PHYSICAL REVIEW E VOLUME 49, NUMBER 6 JUNE 1994

Vibration modes of a gap soliton in a nonlinear optical medium

Boris A. Malomed and Richard S. Tasgal
Department of Applied Mathematics, School of Mathematical Sciences, Tel Aviv University, Ramat Aviv 69978, Israel
(Received 23 November 1993)

We analyze the dynamics of small internal vibrations in a two-component gap soliton. The general model considered describes at least three different nonlinear optical systems: a pair of waves coupled by the Bragg scattering in a medium with a periodic grating, a twisted birefringent fiber, and a dual-core asymmetric coupler. In all the cases the material dispersion of the medium is neglected, but an effective dispersion is induced by the linear coupling between the two modes. Employing the averaged Lagrangian variational technique we derive a system of ordinary differential equations which approximates the dynamics of the gap soliton. We find three oscillation modes, which are composed of (mixtures of) dilation-contraction of each component's width, and a relative translation of the two components. At certain values of the parameters the analysis yields spurious instabilities, which is a novel failure of the averaged Lagrangian variational technique.

PACS number(s): 42.81.Dp, 03.40.Kf

I. INTRODUCTION

In the currently most familiar systems which support solitons, such as the Korteweg-de Vries equation and the nonlinear Schrödinger equation, the solitons' existence results from a balance between nonlinearity and material dispersion. By contrast, a system which supports solitons without a *material* dispersion term (though not to say no dispersion at all) is

$$iu_t + iu_x + (\sigma|u|^2 + |v|^2)u + v = 0, \tag{1a}$$

$$iv_t - iv_x + (|u|^2 + \sigma|v|^2)v + u = 0. \tag{1b}$$

[Material dispersion would additively contribute the terms u_{xx} and v_{xx} to Eqs. (1a) and (1b), respectively. Thus changed, the system would be classified as a pair of coupled nonlinear Schrödinger equations.] For conciseness, we will refer to the fields u and v as polarization components on account of the relevance of Eqs. (1) to a birefringent optical fiber, though, as detailed below, the equations also describe other physical systems. A name for Eqs. (1) does not appear to have been settled on in the literature, except for the case $\sigma = 0$, in which Eqs. (1) represent the one-dimensional massive Thirring model ("MTM") [1–3]. We follow the convention of calling stable solitary waves solitons, even when the system is not completely integrable, and the solitary waves are, therefore, not true solitons.

Equations (1) may describe, in addition to the MTM, at least three different nonlinear optical systems (in their appropriate limits): a twisted birefringent optical fiber, an optical fiber with a periodic grating (which seems the most interesting application), and a nonlinear asymmetric coupler. Note that when Eqs. (1) are regarded as a model of the twisted birefringent fiber or of the asymmetric coupler, the meaning of the variables t and x is reversed: the former becomes a propagation distance and the latter becomes the so-called reduced time. Below, we describe these three physical systems in order.

Optical fibers are today the prototypical physical system described by nonlinear Schrödinger equations [4]. The Kerr effect accounts for the cubic nonlinearity. A birefringence may cause the fibers to have different group velocities in each of two polarization modes — the source of the terms $+iu_x$ and $-iv_x$ in the nondimensionalized Eqs. (1) [5]. A third order susceptibility produces the nonlinearities; the coefficient of self-phase modulation may be in the range $2/3 \le \sigma^{-1} \le 2$ [5]. If the birefringent fiber is, in addition, periodically twisted, this gives rise to the linear cross-coupling terms in Eqs. (1). We presume that the proper dispersion in each polarization, as well as a phase-velocity difference between the polarizations, are insignificant, along with absorption, inhomogeneities, etc. (This dispersionless approximate model of the twisted birefringent fiber was discussed in some detail in the review paper [11].)

In a periodically modulated nonlinear fiber (i.e., with a periodic grating), Eqs. (1) describe a pair of counterpropagating waves coupled through the Bragg scattering (represented by the linear cross-coupling terms [6–8]) induced by the grating. We assume the usual approximation, in which the proper dispersion of each wave mode is neglected. "Gap solitons" in nonlinear fibers with a grating have recently attracted a great deal of attention (see, e.g., Refs. 6–8 and references therein). These are the gap solitons whose internal dynamics is analyzed in this work.

A nonlinear dual-core coupler consists of two fibers juxtaposed so that radiation within each fiber overlaps the other one, which causes an energy oscillation between the two cores [9–11]. As usual, the Kerr effect produces a nonlinearity in each mode. In the framework of the model considered, we neglect the proper dispersion of each core, as in the original paper [9]. There is not necessarily (depending on the implementation) any significant nonlinear cross-coupling; in terms of Eqs. (1), this means the parameter σ formally ranges up to the limit $\sigma \to \infty$. The group-velocity difference terms, $+u_x$ and $-v_x$, may be present in Eqs. (1) if the two cores of the coupler are different (which is why we speak of the coupler as being asymmetric). However, in the model of the asymmetric coupler corresponding to Eqs. (1), we neglect differences in all other parameters, such as the phase-velocities and nonlinear coefficients.

The (classical) massive Thirring model, $\sigma = 0$, is the only Lorentz invariant case of Eqs. (1). The Lorentz invariance is manifest when the equations are written in the covariant form:

$$(-i\partial_\nu \gamma^\nu + 1)\psi + \gamma^\nu \psi(\bar{\psi}\gamma_\nu \psi) = 0,$$

where

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \bar{\psi} = \psi^+ \gamma^0.$$

The physics of this model is far afield from the subject of this paper, and current interest in applications is slim, so rather than give a description here we refer readers to Ref. 1.

Whereas a pair of coupled nonlinear Schrödinger equations can support solitons in one of the polarization components without the "assistance" of the other, solitons in Eqs. (1) can only exist through the interaction of both polarization components. In the small amplitude limit Eqs. (1) reduce to the single Klein-Gordon equation with unit mass, $(\partial_t^2 - \partial_x^2 + 1)u = 0$, which is dispersive. On the other hand, were the linear cross terms or birefringence (i.e,. group-velocity difference) absent, the system would have zero linear dispersion, and the nonlinearity would cause a pulse to compress and ultimately blow up. It turns out that dispersion and nonlinearity will find a balance, supporting solitary waves. When $\sigma = 0$, Eqs. (1) are completely integrable, and the solitary waves are true solitons [2,3]. When $\sigma \neq 0$, though Eqs. (1) are not integrable, solitary waves which are continuously deformable into the $\sigma = 0$ solitons are known in exact form and are stable [6,7]. The solitary waves have a frequency which is forbidden for the linearized system because of the Bragg reflection [in the case when Eqs. (1) apply to the nonlinear optical fiber bearing a periodic grating]. The solitary waves supported by Eqs. (1) are hence variously called Bragg solitons, gap solitons, and self-induced transparency solitons. We will generally use the term "gap solitons."

