Variational approach to approximate propagation of Gaussian pulses in a Langmuir plasma

J. C. Bhakta
Department of Mathematics, University of Chittagong, Chittagong, Bangladesh
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The approximate propagation of Gaussian pulses in a Langmuir plasma described by Zakharov equations is considered. By using the Rayleigh-Ritz optimization method based on trial functions, a set of ordinary differential equations for the pulse parameters has been derived. In the static limit, previous results are regained. In the nonstatic description, the periodic time of oscillation for the width of the pulse is reduced from that in the static limit.

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I. INTRODUCTION

Zakharov equations [1] describe the interaction of averaged high-frequency Langmuir waves with low-frequency plasma density fluctuations. The coupling between the high- and low-frequency oscillations is due to the ponderomotive force. This force associated with the Langmuir oscillations drives the low-frequency density fluctuations which in turn trap the Langmuir oscillations in them. Excellent review articles are available (e.g., [2–4]). Zakharov equations have uniform solutions as well as stationary solitary-wave solutions. Numerical investigations of these equations were done by many authors (e.g., [5–8]).

The evolution of pulses which are initially nonsolitary has been investigated by using the inverse-scattering method for nonlinear many-field equations [9–12]. But for the case of Zakharov equations no inverse-scattering transform exists. Therefore approximate analytical solutions for the evolution of pulses are necessary to summarize information and to give physical insight in the numerical results. In this respect different methods have been suggested, such as the moment method [13–15] and a variational method based on Rayleigh-Ritz optimization [16–18]. If pulses are sech-shaped with nonsoliton initial conditions and the fields governed by the Zakharov equations, then it is shown [18] by using a variational approach that plasmon number and momentum remain conserved and the Hamiltonian of the system remains almost conserved for small values of the pulse velocity and b where b is the coefficient of x^2 in the chirp factor.

In this paper we wish to investigate the evolution of Gaussian-shaped pulses connected with the Zakharov equations by applying the variational Rayleigh-Ritz optimization method. The same problem connected to the nonlinear Schrödinger (NLS) equation was considered in [19]. In Sec. II we derive the evolution equations for the solution parameters of pulses. Analytical solutions for special cases are given in Sec. III. In Sec. IV, integrals of motion are given. Numerical results and discussion are given in Secs. V and VI, respectively.

II. ORDINARY DIFFERENTIAL EQUATIONS FOR PULSE PARAMETERS

The coupling between high-frequency Langmuir waves and the associated low-frequency ion density perturbation is described in one dimension by the Zakharov equations

$$\begin{aligned} &iE_t+E_{xx}=nE\ ,\\ &n_{tt}-n_{xx}=|E|^2_{xx}\ , \end{aligned} \tag{1}$$

where t, x, E, and n are the dimensionless time, distance, slowly varying envelope of the high-frequency Langmuir field, and low-frequency density perturbation, respectively [2]. In order to investigate the propagation of the slowly varying pulse envelope determined by (1) we use the initial forms of the pulses as

$$\begin{aligned} &E=A_0\exp(-x^2/2a_0^2)\exp[i(b_0x^2+c_0x)]\ ,\\ &n=n_0\exp(-x^2/a_0^2)\ . \end{aligned} \tag{2}$$

We have used the Rayleigh-Ritz optimization method based on chosen trial functions [19,20] to study the propagation of pulses. The field equations (1) may be derived from a variational problem:

$$\delta\int\int L\,dt\,dx=0\ . \tag{3}$$

Here the Lagrange field density L corresponding to (1) is

$$L=\frac{i}{2}(EE_t^*-E^*E_t)+E_x^*E_x-\tfrac{1}{2}[(E^*E-u_t)^2-u_x^2]\ , \tag{4}$$

and the ion flux density u is given by

$$u_t=n+|E|^2\ . \tag{5}$$

In order to describe the evolution of pulses subject to the initial conditions (2), we use as trial functions

$$\begin{aligned} &E^T=A\exp\left[\frac{(x-x_1)^2}{2a^2}\right]\exp[i(bx^2+cx)]\ ,\\ &u^T=B\exp[-(x-x_1)^2/a^2]\ , \end{aligned} \tag{6}$$

