the functions Ψ_ν and Γ_ν are subject to the periodicity conditions

$$\Psi_\nu(q_1(t+\tau),\ldots)=\Psi_\nu(q_1(t),\ldots) \tag{D6}$$

and

$$\Gamma_\nu(q_1(t+\tau),\ldots)=\Gamma_\nu(q_1(t),\ldots)+2\pi n_\nu\ , \tag{D7}$$

n_ν being *any integer number,* i.e., $n_\nu=0\pm1,\ldots$. This then determines C_ν for the PPM from Eq. (D4) as

$$C_\nu=\frac{1}{\langle\alpha_2\rangle}\left[\langle\alpha_1\rangle+\frac{2\pi}{\tau}n_\nu\right]\ , \tag{D8}$$

where, explicitly, from Eq. (34),

$$\alpha_1=\frac{1}{2\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}}\left[2\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\mathbf{k}_{23}\cdot\mathbf{v}+k_2\frac{\partial f_\nu^{(0)}}{\partial p_{\nu 2}}+k_3\frac{\partial f_\nu^{(0)}}{\partial p_{\nu 3}}\right]$$

$$=\frac{1}{2\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}}\left[2\mathbf{k}_{23}\cdot\frac{1}{m_\nu}\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right|_{\mathbf{x}}-k_2\frac{\partial f_\nu^{(0)}}{\partial p_{\nu 2}}-k_3\frac{\partial f_\nu^{(0)}}{\partial p_{\nu 3}}\right]$$

$$=\frac{1}{2\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}}\left[\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\mathbf{k}_{23}\cdot\mathbf{v}+\mathbf{k}_{23}\cdot\frac{1}{m_\nu}\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right|_{\mathbf{x}}\right], \tag{D9}$$

$$\alpha_2=\frac{1}{2\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}}\Psi_\nu^2 . \tag{D10}$$

On the other hand, for particles with nonperiodic motion, Eq. (D1) imposes no restriction on C_ν. $C_\nu(\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3})$ can be chosen *arbitrarily* for the PNPM.

APPENDIX E: COORDINATE SYSTEM $t,\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3}$ IN q_1-v_i SPACE

It is useful to introduce coordinates which make discussion of the expression for the perturbation energy particularly simple. Let the new coordinates be defined by the following transformations:

$$\begin{bmatrix} q_1\\ v_1\\ v_2\\ v_3\end{bmatrix}\Longrightarrow\begin{bmatrix} t=\int_{q_{10}}^{q_1}\frac{d\hat{q}_1}{\dot{q}_1}\\ \mathcal{H}_\nu\\ p_{\nu 2}\\ p_{\nu 3}\end{bmatrix} . \tag{E1}$$

Therefore one has

$$\frac{\partial t}{\partial q_1}=\frac{1}{\dot{q}_1},\quad \frac{\partial t}{\partial v_1}=0,\quad \frac{\partial t}{\partial v_2}=0,\quad \frac{\partial t}{\partial v_3}=0 . \tag{E2}$$

From the expressions for the particle energy, viz.,

$$\mathcal{H}_\nu=\frac{m_\nu}{2}g^{ij}(q_1)v_iv_j+e_\nu\Phi^{(0)}(q_1) , \tag{E3}$$

and for the canonical momenta, viz.,

$$p_{\nu\kappa}=m_\nu v_\kappa+\frac{e_\nu}{c}A_\kappa^{(0)}(q_1) , \tag{E4}$$

one obtains

$$\frac{\partial\mathcal{H}_\nu}{\partial q_1}=\frac{m_\nu}{2}\frac{\partial g^{ij}}{\partial q_1}v_iv_j+e_\nu\frac{\partial\Phi^{(0)}}{\partial q_1} , \tag{E5}$$

$$\frac{\partial\mathcal{H}_\nu}{\partial v_1}=\frac{m_\nu}{2}[g^{1j}v_j+g^{1i}v_i]$$
$$=\frac{m_\nu}{2}[v^1+v^1]$$
$$=m_\nu v^1=m_\nu\dot{q}_1 , \tag{E6}$$