Beyond knowing the gap solitons' form in the steady state, an important subject is their vibrations. Vibrations are interesting, first, in themselves because they are fundamental dynamical effects that a solitary wave will exhibit when it is not precisely at equilibrium. Second, vibrations are intimately related to the important fact that, in the nonintegrable case, soliton-soliton collisions are inelastic: direct numerical simulations demonstrate that collisions induce internal oscillations in the recoiling solitons [6]. In order to analyze aspects of inelasticity in soliton-soliton collisions, it is necessary, as an intermediate step, to understand gap solitons' dynamics in isolation.

Gap soliton oscillations have as yet been little studied. One can say a few things about oscillations based on general properties of integrable systems. In the MTM case ($\sigma = 0$), an initially vibrating gap soliton will gradually shed its oscillation energy, until it stabilizes. For $\sigma \neq 0$, by extension, small oscillations of the gap soliton will be slowly radiated away too. The inverse scattering method through which this result is obtained in the exactly integrable case ($\sigma = 0$), is not very useful for studying the oscillations' form. Aceves and Wabnitz [6] complemented their discovery of the exact solitary wave solutions to Eqs. (1) with numerical simulations in which collision-induced vibrations may be seen. Though vibrations were clearly present, they were not the focus of Ref. 4 and received negligible attention therein. De Sterke and Sipe [8] studied Eqs. (1) numerically on a finite interval. The phenomena studied in Ref. 8 depend strongly on the boundary conditions, which were different than ours, and on the presence of a large radiative (non-soliton) energy component, so the results do not give the same insight into small perturbations of solitons as does our work. To our knowledge, there is no other published work on gap soliton oscillations; as for analytic study of gap soliton oscillations, there has been none at all.

Analytic results on internal soliton vibrations are generally obtained by the averaged Lagrangian variational technique [12–14], which is as follows. The exact governing (partial differential) equations may be derived from a Lagrangian density. The solitary wave is approximated by an ansatz with a small number of time dependent parameters, which is substituted into the Lagrangian density. The Lagrangian is integrated over the spacial variable, leaving only a temporal dependence. From this averaged Lagrangian, one may derive a finite number of ordinary differential equations which should approximate the system's actual dynamics. It is then generally useful to look at the small perturbation limit, and perform a vibrational mode analysis.

Gap solitons provide a novel test of the averaged Lagrangian variational technique in several respects. The governing equations (1) contain no explicitly expressed derivatives of second or higher order (zero material dispersion). The dispersion results from the linear coupling of the two components (in combination with the group velocity difference); solitary waves cannot exist if exactly one of the polarization components is empty. To our knowledge, the averaged Lagrangian variational technique has been applied to no system in which either of these characteristics hold.

We perform herein a small vibrational analysis of the gap soliton by the averaged Lagrangian variational technique. The analysis yields three oscillation modes. In general, the dilation-contraction of each polarization component's width and the relative translation of the polarization components are mixed in each mode. However, in the MTM case ($\sigma = 0$), as well as for solitons in the rest frame ($\sigma \neq 0$ is not Lorentz invariant), the mode in which the polarization components' dilation-contractions are π radians out of phase decouples from the other two vibrational modes. For wide, flat pulses, all three oscillation modes are stable; as the pulses narrow, the formal analysis gives instabilities, even in the completely integrable MTM case ($\sigma = 0$). Since MTM solitons are known to be *always* stable, the instabilities are spurious. To our knowledge, this is the first known "catastrophic" failure—failure to give qualitatively correct results—of the averaged Lagrangian variational technique.

The rest of the paper is organized as follows. In Sec. II, we introduce the soliton's ansatz which is inserted into the variational equations, and discuss the meaning of its free parameters. In Sec. III, effective equations of motion for those parameters are derived from the Lagrangian corresponding to Eqs. (1). (The complete form of the equations is very cumbersome, so the detailed and explicit expression of the equations is relegated to the Appendix.) Next, in Sec. III, we use these equations to develop a general analysis of the small internal vibrations of the soliton and find the corresponding eigenfrequencies. Complex values of the eigenfrequencies produced by the variational technique give spurious instabilities, which is discussed. In Sec. IV, we analyze in more detail the special case of a small-amplitude, broad soliton. In this case, we obtain simple expressions for the eigenfrequencies, which are free of the spurious instabilities. Another special case, corresponding to the asymmetric dual-core coupler [Eqs. (1) in the limit $\sigma = \infty$] is considered in Sec. V. Conclusions and summary are in Sec. VI.

II. THE VARIATIONAL ANSATZ

Our choice of ansatz to approximate the oscillating gap soliton must be guided by the form of the undisturbed gap soliton. As noted above, gap solitons under Eqs. (1) are known in the exact form [6]:

$$u = \alpha\sqrt{\gamma(1+\rho)}\ \sin Q\ \mathrm{sech}\left(\xi \sin Q - \frac{i}{2}Q\right) \times \exp\left\{ i\left[\phi - \tau\cos Q + 4\sigma\alpha^2\gamma^2\rho\tan^{-1} \times \left(\tan\frac{Q}{2}\tanh(\xi\sin Q)\right)\right]\right\}, \tag{2a}$$

$$v = -\alpha\sqrt{\gamma(1-\rho)}\ \sin Q\ \mathrm{sech}\left(\xi \sin Q + \frac{i}{2}Q\right) \times \exp\left\{ i\left[\phi - \tau\cos Q + 4\sigma\alpha^2\gamma^2\rho\tan^{-1} \times \left(\tan\frac{Q}{2}\tanh(\xi\sin Q)\right)\right]\right\}, \tag{2b}$$

where

$$\alpha^{-2} \equiv 1 + \sigma\gamma^2(1+\rho^2),\ \gamma \equiv (1-\rho^2)^{-1/2},$$
$$\tau \equiv \gamma(t-\rho x),\ \xi \equiv \gamma(x-\rho t),$$

and $Q, \rho, \phi = \text{const}$, with $|\rho| < 1$, and $0 < Q < \pi$. The global phase, ϕ, is trivial. The velocity, ρ, is nontrivial when $\sigma \neq 0$, where the system is not integrable and not Lorentz invariant. The parameter Q determines the gap soliton's width [$(\sin\frac{Q}{2})^{-1}$] and amplitude.