where A, B, a, b, c, and x_1 are time-dependent parameters. The above form of widths and equal displacement x_1 is introduced in the trial functions to simplify the calculations. The behaviors of A, B, a, b, c, and x_1 are determined by the reduced variational problem [21]

$$\delta \int \langle L \rangle dt = 0 , \qquad (7)$$

where

$$\langle L \rangle = \int_{-\infty}^{\infty} L(E^T, u^T) dx . \qquad (8)$$

Using the solutions (6) for E^T and u^T, we can write (8) in the form

$$\langle L \rangle = \frac{i\sqrt{\pi}}{2}(AA_t^* - A^*A_t)a + \frac{\sqrt{\pi}}{2a}A^*A + \sqrt{\pi}A^*Aa[(b_t+4b^2)(\tfrac{1}{2}a^2+x_1^2)+(4bc+c_t)x_1+c^2]$$
$$-\frac{\sqrt{\pi}}{2\sqrt{2}}[a(A^*A-B_t)^2+\frac{B^2}{a}(x_{1t}^2+\tfrac{3}{4}a_t^2-1)-Ba_t(A^*A-B_t)] . \qquad (9)$$

For arbitrary variations δA, δA^*, δB, δa, δb, δc, and δx_1, (7) gives seven equations, which can be reduced to three coupled equations for a, B, and x_1:

$$\alpha_1 a_{tt} + \beta_1 B_{tt} + \gamma_1 = 0 , \qquad (10)$$

$$\alpha_2 a_{tt} + \beta_2 B_{tt} + \gamma_2 = 0 , \qquad (11)$$

$$x_{1t} = 2c_0 N^2 a/(N^2 a + \sqrt{2}B^2) , \qquad (12)$$

where

$$\alpha_1 = B, \quad \alpha_2 = \frac{1}{4}\left[N^2 + \frac{3}{\sqrt{2}}\frac{B^2}{a}\right] ,$$
$$\beta_1 = 2a, \quad \beta_2 = \frac{1}{2\sqrt{2}}B ,$$
$$\gamma_1 = 2a_t B_t - \frac{3}{2}\frac{Ba_t^2}{a} + \frac{N^2 a_t}{a} - \frac{2Bx_{1t}^2}{a} + \frac{2B}{a} , \qquad (13)$$
$$\gamma_2 = \frac{3}{2\sqrt{2}}\frac{BB_t a_t}{a} - \frac{3}{8\sqrt{2}}\frac{B^2 a_t^2}{a^2} - \frac{N^2}{2\sqrt{2}}\frac{B_t}{a}$$
$$+\frac{1}{a^3}\left[\frac{1}{2\sqrt{2}}(N^4 + B^2 x_{1t}^2 - B^2)a - N^2\right] ,$$

and

$$N^2 = |A_0|^2 a_0 . \qquad (14)$$

The other parameters are determined as follows:

$$|A|^2 = N^2/a , \qquad (15)$$

$$b = \frac{1}{4}\frac{d}{dt}(\ln a) , \qquad (16)$$

$$c = \tfrac{1}{2}a(x_1/a)_t , \qquad (17)$$

$$\frac{d}{dt}(\arg A) = -(b_t+4b^2)(\tfrac{1}{2}a^2+x_1^2) - c_t x_1 - \frac{1}{2a^2}$$
$$-4bcx_1 - c^2 - \frac{1}{\sqrt{2}}(B_t - N^2/a)$$
$$-Ba_t/2\sqrt{2}a , \qquad (18)$$

where we have used the following initial values:

$$B(t=0)=0, \quad x_1(t=0)=0, \quad c(t=0)=c_0 . \qquad (19)$$

The advantage for choosing such an initial value of B is to ensure that the initial ion density perturbation, n, is Gaussian. The second initial condition is used for making initial pulses symmetric about the line $x=0$. Analytical solutions for the pulse parameters in a special case and numerical solutions are given in Secs. III and V, respectively.