$$\frac{\partial\mathcal{H}_\nu}{\partial v_2}=m_\nu\dot{q}_2,\quad \frac{\partial\mathcal{H}_\nu}{\partial v_3}=m_\nu\dot{q}_3 , \tag{E7}$$

$$\frac{\partial p_{\nu 2}}{\partial q_1}=\frac{e_\nu}{c}\frac{\partial A_2^{(0)}}{\partial q_1},\quad \frac{\partial p_{\nu 2}}{\partial v_1}=0 ,\quad \frac{\partial p_{\nu 2}}{\partial v_2}=m_\nu,\quad \frac{\partial p_{\nu 2}}{\partial v_3}=0 , \tag{E8}$$

$$\frac{\partial p_{\nu 3}}{\partial q_1}=\frac{e_\nu}{c}\frac{\partial A_3^{(0)}}{\partial q_1},\quad \frac{\partial p_{\nu 3}}{\partial v_1}=0 ,\quad \frac{\partial p_{\nu 2}}{\partial v_2}=0,\quad \frac{\partial p_{\nu 2}}{\partial v_3}=m_\nu . \tag{E9}$$

These relations enable one to calculate the Jacobian of the transformation,

$$\Delta=\frac{\partial(t,\mathcal{H}_\nu,p_{\nu 2},p_{\nu 3})}{\partial(q_1,v_1,v_2,v_3)}=\begin{vmatrix}\frac{\partial t}{\partial q_1}&\frac{\partial t}{\partial v_1}&\frac{\partial t}{\partial v_2}&\frac{\partial t}{\partial v_3}\\ \frac{\partial\mathcal{H}_\nu}{\partial q_1}&\frac{\partial\mathcal{H}_\nu}{\partial v_1}&\frac{\partial\mathcal{H}_\nu}{\partial v_2}&\frac{\partial\mathcal{H}_\nu}{\partial v_3}\\ \frac{\partial p_{\nu 2}}{\partial q_1}&\frac{\partial p_{\nu 2}}{\partial v_1}&\frac{\partial p_{\nu 2}}{\partial v_2}&\frac{\partial p_{\nu 2}}{\partial v_3}\\ \frac{\partial p_{\nu 3}}{\partial q_1}&\frac{\partial p_{\nu 3}}{\partial v_1}&\frac{\partial p_{\nu 3}}{\partial v_2}&\frac{\partial p_{\nu 3}}{\partial v_3}\end{vmatrix} . \tag{E10}$$

One obtains

$$\Delta=m_\nu^3 , \tag{E11}$$

and the new coordinate system is therefore well defined.

Taking into account Eq. (B7), one obtains from the relations

$$dt\,d\mathcal{H}_\nu dp_{\nu 2}dp_{\nu 3}=\Delta dq_1dv_1dv_2dv_3 \tag{E12}$$

and

$$d^3v=\frac{\partial \mathbf{v}}{\partial v_1}\cdot\frac{\partial \mathbf{v}}{\partial v_2}\times\frac{\partial \mathbf{v}}{\partial v_3}dv_1dv_2dv_3$$
$$=\nabla q_1\cdot\nabla q_2\times\nabla q_3 dv_1dv_2dv_3$$
$$=\frac{1}{J(q_1)}dv_1dv_2dv_3\ , \qquad (E13)$$

the volume element in q_1-$\mathbf{v}$ space,

$$dq_1d^3v=\frac{1}{J(q_1)m_\nu^3}dt\,d\mathcal{H}_\nu dp_{\nu 2}dp_{\nu 3}\ . \qquad (E14)$$