That exact solutions are known, both simplifies and restricts the choice of ansatz. Since we wish our approximation to match the exact solution in the steady state, we choose an ansatz of the same form as Eqs. (2), but replace the constants with parameters that can vary with time, or functions of those parameters. Our choice is

$$u = \eta_u\alpha\sqrt{\gamma(1+\rho)}\ \sin(Q+\Delta Q)\mathrm{sech}\left[(\xi+\Delta\xi)\sin(Q+\Delta Q) - \frac{i}{2}(Q+\Delta Q)\right] \times \exp\left[-i\left(a_u + b_u(\xi+\Delta\xi) + \frac{1}{2}c_u\sin(Q/2)(\xi+\Delta\xi)^2 - 4\sigma\alpha^2\gamma^2\rho\tan^{-1}\{\tan(Q/2)\tanh[(\xi+\Delta\xi)\sin Q]\}\right)\right], \tag{3a}$$

$$v = -\eta_v\alpha\sqrt{\gamma(1-\rho)}\ \sin(Q-\Delta Q)\mathrm{sech}\left[(\xi-\Delta\xi)\sin(Q-\Delta Q) + \frac{i}{2}(Q-\Delta Q)\right] \times \exp\left[-i\left(a_v + b_v(\xi-\Delta\xi) + \frac{1}{2}c_v\sin(Q/2)(\xi-\Delta\xi)^2 - 4\sigma\alpha^2\gamma^2\rho\tan^{-1}\{\tan(Q/2)\tanh[(\xi-\Delta\xi)\sin Q]\}\right)\right], \tag{3b}$$

where η_u, η_v, Q, ΔQ, $\Delta\xi$, a_u, a_v, b_u, b_v, c_u, and c_v are functions of τ. This ansatz lets us independently vary, for each polarization component, the central position, pulse width, amplitude, constant phase, carrier frequency, and frequency chirp. Under the obeisance of the effective Lagrangian for Eqs. (1), this gap soliton model may exhibit transfer of energy between the two components, independent contraction-dilation of each of the two polarization components, relative translations of the polarization components, and internal energy storage by the frequency chirp. Not supported is radiation and anything but simple changes in the pulse shapes.

III. EQUATIONS OF MOTION

Substituting Eqs. (3) into the Lagrangian density from which the governing equations may be derived,

$$\mathcal{L} = \frac{i}{2}[u^*(\partial_t + \partial_x)u - u(\partial_t + \partial_x)u^*$$
$$+v^*(\partial_t - \partial_x)v - v(\partial_t - \partial_x)v^*]$$
$$+\frac{\sigma}{2}|u|^4 + \frac{\sigma}{2}|v|^4 + |u|^2|v|^2 + u^*v + uv^*, \quad (4)$$

integrating the result over the variable x from $-\infty$ to $+\infty$, and taking the variation with respect to each of the parameters gives a simplified system of equations of motion. We do not write it in its complete form because it is cumbersome and does not provide insight by itself.

There is exactly one physically distinct fixed point, which corresponds precisely to the exact solitary-wave solution given by Eqs. (2):

$$\eta_u = \eta_v = 1, \quad Q = \text{const},$$
$$a_u = a_v = \tau \cos Q + \text{const},$$
$$b_u = b_v = c_u = c_v = \Delta Q = \Delta\xi = 0.$$

Linearizing the general dynamical equations about the fixed point, we arrive at a system with six degrees of freedom, here expressed as one second and one fourth order equation for $\frac{\Delta Q}{Q}$ and $\frac{Q-Q_0}{Q}$ (Q_0 is the steady-state value of Q):

$$\left(p_1\frac{d^2}{d\tau^2} + p_2\right)\frac{\Delta Q}{Q} = (4\sigma\alpha^2\gamma^2\rho)\frac{Q-Q_0}{Q}, \quad (5a)$$

$$\left(p_3\frac{d^4}{d\tau^4} + p_4\frac{d^2}{d\tau^2} + p_5\right)\frac{Q-Q_0}{Q} = (4\sigma\alpha^2\gamma^2\rho)\frac{\Delta Q}{Q}. \quad (5b)$$

Note that the derivatives in Eqs. (5) are with respect to τ, time in the gap soliton's reference frame, defined above. The coefficients p_j are functions of the parameters Q, σ, and ρ. The complete, explicit forms of the coefficients p_j and the dependence of the other ansatz parameters on Q and ΔQ are given in the Appendix.

The essential relationships are as follows. When $\sigma\rho = 0$, in which either the gap soliton is quiescent or the system represents the (completely integrable and Lorentz invariant) MTM case, ΔQ decouples from Q (and from $\Delta\xi$); when $\sigma\rho \neq 0$, ΔQ is slaved to Q. The parameters $\Delta\xi$, $(\eta_u^2+\eta_v^2)$, (a_u+a_v), (b_u-b_v), and (c_u+c_v) are functions of Q (more precisely, of its variation $\frac{Q-Q_0}{Q}$). Note that time derivatives of Q or other parameters serve here as independent parameters. The parameters $(\eta_u^2-\eta_v^2)$, (a_u-a_v), (b_u+b_v), and (c_u-c_v) are functions of ΔQ.

We examine in detail the MTM ($\sigma = 0$) soliton first. It is the simplest limit: the equations are completely integrable, the MTM solitons are true solitons, and there is no nontrivial dependence on any parameter aside from Q. The oscillation frequencies may be obtained from Eqs. (5), taking the right-hand sides of the equations equal to zero,

$$\omega_{\Delta Q}^2 = \frac{p_2}{p_1}\gamma^2, \quad (6a)$$

and

$$\omega_{Q\pm}^2 = \frac{p_4 \pm \sqrt{p_4^2 - 4p_3p_5}}{2p_3}\gamma^2. \quad (6b)$$

The model is stable when all the frequencies, $\omega_{\Delta Q}$, ω_{Q+}, and ω_{Q-} are real. Note that the frequencies are functions of the single parameter Q; there is no dependence on the velocity, ρ, save for the time dilation. This conforms to expectations, as the system is Lorentz invariant when $\sigma = 0$.

The real and imaginary parts of the frequencies are shown in Fig. 1. There are values of the gap soliton width at which the vibration frequencies are purely imaginary, which implies a formal (and spurious — see below) instability. We emphasize that these formal instabilities occur even for the MTM case, $\sigma = 0$, which is completely integrable. Since the MTM solitons are known to be stable, the instability is certainly an artifact produced by the variational approximation, indicating a failure of the averaged Lagrangian variational technique in a certain parametric range.

The first instability appears in the ΔQ mode at roughly $Q = 0.3\pi$. Past that point, the oscillation frequencies alternate between stability and instability, with the frequencies going asymptotically to infinity at several points. The borders between the stable and unstable regions may in principle be determined from Eqs. (6). It seems very unlikely that the results will hold true past the point of the first instability.

The lengthiness of the expressions makes it difficult to find simple heuristic accounts of the vibrational dynamics. We can nevertheless account — within the context of the averaged Lagrangian variational technique — for some of the most prominent features of the graphs of the frequencies: the asymptotes. Dispersion in Eqs. (1) results from the group velocity difference combined with the linear cross coupling, and it is balanced by the nonlinearity. In the averaged Lagrangian, the linear cross coupling gives rise to functions of the phase perturbation terms (a, constant; b, linear; and c, quadratic), multiplied by functions of Q, which change sign at $Q = (1/\sqrt{3})\pi \approx 0.58\pi$ and $Q = \sqrt{1-\sqrt{8/15}}\pi \approx 0.52\pi$. The group-velocity difference terms give rise to phase perturbation terms a, b, and c, or their time derivatives multiplied by functions of the fundamental soliton parameter Q which do not change sign over the interval $0 < Q < \pi$. Thus, the effect of the dispersion *in the averaged Lagrangian* undergoes a sign change. The nonlinearity is a sum of absolute values, so it does not directly affect the phase terms, nor do any other terms counterbalance the sign changes. So one would expect a transition from stability to instability in the region of the sign changes, $Q \approx 0.52\pi$ and $Q \approx 0.58\pi$.