III. SPECIAL SOLUTIONS

In general, the exact solutions of (10)–(12) and (15)–(18) cannot be determined analytically. However, analytical solutions may be obtained for a particular case. Zakharov equations reduce to the NLS equation in the static limit for which $n = -|E|^2$. In this case the Lagrange field density expression (4) will not involve u and the constraint due to arbitrary variation δB will be absent. Then (11) becomes

$$a_{tt} = (4 - \sqrt{2}N^2 a)/a^3 . \qquad (20)$$

The periodic solution for the width of the pulse is given by

$$[\sqrt{2}N^2 a_0^2 a - (\sqrt{2}N^2 a_0 - 2 - 8b_0^2 a_0^4)a^2 - 2a_0^2]^{1/2}$$
$$+\frac{N^2 a_0^2}{(2N^2 a_0 - 2\sqrt{2} - 8\sqrt{2}b_0^2 a_0^4)^{1/2}}\left[\cos^{-1}\left\{\frac{(2N^2 a_0 - 2\sqrt{2} - 8\sqrt{2}b_0^2 a_0^4)a - N^2 a_0^2}{a_0(N^4 a_0^2 - 4\sqrt{2}N^2 a_0 + 8 + 32b_0^2 a_0^4)^{1/2}}\right\}\right.$$
$$\left.-\cos^{-1}\left\{\frac{N^2 a_0 - 2\sqrt{2} - 8\sqrt{2}b_0^2 a_0^4}{(N^4 a_0^2 - 4\sqrt{2}N^2 a_0 + 8 + 32b_0^2 a_0^4)^{1/2}}\right\}\right] = (2N^2 a_0 - 2\sqrt{2} - 8\sqrt{2}b_0^2 a_0^4)t/a_0 , \qquad (21)$$

while the other parameters are

$$c = c_0 = 0, \quad x_1 = 0 , \qquad (22)$$

together with (16) and (18) for b and $\arg A$, respectively, provided that

$$|b_0| \equiv |b(t=0)| < \frac{1}{2a_0^2}\left[\frac{N^2 a_0}{\sqrt{2}} - 1\right]^{1/2} , \qquad (23)$$

and

$$|A(t=0)|a_0 > 2^{1/4} . \qquad (24)$$

The period of oscillation of the width is

$$T = \pi N^2 a_0^3 (\sqrt{2} N^2 a_0 - 8 b_0^2 a_0^4 - 2)^{-3/2} . \qquad (25)$$

This special case has recently been considered by Anderson [16] with the dynamics governed by the NLS equation. Our results correspond to those results with appropriate changes in notation.

IV. INTEGRALS OF MOTION

Zakharov equations possess three integrals of motion for the one-dimensional case. These are the number of plasmons

$$I = \int_{-\infty}^{\infty} |E|^2 dx , \qquad (26)$$

the momentum

$$P = \int_{-\infty}^{\infty} \left[\frac{i}{2}(E E_x^* - E^* E_x) - n u_x \right] dx , \qquad (27)$$

and the energy

$$H = \int_{-\infty}^{\infty} [\tfrac{1}{2} E^{*2} E^2 - E_x^* E_x - \tfrac{1}{2}(u_x^2 + u_t^2)] dx . \qquad (28)$$

If the fields are given as in (6), then the above integrals of motion become

$$I = \sqrt{\pi} |A|^2 a , \qquad (29)$$

$$P = \frac{1}{2} \sqrt{\pi} x_{1t} \left[|A|^2 a + \frac{\sqrt{2} B^2}{a} \right] , \qquad (30)$$

$$H = -\frac{1}{8} \sqrt{\pi} N^2 \left[\frac{4}{a^2} + a_t^2 + 2 x_{1t}^2 \right] + \frac{\sqrt{\pi}}{2\sqrt{2} a} [N^4 - B^2(1 + x_{1t}^2) - B_t^2 a^2 - B B_t a a_t - \tfrac{3}{4} B^2 a_t^2] . \qquad (31)$$

