Stochastic resonance as crisis

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Based on approximate methods of nonlinear oscillations and experimental studies, we show the evidence that stochastic resonance in Duffing's equation can be considered as an example of crisis.

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Stochastic resonance is a phenomenon which occurs in multistable [1–9] and monostable [10] systems driven by a sum of a noise signal and a weak periodic signal. Under appropriate conditions a weak periodic signal can be amplified by a noise signal. This amplification can be observed by the measurement of the signal-to-noise ratio [6–9], stochastic amplification factor, [10,11] or spectrum correlation function [12]. Stochastic resonance has been studied both theoretically and experimentally in a variety of contexts such as, for example, meteorology [2], optical systems including lasers [3,4], electronic circuits [5,6,9], a magnetoelastic ribbon [7], sensory neurons [8], etc. In all these examples the theory describing stochastic resonance is statistical.

Recently, Carroll and Pecora [13] and Ippen, Lindner, and Ditto [14] showed that stochastic resonance can be observed when chaos, rather than noise, is used as a nonperiodic component of a driving force. In this case, stochastic resonance occurs in a completely deterministic system and can be described using a dynamical, rather than statistical approach. Although not all systems for which stochastic resonance appears can be envisaged as exhibiting crises, in [13] authors showed that crises may give rise to stochastic resonance behavior. First, Carroll and Pecora [13] studied the system forced only by chaotic force and considered crisis due to the collision of stable manifold of an unstable periodic orbit with the unstable manifold of a chaotic attractor. Previous scaling results available on chaotic intermittency near the crisis point [15] gave them a quantitative handle which replaces the traditional stochastic description. They then considered the effect of a small added periodic signal and deduced the onset of stochastic resonance phenomenon.

In this paper we show that the dynamical description of stochastic resonance is also possible in the systems driven in a traditional way, i.e., by a sum of a noise signal and a weak periodic signal. We give experimental and approximate analytical evidence that stochastic resonance can be caused by noise-induced crisis. In our case it will be the simplest form of crisis, i.e., the collision of stable and unstable periodic orbits. We consider a particular yet representative case of Duffing's equation driven by band-limited white noise plus a weak periodic force,

$$\ddot{x}+a\dot{x}+x+x^3=f(t)+A\cos\Omega t\ , \tag{1}$$

where a, A, and Ω are constants. $f(t)$ is a zero-mean stochastic process with a spectral density

$$s(\nu)=\begin{cases}\delta/(\nu_{\max}-\nu_{\min})\ , & \nu\in(\nu_{\min},\nu_{\max})\\ 0\ , & \nu\notin(\nu_{\min},\nu_{\max})\ ,\end{cases} \tag{2}$$

where δ is a noise intensity and $[\nu_{\min},\nu_{\max}]$ is an interval of considered frequencies. Equation (1) is a single-well system which is the first example of a monostable system exhibiting stochastic resonance phenomenon [10]. We took it in this form to allow comparison of our results with that of [10]. In our system we considered bandlimited white noise instead of ordinary white noise to allow some analytical approximations [16,17] which will be described later.

First consider Eq. (1) without random signal $[f(t)=0]$. In this case one can find an approximate $2\pi/\Omega$ periodic solution in the form

$$x(t)=C_0\cos(\Omega t+\phi_0)\ , \tag{3}$$

where the constants C_0 (amplitude of oscillations) and ϕ_0 (phase difference between oscillations and periodic forcing) can be estimated by intersecting Eq. (3) into Eq. (1) and applying, for example, the harmonic balance method [16,17]. It is well known that the typical relations between C_0, ϕ_0 and Ω have the form shown in Figs. 1 and 2. Although the application of the described method requires numerical calculations in the theory of nonlinear oscillations it is called an analytical one [16,18]. For Ω between Ω_1 and Ω_2 three different periodic solutions of the formula (3) are possible. Two of them, which are indicated by solid lines in Fig. 1, are stable. The third one (dashed line) is unstable [18]. At points Ω_1 and Ω_2 our system exhibits crises as one of the stable solutions collides with an unstable one. As the result of crises we observe increment or decrement of the amplitude C_0. It has to be noted here that this described simple form of crises is also known as a jump phenomenon [18]. In the rest of

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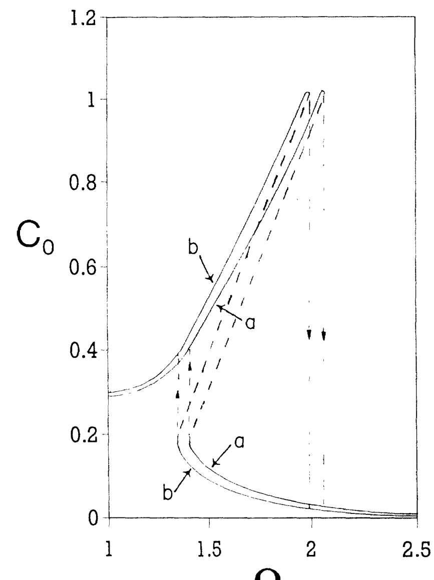