A similar analysis may be performed for the general system, in which $\sigma \neq 0$ and the Bragg soliton is moving, $\rho \neq 0$. The eigenfrequencies of the small vibrations are

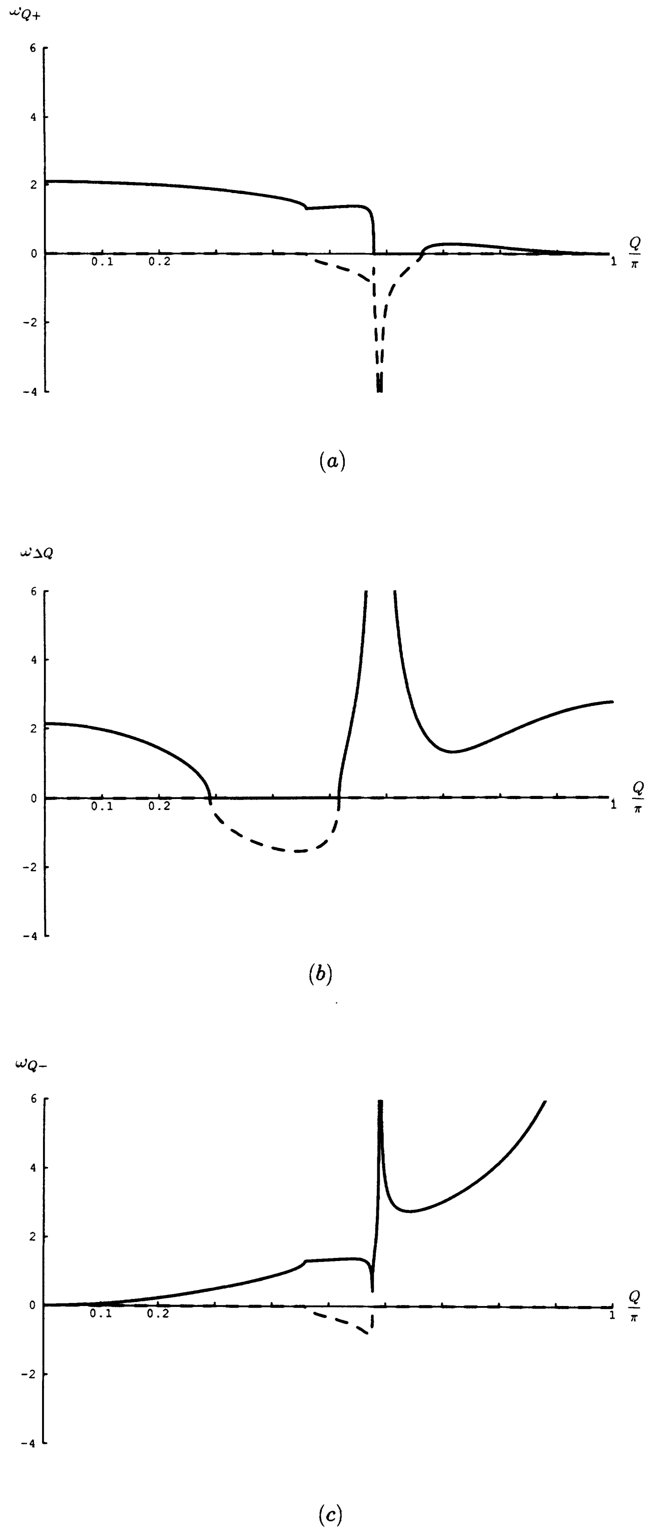

FIG. 1. The vibration frequencies in the massive Thirring model ($\sigma = 0$, $\rho = 0$): (a) $\omega_{\Delta Q}$ vs Q; (b) ω_{Q+} vs Q; (c) ω_{Q-} vs Q. In this figure and in Fig. 2, below, the real parts of the frequencies [which are not associated with the (spurious) instabilities] are drawn as solid lines above the Q axis, and the imaginary parts of the frequencies are drawn as dashed lines below the Q axis [the presence of a nonzero imaginary frequency indicates a (spurious) instability].

determined by the single sixth order equation

$$
\begin{aligned}
0 = \Bigg[& p_1 p_3 \frac{d^6}{d\tau^6} + (p_1 p_4 + p_2 p_3) \frac{d^4}{d\tau^4} \\
& + (p_1 p_5 + p_2 p_4) \frac{d^2}{d\tau^2} + p_2 p_5 \\
& - (4\sigma\alpha^2\gamma^2\rho)^2 \Bigg] \left(\frac{Q - Q_0}{Q} \right).
\end{aligned} \tag{7}
$$

When $\sigma\rho \neq 0$, $\frac{\Delta Q}{Q}$ is a function of $\frac{Q-Q_0}{Q}$:

$$
\frac{\Delta Q}{Q} = \frac{4\sigma\alpha^2\gamma^2\rho}{-p_1\omega^2 + p_2} \frac{Q - Q_0}{Q}, \tag{8}
$$

where ω is a frequency determined by Eq. (7). The three squared frequencies give three distinct dependencies of $\frac{\Delta Q}{Q}$ on $\frac{Q-Q_0}{Q}$, which distinguishes the corresponding three eigenvectors.

IV. GAP SOLITONS IN THE SMALL-AMPLITUDE LIMIT

For small values of the parameter Q (small-amplitude and large width gap solitons), the averaged Lagrangian variational technique produces only real oscillation frequencies. We encounter no spurious instabilities. The three squared oscillation frequencies are

$$
\begin{aligned}
\omega_{\Delta Q}^2 = & \frac{16(3+\pi^2)}{45} \\
& - \left(\frac{80 + 192\alpha^2}{675}\pi^4 + \frac{124}{45}\pi^2 + 4 \right) \left(\frac{Q}{\pi} \right)^2, \\
\omega_{Q+}^2 = & \left(\frac{2\pi}{3} \right)^2 + \left(4 - \frac{48\alpha^2 + 20}{135}\pi^2 \right) Q^2, \\
\omega_{Q-}^2 = & \left(\frac{6Q^2}{\pi^2} \right)^2 \left[1 + \left(\frac{4\alpha^2 + 1}{5}\pi^2 - 12 \right) \left(\frac{Q}{\pi} \right)^2 \right],
\end{aligned}
$$

which works out to approximately

$$
\begin{aligned}
\omega_{\Delta Q} &= 2.14 - (1.01 + 0.66\alpha^2)Q^2, \\
\omega_{Q+} &= 2.09 + (0.61 - 0.84\alpha^2)Q^2, \\
\omega_{Q-} &= 0.61Q^2[1 - (0.51 - 0.40\alpha^2)Q^2].
\end{aligned}
$$