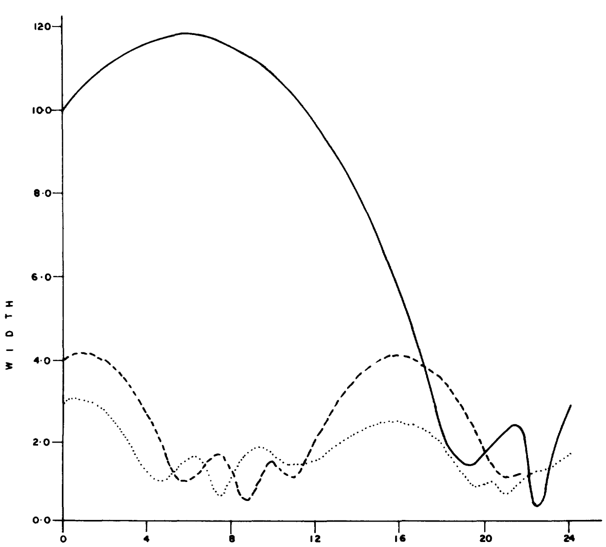

FIG. 1. Width versus time for $|A_0|=1$, $x_1(0)=0$, $B(0)=0$, $b_0=0.017$, $c_0=0.1$, $n_0=-1$: . . . ., $a_0=3$; — — —, $a_0=4$; ——, $a_0=10$.

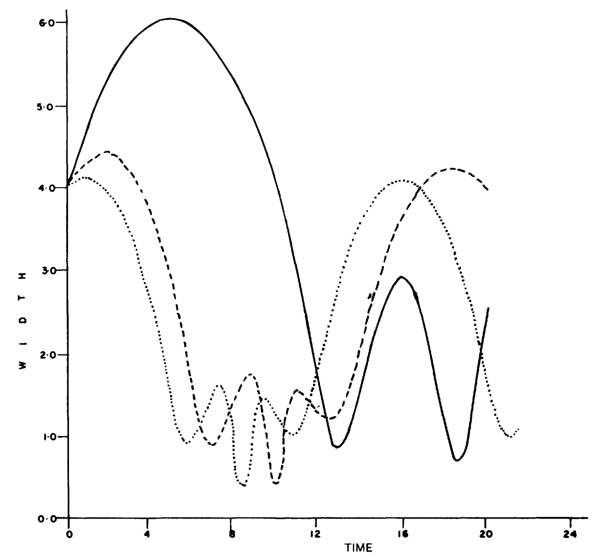

FIG. 2. Width versus time for $|A_0|=1$, $x_1(0)=0$, $B(0)=0$, $n_0=-1$, $c_0=0.1$, $a_0=4$: ——, $b_0=0.06$; — — —, $b_0=0.03$; . . . ., $b_0=0.017$.

By using (12) and (15), we may find that I and P are conserved for solutions but H is not conserved.

V. NUMERICAL RESULTS

The evolution equations of the pulse parameters are determined by Eqs. (10)–(12) and (16)–(18). In the static limit we have $n=-|E|^2$ for which $B=0$ and the solutions of the other parameters are given by (15), (16), (18), (21), and (22). General solutions of the equations of the pulse parameters have been investigated numerically by using the Runge-Kutta method. The difference between these numerical solutions and the analytical solutions in the static limit may be interpreted as the effect of the

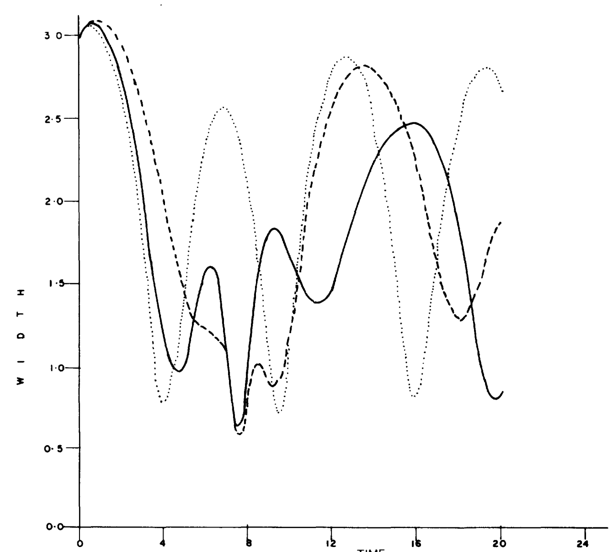

FIG. 3. Width versus time for $|A_0|=1$, $x_1(0)=0$, $B(0)=0$, $c_0=0.1$, $b_0=0.017$, $a_0=3$: ——, $n_0=-1$; — — —, $n_0=-0.8$; . . . ., $n_0=-1.2$.