FIG. 1. $C_0(\Omega)$—resonance curves of Eq. (1): $a=0.022$, $A=0.02$; a, $f(t)=0$, $\Omega_1=1.345$, $\Omega_2=1.982$; b, $\delta=0.39$, $\Omega_1=1.413$, $\Omega_2=2.083$.

this paper we consider only a crisis at Ω_1 as this one can be connected with stochastic resonance phenomenon.

Now consider a complete form of Eq. (1) (with periodic and random forcing). Random forcing of the form of band-limited white noise (it realizations) can be approximated by a sum of N harmonics,

$$f(t)=\delta\sum_{i=1}^{N}\cos(\nu_i+\phi_i)\ , \tag{4}$$

where ν_i and ϕ_i are independent random variables [16]. Frequencies (ν_i) are described by where $\Delta\nu=(\nu_{max}-\nu_{min})/N$ and ϕ_i are independent random variables with uniform

$$\nu_i=(i-1/2)\Delta\nu+\delta\nu_i+\nu_{min}$$

distribution on the interval $[0,2\pi]$. For a given realization of random forcing $f(t)$, the above approximation allowed us to consider Eq. (1) as a deterministic system. Following an analytical approach introduced in [17] one can assume the solution of Eq. (1) in the form

$$x(t)=C_0\cos(\Omega t+\phi_0)+\sum_{i=1}^{N}C_i\cos(\nu_i t+\phi_i)\ , \tag{5}$$

where C_i and ϕ_i $(i=0,1,\ldots,N)$ are constant. Inserting Eq. (5) into Eq. (1) it is possible to compute all constants in Eq. (5). Typical results for $a=0.022$, $A=0.02$, $N=50\,000$, and $\nu_{min}=0$, $\nu_{max}=100$ are shown in Figs. 1 and 2 (curves b). Comparing curve a with curve b in Fig. 1 one finds that an addition of noise slightly changes the $C_0(\Omega)$ and $\phi_0(\Omega)$ plots but for some values of Ω (for example, $\Omega=1.35$; $C_0=0.16$ on the lower branch of curve a and $C_0=0.35$ on the upper branch of curve b) we can observe a significant increase of amplitude C_0 as the result of the shift of crisis parameter Ω. In Fig. 3 we present the variation of amplitude C_0 and phase ϕ_0 for different noise intensity δ and constant frequency of periodic signal $\Omega=1.35$. To allow the comparison of our results with that of [10] we plotted the squared stochastic amplification factor

$$S^2(\delta)=[C_0(\delta)/C_0(0)]^2$$

versus noise intensity δ rather than simply $C_0(\delta)$ versus δ. The solid line represents the results of described analytical approximation while the dots represent experimental results of an electronic model of Eq. (1) designed, constructed, and operated according to standard practice. The model was driven with a quasi white noise from an external noise generator and with a weak periodic

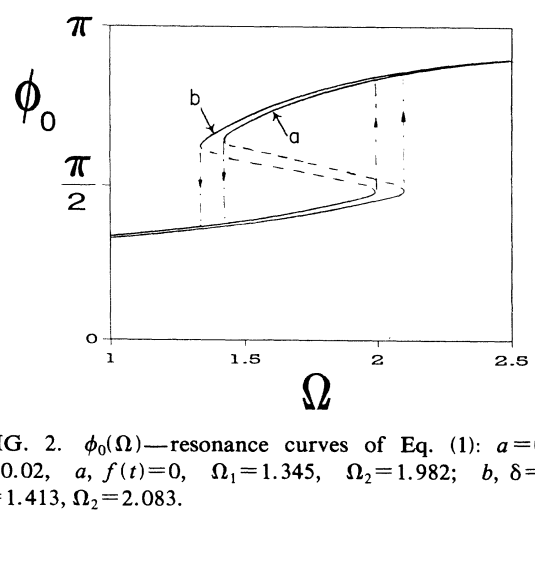

FIG. 2. $\phi_0(\Omega)$—resonance curves of Eq. (1): $a=0.022$, $A=0.02$, a, $f(t)=0$, $\Omega_1=1.345$, $\Omega_2=1.982$; b, $\delta=0.39$, $\Omega_1=1.413$, $\Omega_2=2.083$.