APPENDIX F: NEGLECT OF THE ELECTROSTATIC ENERGY TERM

The contribution of the electrostatic energy term

$$\frac{1}{8\pi}\int d^3x\,\delta E^2 \qquad (F1)$$

has been neglected. To justify this, let us consider the perturbed electric charge density $\delta\rho$. Generally, the charge density is

$$\rho=\sum_\nu e_\nu\int f_\nu d^3v\ , \qquad (F2)$$

and the perturbed charge density is

$$\delta\rho=\sum_\nu e_\nu\int \delta f_\nu d^3v\ . \qquad (F3)$$

The perturbation in the distribution function is given by

$$\delta f_\nu=\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{p}}\cdot\delta\mathbf{x}_\nu+\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{p}_\nu}\right|_{\mathbf{x}}\cdot\delta\mathbf{p}_\nu\ , \qquad (F4)$$

with $\mathbf{p}_\nu=m_\nu\mathbf{v}+(e_\nu/c)\mathbf{A}^{(0)}$ the canonical momentum of species ν, given by Eq. (16). It therefore follows that

$$\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right|_{\mathbf{x}}=m_\nu\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{p}_\nu}\right|_{\mathbf{x}}\ , \qquad (F5)$$

$$\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{v}}=\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{p}}+\left.\frac{\partial p_{\nu i}}{\partial \mathbf{x}}\right|_{\mathbf{v}}\left.\frac{\partial f_\nu^{(0)}}{\partial p_{\nu i}}\right|_{\mathbf{x}}$$
$$=\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{p}}+\frac{e_\nu}{c}\frac{\partial A_i^{(0)}}{\partial \mathbf{x}}\left.\frac{\partial f_\nu^{(0)}}{\partial p_{\nu i}}\right|_{\mathbf{x}}\ , \qquad (F6)$$

$$\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{p}}=\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{v}}-\frac{e_\nu}{c}\frac{\partial A_i^{(0)}}{\partial \mathbf{x}}\left.\frac{\partial f_\nu^{(0)}}{\partial p_{\nu i}}\right|_{\mathbf{x}}\ ,$$
$$=\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{v}}-\frac{e_\nu}{m_\nu c}\frac{\partial A_i^{(0)}}{\partial \mathbf{x}}\left.\frac{\partial f_\nu^{(0)}}{\partial v_i}\right|_{\mathbf{x}}\ . \qquad (F7)$$

The perturbations $\delta\mathbf{x}_\nu$ and $\delta\mathbf{p}_\nu=m_\nu\delta\mathbf{v}+(e_\nu/c)\delta\mathbf{A}+(e_\nu/c)(\delta\mathbf{x}_\nu\cdot\nabla)\mathbf{A}^{(0)}$ are given by

$$\delta\mathbf{x}_\nu=\left.\frac{\partial G_\nu}{\partial \mathbf{p}_\nu}\right|_{\mathbf{x}}$$
$$=\frac{1}{m_\nu}\left.\frac{\partial G_\nu}{\partial \mathbf{v}}\right|_{\mathbf{x}}\ , \qquad (F8)$$

$$\delta\mathbf{p}_\nu=m_\nu\delta\mathbf{v}+\frac{e_\nu}{c}\delta\mathbf{A}+\frac{e_\nu}{c}(\delta\mathbf{x}_\nu\cdot\nabla)\mathbf{A}_{(0)}$$
$$=-\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{\mathbf{p}}$$
$$=-\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{\mathbf{v}}+\frac{e_\nu}{m_\nu c}\frac{\partial A_i^{(0)}}{\partial \mathbf{x}}\left.\frac{\partial G_\nu}{\partial v_i}\right|_{\mathbf{x}}\ . \qquad (F9)$$

Employing the relations above, one obtains δf_ν as a function of $\mathbf{x}$ and $\mathbf{v}$:

$$\delta f_\nu=\frac{1}{m_\nu}\left[\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}+\frac{e_\nu}{m_\nu c}\left[\mathbf{B}^{(0)}\times\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right]\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}-\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]\ , \qquad (F10)$$

which, with Eq. (7) taken into account, yields

$$\delta f_\nu=\frac{1}{m_\nu}\left[\mathbf{F}_\nu^{(0)}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}-\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]\ . \qquad (F11)$$

With Eqs. (B10), (18), (20), and (21) taken into account, δf_ν can be expressed as

$$\delta f_\nu=-(d_\nu G_\nu)\left.\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\right|_{p_{\nu\kappa}}$$
$$-\left[\frac{\partial \mathbf{x}}{\partial q_\kappa}\cdot\left.\frac{\partial G_\nu}{\partial_{\mathbf{x}}}\right|_{v_i}\right]\frac{\partial f_\nu^{(0)}}{\partial p_{\nu\kappa}}\ . \qquad (F12)$$