The corresponding eigenvectors are defined by the relations

$$
\begin{aligned}
(\eta_u^2 - \eta_v^2) &= -2\frac{\Delta Q}{Q}, \\
(a_u - a_v) &= \frac{5}{16} \frac{d}{d\tau} \frac{\Delta Q}{Q}, \\
(c_u - c_v) &= -\frac{15Q}{\pi^2} \frac{d}{d\tau} \frac{\Delta Q}{Q}, \\
(b_u + b_v) &= \frac{5}{4} \left(\frac{d^2}{d\tau^2} + \frac{16(3+\pi^2)}{45} \right) \frac{\Delta Q}{Q},
\end{aligned}
$$

and

$$(\eta_u^2+\eta_v^2)Q \equiv 2Q_0 = \text{const},$$

$$\Delta\xi = -\frac{\pi^2}{12Q^2}\frac{d}{d\tau}\frac{Q-Q_0}{Q},$$

$$(b_u-b_v) = -\frac{1}{2}\frac{d^2}{d\tau^2}\frac{Q-Q_0}{Q},$$

$$(c_u+c_v) = \frac{1}{Q}\left(\frac{d^3}{d\tau^3}+\frac{4\pi^2}{9}\frac{d}{d\tau}\right)\frac{Q-Q_0}{Q},$$

$$\frac{d}{d\tau}(a_u+a_v) - 2\cos Q_0$$

$$= \frac{\pi^2}{48Q^2}\left(\frac{d^4}{d\tau^4}+\frac{4\pi^2}{9}\frac{d^2}{d\tau^2}\right)\frac{Q-Q_0}{Q}.$$

The eigenvector of the $\omega_{\Delta Q}$ mode in the small Q limit is dominated by the oscillations of $\frac{\Delta Q}{Q}$, $(\eta_u^2-\eta_v^2)$, and (a_u-a_v), while the other components, (c_u-c_v) and (b_u+b_v), are smaller by a factor of Q. The ω_{Q+} eigenvector is dominated by the $\Delta\xi$ and (a_u+a_v) oscillations. The Q^2 slow oscillation mode ω_{Q-} has the same dominant terms as ω_{Q+}, $\Delta\xi$ and (a_u+a_v), but in a different ratio.

V. THE NONLINEAR ASYMMETRIC COUPLER

The asymmetric dual-core nonlinear coupler is described by the system of two equations [11]

$$iu_z + iu_s + |u|^2u + v = 0, \tag{9a}$$

$$iv_z - iv_s + |v|^2v + u = 0, \tag{9b}$$

where z and s are, respectively, the propagation distance and the so-called reduced time. As detailed in Sec. I, we neglect the proper dispersion in both cores of the coupler, but do take into account the fact that, due to an asymmetry between the cores, they may have a group-velocity difference, accounted for by the second linear terms in Eqs. (9).

Making in Eqs. (1) the substitution

$$u = \frac{1}{\sqrt{\sigma}}u', \quad v = \frac{1}{\sqrt{\sigma}}v',$$

$$t = z, \quad x = s,$$

and then taking $\sigma \to \infty$, one recovers the nonlinear coupler equations (9). The asymmetric nonlinear coupler is, therefore, a limiting case of Eqs. (1), and we may apply to it the results herein. In the appropriate limit we have $\alpha^2 = \left(\frac{1}{\sigma}\right)\frac{1-\rho^2}{1+\rho^2}$, $4\sigma\alpha^2\gamma^2\rho = \frac{4\rho}{1+\rho^2}$, and the coupler soliton is

$$u' = \frac{(1+\rho)^{3/4}(1-\rho)^{1/4}}{(1+\rho^2)^{1/2}}\sin Q\operatorname{sech}\left(\xi\sin Q - \frac{i}{2}Q\right)$$
$$\times\exp\left\{i\left[\phi - \tau\cos Q + \frac{4\rho}{1+\rho^2}\tan^{-1}\left(\tan\frac{Q}{2}\tanh(\xi\sin Q)\right)\right]\right\},$$

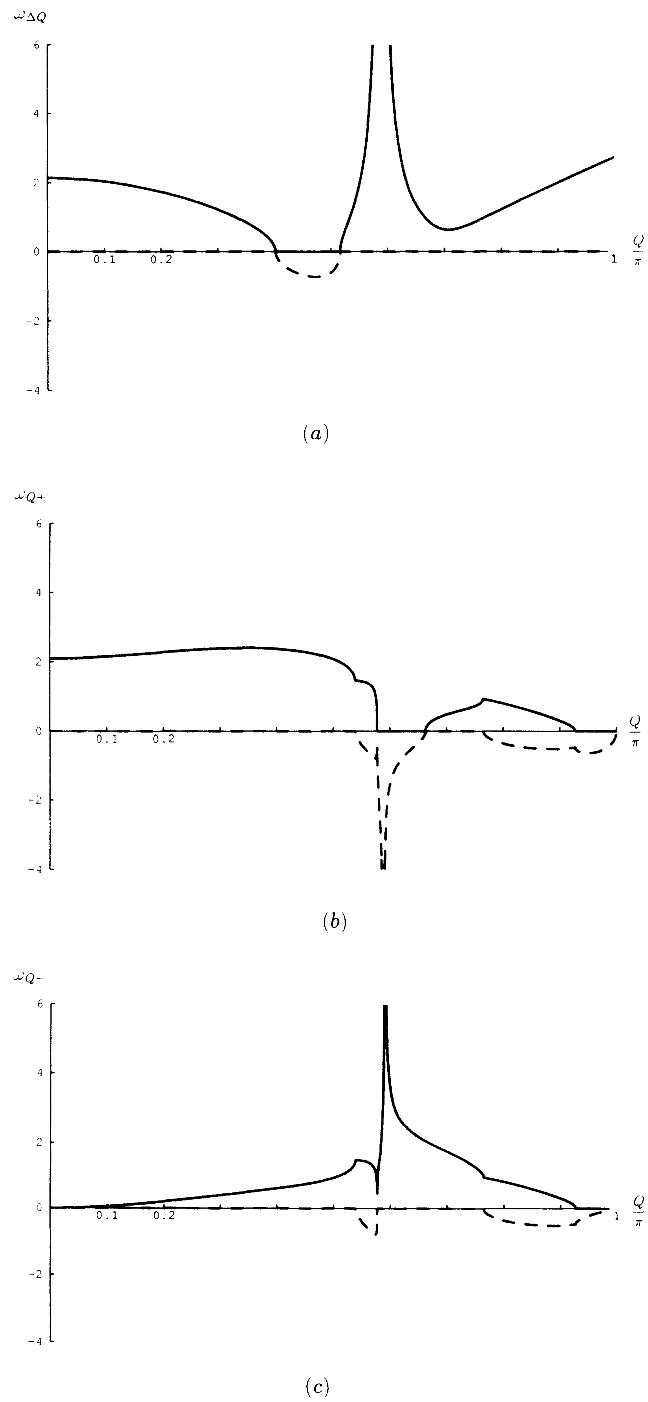

FIG. 2. The nonlinear asymmetric coupler ($\sigma = \infty$, $\rho = 0$): (a) $\omega_{\Delta Q}$ vs Q; (b) ω_{Q+} vs Q; (c) ω_{Q-} vs Q.