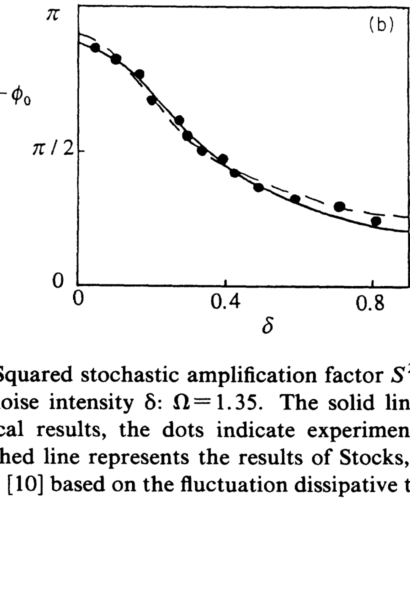

FIG. 3. Squared stochastic amplification factor S^2 (a) and ϕ_0 (b) versus noise intensity δ: $\Omega=1.35$. The solid line indicates our analytical results, the dots indicate experimental results, and the dashed line represents the results of Stocks, Stein, and McClintock [10] based on the fluctuation dissipative theorem.

force from a frequency synthesizer. The resultant fluctuation $x(t)$ was collected by a 12-bit data-acquisition system in a microcomputer and ensemble averaged to yield $\langle x(t) \rangle$. To estimate C_0 the fast Fourier transform transformation of $\langle x(t) \rangle$ was taken.

It is immediately evident from Fig. 3(a) that $S^2(\delta)$ at first increases rapidly with noise intensity, but then passes through a maximum and decreases again, albeit more slowly. As the periodic frequency Ω is beyond the natural frequency of the system $-\phi_0$ is close to π but as δ increases $-\phi_0$ decreases and at resonance and passes through $\pi/2$ and later decreases much slowly as shown in Fig. 3(b). This behavior of the stochastic amplification factor and phase difference ϕ_0 [19] indicates the presence of stochastic resonance phenomenon.

If one considers C_0 and ϕ_0 curves for different δ like those of Fig. 1 it is possible to obtain $C_0(\delta)$ dependence as shown in Fig. 4. This plot allows us to describe the results of Fig. 3(a) in the following way. The initial increase of $S^2(\delta)$ is an effect of the noise-induced increment of C_0 on the lower branch of the resonant curve, while its further increase is the result of crisis at $\delta=0.37$ and a jump to the upper branch of the resonant curve. The decrease of $S^2(\delta)$ for higher values of δ can be understood as the effect of the noise-caused decrement of C_0 on the upper branch of a resonant curve. In a similar way one can describe the behavior of ϕ_0 at stochastic resonance.

Good agreement of our analytical model results with the experiment as well as with the theoretical prediction obtained by Stocks, Stein, and McClintock [10] from the fluctuation dissipation theorem (dashed line in Fig. 3) proves that the described stochastic resonance is an example of crisis. It has to be noted here that although the results of [10] were obtained for the random force of the white noise form they can be compared with our results as we considered relatively large interval of frequencies ν in our band-limited white noise (4).

In conclusion, the application of approximate analytical methods of nonlinear oscillations allows one to examine stochastic resonance as a dynamical phenomenon. We found evidence that stochastic resonance in Duffing's equation can be considered as an example of crisis. We also applied the same approach to the other stochastic resonance cases in this equation and found that also subharmonic stochastic resonances, when the response component of Ω/n frequency is amplified, are caused by crises. These results will be published later.

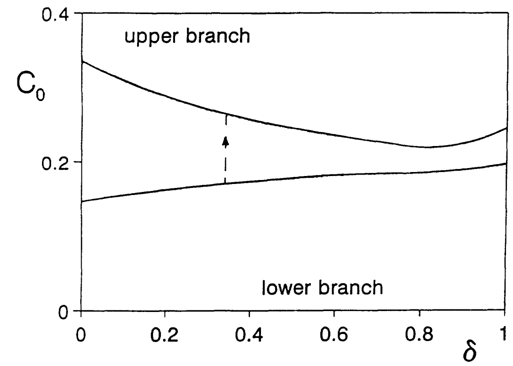

FIG. 4. C_0 versus noise intensity δ: $\Omega=1.35$.

We hope that the present results can be generalized to the other systems and give additional support to the idea of Carroll and Pecora [13] that stochastic resonance is a particular case of crisis.

It should be added here that the dynamical model of stochastic resonance phenomenon as well as the linear response of Dykman *et al.* [5] allows to explain it by the same mechanism both in multistable and monostable systems while in the most statistical approaches [6,10] two different physical mechanisms have to be applied.

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