Since

$$\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{v_i}=\nabla q_\lambda\left.\frac{\partial G_\nu}{\partial q_\lambda}\right|_{v_i}\ , \qquad (F13)$$

one obtains

$$\frac{\partial \mathbf{x}}{\partial q_\kappa}\cdot\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{v_i}=\left.\frac{\partial G_\nu}{\partial q_\kappa}\right|_{v_i}\ . \qquad (F14)$$

With G_ν given by Eqs. (23) and (29), the following relations can be derived:

$$d_\nu G_\nu=\tfrac{1}{2}(d_\nu\Psi_\nu)[e^{i[\Gamma_\nu+k_2q_2+k_3q_3]}+e^{-i[\Gamma_\nu+k_2q_2+k_3q_3]}]+\frac{i}{2}\Psi_\nu[d_\nu\Gamma_\nu+\mathbf{k}_{23}\cdot\mathbf{v}][e^{i[\Gamma_\nu+k_2q_2+k_3q_3]}-e^{-i[\Gamma_\nu+k_2q_2+k_3q_3]}] \qquad (F15)$$

and

$$\left.\frac{\partial G_\nu}{\partial q_\kappa}\right|_{v_i}=\frac{i}{2}k_\kappa\Psi_\nu[e^{i[\Gamma_\nu+k_2q_2+k_3q_3]}-e^{-i[\Gamma_\nu+k_2q_2+k_3q_3]}],\quad \kappa=2,3\ , \qquad (F16)$$

which, together with Eqs. (F3) and (F12), yield

$$\delta\rho = -\sum_\nu \frac{e_\nu}{2}\int d^3v \left\{ \frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}[d_\nu \Psi_\nu][e^{i[\Gamma_\nu + k_2 q_2 + k_3 q_3]} + e^{-i[\Gamma_\nu + k_2 q_2 + k_3 q_3]}] + i\Psi_\nu \left[\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}[d_\nu \Gamma_\nu + \mathbf{k}_{23}\cdot\mathbf{v}] + k_2 \frac{\partial f_\nu^{(0)}}{\partial p_{\nu 2}} + k_3 \frac{\partial f_\nu^{(0)}}{\partial p_{\nu 3}} \right] [e^{i[\Gamma_\nu + k_2 q_2 + k_3 q_3]} - e^{-i[\Gamma_\nu + k_2 q_2 + k_3 q_3]}] \right\} . \tag{F17}$$

Taking $d_\nu \Psi_\nu = 0$, i.e., $\Psi_\nu = \Psi_\nu(\mathcal{H}_\nu, p_{\nu 2}, p_{\nu 3})$, does not have any influence whatsoever on the results obtained in Sec. V. In this case, the perturbed charge density is

$$\delta\rho = -\sum_\nu \frac{i}{2} e_\nu \int d^3v\, \Psi_\nu \left[\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}[d_\nu \Gamma_\nu + \mathbf{k}_{23}\cdot\mathbf{v}] + k_2 \frac{\partial f_\nu^{(0)}}{\partial p_{\nu 2}} + k_3 \frac{\partial f_\nu^{(0)}}{\partial p_{\nu 3}} \right] [e^{i[\Gamma_\nu + k_2 q_2 + k_3 q_3]} - e^{-i[\Gamma_\nu + k_2 q_2 + k_3 q_3]}] . \tag{F18}$$

The perturbed charge density $\delta\rho$ can be made zero since the expressions for $\delta^2 H$ only contain $\Psi_\nu^2, (d_\nu \Psi_\nu)^2$. Ψ_ν is chosen localized in $\mathcal{H}_\nu$ and $p_{\nu\kappa}$. The distribution of *signs* in Ψ_ν is free. For instance, one can take Ψ_ν piecewise continuous in $\mathcal{H}_\nu$ and $p_{\nu\kappa}$, with changing signs so that positive and negative contributions to $\delta\rho$ balance each other.

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