$$v' = -\frac{(1+\rho)^{1/4}(1-\rho)^{3/4}}{\sqrt{1+\rho^2}} \sin Q \operatorname{sech}\left(\xi \sin Q + \frac{i}{2}Q\right)$$
$$\times \exp\left\{ i \left[\phi - \tau \cos Q + \frac{4\rho}{1+\rho^2} \tan^{-1}\left(\tan \frac{Q}{2} \tanh(\xi \sin Q) \right) \right] \right\},$$

where the parameters τ and ξ are functions of z and s rather than of t and x.

For the asymmetric nonlinear coupler, the soliton's vibrational dynamics are roughly like those of the gap soliton. In our calculations, we simply apply Eqs. (5), taking $\sigma \to \infty$. Figure (2) shows the three oscillation frequencies for a stationary soliton. The graphs of all three oscillation mode frequencies, $\omega_{\Delta Q}$, ω_{Q+}, and ω_{Q-}, are qualitatively similar to the MTM case, with only moderate quantitative differences in the frequencies and the critical values of Q corresponding to the separatrices between stability and the spurious instability.

VI. CONCLUSION

The averaged Lagrangian variational technique is a standard tool for investigating free [12,13] or driven [14] oscillations of a solitary wave. Applying the technique to gap solitons governed by Eqs. (1), we found three degrees of freedom in which the gap soliton can oscillate. Throughout, the total energy is conserved, i.e., to first order, $(\eta_u^2 + \eta_v^2)Q$ remains constant. A caveat: the analysis *assumed* zero energy was radiated in the course of the oscillations, so any effects that involve radiative decay are not captured.

As the parameter Q, which determines the mean width of the soliton, $(\sin \frac{Q}{2})^{-1}$, oscillates, so do $\frac{d}{d\tau}(a_u + a_v)$, the time derivative of the global phase; $(b_u - b_v)$, the antisymmetric spacial frequency; $(c_u + c_v)$, the symmetric frequency chirp; and $\Delta\xi$, the distance between the centers of the two components. The first two move in phase with the Q oscillation, the last two, out of phase.

As ΔQ, which determines the difference in the widths of the two components, oscillates, so do $(a_u - a_v)$, the phase difference between the polarization components; $(c_u - c_v)$, the antisymmetric frequency chirp; $(b_u + b_v)$, the symmetric spatial frequency; and $(\eta_u^2 - \eta_v^2)$, antisymmetric increase and decrease in the amplitudes of the polarization components. The first two oscillate out of phase with ΔQ, the next two, in phase.

Whether or not Q and ΔQ couple to each other depends on whether the system is at the MTM limit ($\sigma = 0$) and on whether the gap soliton as a whole is not moving. If either of these conditions holds, Q and ΔQ are independent. ΔQ moves in a two-dimensional phase space, while Q moves in a four-dimensional phase space. If both $\sigma \neq 0$ (in which case the system is not Lorentz invariant) and the soliton is in motion, Q and ΔQ are coupled together, and oscillate within in a six-dimensional phase space.

For small Q — small-amplitude and large-width gap solitons — the system is stable in all of the three degrees of freedom. For pulses of larger amplitude and smaller width, the analysis gives formal instabilities. Some of the causes of the instabilities are evident in a direct examination of the averaged Lagrangian (see Sec. III), while the causes of other instabilities are obscured by the complexity of the equations which govern small vibrations. In any case, the instabilities occur even in the massive Thirring model, which is known to be always stable, so it follows that the instabilities are spurious. These spurious instabilities, represent a "catastrophic" failure of the averaged Lagrangian variational technique as applied to soliton systems.

Since this is a newly discovered failure of the averaged Lagrangian variational technique, it is not entirely clear what confidence can be placed in the results even in the range in which there are no apparent problems. Nonetheless, Sec. IV outlines in detail one limit (small amplitude) in which there is no *a priori* reason to doubt the accuracy. We leave to a later paper a comparison with numerical simulations. In any case, the discovery of spurious instabilities in an application of the technique calls for a deeper justification of the technique in a general form.

ACKNOWLEDGMENTS

A useful discussion with Stefan Wabnitz is appreciated. This work was supported in part by a grant from the Research Office of Tel Aviv University.

APPENDIX

The equations of motion that arise from the averaged Lagrangian variational method, in the small vibrational limit about the fixed point (the fundamental soliton solution) are

$$0 = q_1(\Delta Q/Q) + q_2(\eta_u^2 - \eta_v^2),$$
$$0 = q_3(a_u - a_v) + q_4(c_u - c_v) + q_5\frac{d}{d\tau}(\Delta Q/Q) + q_6\frac{d}{d\tau}(\eta_u^2 - \eta_v^2),$$
$$0 = q_7(a_u - a_v) + q_8(c_u - c_v) + q_9\frac{d}{d\tau}(\Delta Q/Q) + q_{10}\frac{d}{d\tau}(\eta_u^2 - \eta_v^2),$$
$$0 = q_{11}\frac{d}{d\tau}(a_u - a_v) + q_{12}\frac{d}{d\tau}(c_u - c_v) + q_{13}(b_u + b_v) + q_{14}(\Delta Q/Q) + q_{15}(\eta_u^2 - \eta_v^2) + q_{16}4\sigma\alpha^2\gamma^2\rho[(Q - Q_0)/Q],$$
$$0 = q_{17}\frac{d}{d\tau}(a_u - a_v) + q_{18}\frac{d}{d\tau}(c_u - c_v) + q_{19}(b_u + b_v) + q_{20}(\Delta Q/Q) + q_{21}(\eta_u^2 - \eta_v^2) + q_{22}4\sigma\alpha^2\gamma^2\rho[(Q - Q_0)/Q],$$
$$0 = r_1\frac{d}{d\tau}[(Q - Q_0)/Q] + r_2\Delta\xi,$$
$$0 = r_3\frac{d}{d\tau}\Delta\xi + r_4(b_u - b_v),$$

$$0 = r_5(c_u + c_v) + r_6\frac{d}{d\tau}(b_u - b_v) + r_7\frac{d}{d\tau}[(Q - Q_0)/Q] + r_8\Delta\xi,$$

$$0 = r_9\left(\frac{d}{d\tau}(a_u + a_v) - 2\cos Q_0\right) + 4\sigma\alpha^2\gamma^2\rho[r_{10}(\Delta Q/Q) + r_{11}(\eta_u^2 - \eta_v^2)] + r_{12}\frac{d}{d\tau}(c_u + c_v) + r_{13}[(Q - Q_0)/Q] + r_{14}\frac{d}{d\tau}\Delta\xi,$$

$$0 = r_{15}\left(\frac{d}{d\tau}(a_u + a_v) - 2\cos Q_0\right) + r_{16}\frac{d}{d\tau}(c_u + c_v) + r_{17}\frac{d}{d\tau}\Delta\xi.$$