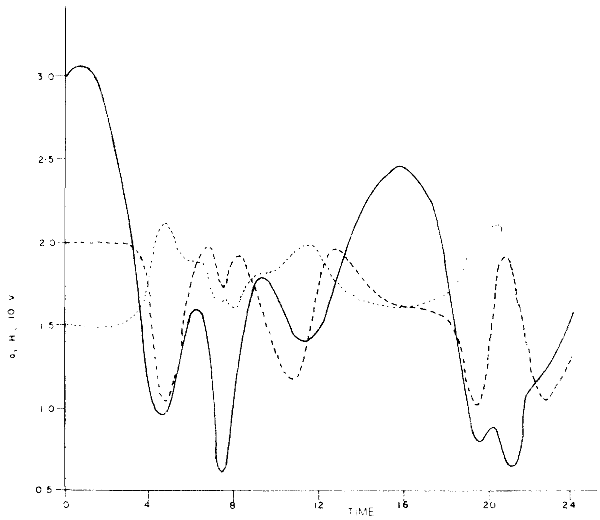

FIG. 4. Width, Hamiltonian, and 10 times velocity of the pulse are plotted against time for $a_0=3$, $|A_0|=1$, $n_0=-1$, $b_0=0.017$, $c_0=0.1$, $B(0)=0$, $x_1(0)=0$: ——, width; — — —, 10 times velocity; · · · ·, Hamiltonian.

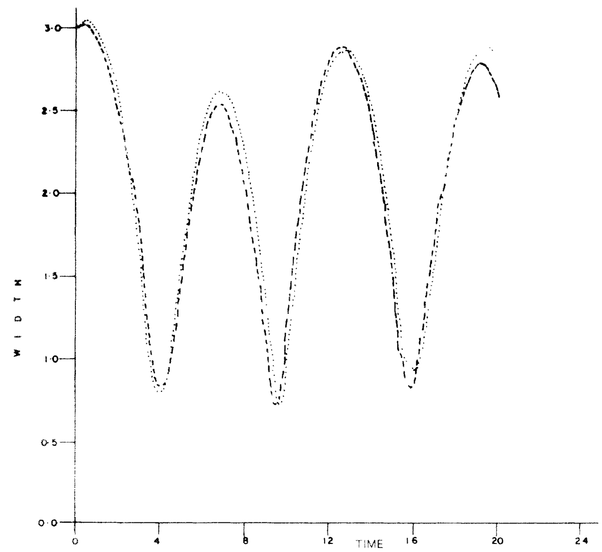

FIG. 5. Width versus time for $|A_0|=1$, $x_1(0)=0$, $B(0)=0$, $b_0=0.017$, $a_0=3$, $n_0=-1.2$: · · · ·, $c_0=0.15$; — — —, $c_0=0.05$.

presence of ion flux density, u, defined via (5). We have put $|A_0|=1$ in our computation.

Figures 1–5 depict the dependence on the initial values of b, a, c, and n_0. Figure 1 corresponds to the evolution of the width for different initial values of a. Figure 2 shows the effects of the initial values of b on the evolution of width. Figure 3 shows the effects of the initial values of n on the evolution of width. Figure 4 shows that the Hamiltonian is not conserved in our solutions, however, almost conserved at the early stage of the evolution of the pulse at which the velocity of the pulse is almost uniform. Figure 5 shows the effects of the initial values of c on the evolution of the width.

VI. DISCUSSION

Using the Rayleigh-Ritz optimization method based on chosen trial functions we have investigated the propagation of pulses for Langmuir waves described by the Zakharov equations. A Gaussian pulse has been used as the initial pulse. From (25) and (21) we may get the period of oscillation, the maximum and minimum values of the width of the pulse in the static limit which are 10.72, 4.45, and 0.38, respectively, for $a_0=4$, $b_0=0.03$, and $|A_0|=1$. On the other hand, for the nonstatic case, the time of oscillation and the maximum value of the pulse are reduced and the minimum value of the width is increased as may be seen from Fig. 2. For our solutions (6), the plasmon number and momentum are conserved but the Hamiltonian is not conserved though the fluctuation is not much. The Hamiltonian is almost conserved if the initial pulse is sech-shaped [18]. If we introduce a certain polynomial in powers of x whose coefficients are time-dependent parameters as the amplitude of the Gaussian pulse (6) instead of A, the situation may be improved. In this case the algebra will be more complicated.

We may conclude that our solution will give an approximate solution of a nonlinear evolution of the pulse.

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