The parameters q_j and r_j are functions of the fundamental soliton parameter Q, the velocity of the soliton ρ, and the coefficient of self-phase modulation σ:

$$q_1 = 2,\ q_2 = 1,\ q_3 = 2,\ q_4 = \tfrac{\pi^2 - 3Q^2}{12\sin^2 Q}\sin\tfrac{Q}{2},\ q_5 = -\tfrac{Q}{\sin Q},\ q_6 = -\tfrac{Q}{2\sin Q},$$
$$q_7 = \tfrac{\pi^2 - 3Q^2}{6},\ q_8 = \tfrac{7\pi^4 - 30\pi^2 Q^2 + 15Q^4}{240\sin^2 Q}\sin\tfrac{Q}{2},\ q_9 = -\tfrac{Q\sin Q}{12}\tfrac{d}{dQ}\tfrac{Q(\pi^2 - Q^2)}{\sin^2 Q},$$
$$q_{10} = -\tfrac{Q(\pi^2 - Q^2)}{24\sin Q},\ q_{11} = 1,\ q_{12} = \tfrac{\pi^2 - Q^2}{24\sin^2 Q}\sin Q/2,\ q_{13} = 1,$$
$$q_{14} = 2\cos Q + (2 - 4\alpha^2)Q\sin Q,\ q_{15} = \cos Q + (3 - 4\alpha^2)(\tfrac{\sin Q}{Q} - \cos Q),$$
$$q_{16} = -2(\tfrac{\sin Q}{Q} - \cos Q),\ q_{17} = 1,\ q_{18} = \tfrac{d}{dQ}\left(\tfrac{Q(\pi^2 - Q^2)}{24\sin^2 Q}\right)\sin\tfrac{Q}{2},\ q_{19} = 1,$$

$$\begin{aligned} q_{20} = {} & -2\alpha^2 Q(\sin Q + Q\cos Q) - \frac{2}{3}\alpha^2 Q\cos^2 Q\left(\frac{\sin^2 Q}{3}\frac{d}{dQ}\frac{1}{\sin Q} + \cos Q\right)\frac{d}{dQ}\frac{1}{\sin Q}\frac{d}{dQ}\frac{Q(\pi^2 - Q^2)}{\sin Q} \\ & + 2\alpha^2 Q\sin^2 Q\left[\frac{\sin^2 Q}{3}\frac{d}{dQ}\frac{1}{\sin Q}\frac{d}{dQ}\frac{1}{\sin Q} + \frac{\cos Q}{3}\frac{d}{dQ}\frac{1}{\sin Q} + (-4\csc^2 Q + 1)\right]\frac{d}{dQ}\frac{Q}{\sin Q} + \left(\frac{4}{\sin^2 Q} - 2\right)\frac{Q}{\sin Q} \\ & + Q\cos^2 Q\left(\frac{2}{3}(\cot^2 Q - 1) + \frac{4}{3}\cos Q\frac{d}{dQ}\frac{1}{\sin Q} + \frac{2}{9}\sin^4 Q\frac{d}{dQ}\frac{1}{\sin Q}\frac{d}{dQ}\frac{1}{\sin Q}\right)\frac{d}{dQ}\frac{Q(\pi^2 - Q^2)}{\sin Q} \\ & + Q\left(2 - \frac{8}{3}\sin^2 Q\cos Q\frac{d}{dQ}\frac{1}{\sin Q} - \frac{2}{3}\sin^4 Q\frac{d}{dQ}\frac{1}{\sin Q}\frac{d}{dQ}\frac{1}{\sin Q}\right)\frac{d}{dQ}\frac{Q}{\sin Q}, \end{aligned}$$

$$q_{21} = \cos Q + (1 - 2\alpha^2)Q\sin Q, q_{22} = -Q\sin Q,$$

$$r_1 = Q\frac{d}{dQ}\left(\frac{\pi^2 - Q^2}{24\sin^2 Q}\right), r_2 = 1,\quad r_3 = 1,\quad r_4 = \frac{\pi^2 - 3Q^2}{6Q\sin Q},$$
$$r_5 = Q\sin\frac{Q}{2}, r_6 = Q,\quad r_7 = -2Q^2\frac{d}{dQ}\left(\frac{\sin Q}{Q} - \cos Q\right),$$

$$\begin{aligned} r_8 = {} & 8\alpha^2\sin^4 Q\left(\frac{\sin^2 Q}{3}\frac{d}{dQ}\frac{1}{\sin Q} + \cos Q\right)\frac{d}{dQ}\frac{1}{\sin Q}\frac{d}{dQ}\frac{Q}{\sin Q} \\ & - 8\sin^4 Q\left(\frac{\sin^2 Q}{3}\frac{d}{dQ}\frac{1}{\sin Q}\frac{d}{dQ}\frac{1}{\sin Q} + 2\cos Q\frac{d}{dQ}\frac{1}{\sin Q} + (\cot^2 Q - 1)\right)\frac{d}{dQ}\frac{Q}{\sin Q}, \end{aligned}$$

$$r_9 = 1,\quad r_{10} = Q\sin Q, r_{11} = \frac{\sin Q}{Q} - \cos Q, r_{12} = \frac{\pi^2 - Q^2}{24\sin^2 Q}\sin\frac{Q}{2},$$

$$r_{13} = 2Q\sin Q - 4\left(\frac{\sin Q}{Q} - \cos Q\right), r_{14} = -2\left(\frac{\sin Q}{Q} - \cos Q\right), r_{15} = 1,$$
$$r_{16} = \frac{d}{dQ}\left(\frac{Q(\pi^2 - Q^2)}{24\sin^2 Q}\right)\sin\frac{Q}{2}, \text{and } r_{17} = -2Q\sin Q.$$

The coefficients p_j of Eqs. (5), which are constructed out of q_j and r_j, are

$$p_1 = (q_3q_8 - q_4q_7)^{-1}(q_{13}q_{22} - q_{16}q_{19})^{-1}\begin{pmatrix} q_{13}q_{17} - q_{11}q_{19} \\ q_{13}q_{18} - q_{12}q_{19}\end{pmatrix}^t\begin{pmatrix} -q_8 & q_4 \\ q_7 & -q_3\end{pmatrix}\begin{pmatrix} q_5 - q_6q_1/q_2 \\ q_9 - q_{10}q_1/q_2\end{pmatrix},$$

$$p_2 = \frac{q_{13}(q_{20} - q_{21}q_1/q_2) - (q_{14} - q_{15}q_1/q_2)q_{19}}{q_{13}q_{22} - q_{16}q_{19}},$$
$$p_3 = \frac{r_1}{r_2}\frac{r_3}{r_4}\frac{r_6}{r_5}\frac{r_9r_{16} - r_{12}r_{15}}{(r_{10} - r_{11}q_1/q_2)r_{15}},$$
$$p_4 = \frac{\frac{1}{r_5}(r_7 - r_8r_1/r_2)(r_9r_{16} - r_{12}r_{15}) + (r_9r_{17} - r_{14}r_{15})r_1/r_2}{(r_{10} - r_{11}q_1/q_2)r_{15}},$$
$$p_5 = \frac{r_{13}}{(r_{10} - r_{11}q_1/q_2)}.$$

(The superscript t in the expression for p_1 indicates matrix transpose.)

The parameters of the Bragg soliton ansatz are interconnected via the following relations, which are expressed in terms of the q_j and r_j parameters defined above. Note that the different eigenfrequencies (where there is more than one) lead to different relations between the ansatz parameters, which creates different eigenvectors. There are five terms slaved to Q, the parameter that describes in-phase contraction-dilation of the two polarization components. Note that the relative translation term $\Delta\xi$ couples to Q. Therefore, symmetric dilation-contraction cannot occur without an antisymmetric (relative) translation oscillation of the polarization component centers:

$$(\eta_u^2 + \eta_v^2)Q \equiv 2Q_0 = \text{const},$$
$$\Delta\xi = -(r_1/r_2)\frac{d}{d\tau}\frac{Q - Q_0}{Q},$$
$$(b_u - b_v) = (r_1/r_2)(r_3/r_4)\frac{d^2}{d\tau^2}\frac{Q - Q_0}{Q},$$
$$(c_u + c_v) = -\left(\frac{r_1r_3r_6}{r_2r_4r_5}\frac{d^3}{d\tau^3} + \frac{r_7 - r_8r_1/r_2}{r_5}\frac{d}{d\tau}\right)\frac{Q - Q_0}{Q},$$
$$\frac{d}{d\tau}(a_u + a_v) - 2\cos Q_0 = \left[\frac{r_1r_3r_6r_{16}}{r_2r_4r_5r_{15}}\frac{d^4}{d\tau^4} + \left(\frac{r_7 - r_8r_1/r_2}{r_5}\frac{r_{16}}{r_{15}} + \frac{r_{17}}{r_{15}}\frac{r_1}{r_2}\right)\frac{d^2}{d\tau^2}\right]\frac{Q - Q_0}{Q}.$$

When $\sigma\rho = 0$, ΔQ is independent of Q, and the behavior of ΔQ is governed by Eq. (5a). But when $\sigma\rho \neq 0$, ΔQ is slaved to Q. (The number of degrees of freedom is the same in both cases, since, when ΔQ is slaved to Q, Q is described by an equation of order higher than when the parameters are independent.) There is a dependence on the eigenfrequency [of which there are three, defined by Eq. (7)]:

$$\frac{\Delta Q}{Q} = \frac{4\sigma\alpha^2\gamma^2\rho}{-\omega^2 p_1 + p_2}\frac{Q - Q_0}{Q}.$$

The final equations describe the dependence of four parameters on the antisymmetric contraction-dilation (ΔQ),

$$(\eta_u^2 - \eta_v^2) = -(q_1/q_2)(\Delta Q/Q),$$
$$(a_u - a_v) = \frac{-(q_5 - q_6q_1/q_2)q_8 + q_4(q_9 - q_{10}q_1/q_2)}{q_3q_8 - q_4q_7}\frac{d}{d\tau}(\Delta Q/Q),$$
$$(c_u - c_v) = \frac{-q_3(q_9 - q_{10}q_1/q_2) + (q_5 - q_6q_1/q_2)q_7}{q_3q_8 - q_4q_7}\frac{d}{d\tau}(\Delta Q/Q),$$
$$(b_u + b_v) = (q_3q_8 - q_4q_7)^{-1}(q_{13}q_{22} - q_{16}q_{19})^{-1}\begin{pmatrix} q_{11}q_{22} - q_{16}q_{17} \\ q_{12}q_{22} - q_{16}q_{18} \end{pmatrix}^t \begin{pmatrix} q_8 & -q_4 \\ -q_7 & q_3 \end{pmatrix} \begin{pmatrix} q_5 - q_6q_1/q_2 \\ q_9 - q_{10}q_1/q_2 \end{pmatrix}\frac{d^2}{d\tau^2}\frac{\Delta Q}{Q}$$
$$-(q_{13}q_{22} - q_{16}q_{19})^{-1}[(q_{14}q_{22} - q_{16}q_{20}) - (q_{15}q_{22} - q_{16}q_{21})q_1/q_2]\frac{\Delta Q}{Q}.$$

[1] W. E. Thirring, Ann. Phys. (NY) **3**, 91 (1958).
[2] E. A. Kuznetsov and A. V. Mikhailov, Teor. Mat. Fiz. **30**, 193 (1977).
[3] D. J. Kaup and A. C. Newell, Lett. Nuovo Cimento **20**, 325 (1977).
[4] G. P. Agrawal, *Nonlinear Fiber Optics* (Academic, San Diego, 1989).
[5] C.R. Menyuk, IEEE J. Quantum Electron. **QE-23**, 174 (1987); **QE-25**, 2674 (1989).
[6] A. B. Aceves and S. Wabnitz, Phys. Lett. A **141**, 37 (1989).
[7] D. N. Christodoulides and R. I. Joseph, Phys. Rev. Lett. **62**, 1746 (1989).
[8] C. M. De Sterke and J. E. Sipe, Phys. Rev. A **42**, 2858 (1990).
[9] S. M. Jensen, IEEE J. Quantum Electron. **18**, 158 (1982).
[10] S. Trillo, S. Wabnitz, E.M. Wright, and G.I. Stegeman, Opt. Lett. **13**, 672 (1990); C. Paré and M. Florjańczyk,

Phys. Rev. A **41**, 6287 (1990).
[11] M. Romangoli, S. Trillo, and S. Wabnitz, Opt. Quantum Electron. **24**, S1237 (1992).
[12] D. Anderson, Phys. Rev. A **27**, 313 (1983); D. Anderson, M. Lisak, and T. Reichel, J. Opt. Soc. Am. B **5**, 207 (1988).
[13] T. Ueda and W. L. Kath, Phys. Rev. A **42**, 563 (1990).
[14] B. A. Malomed, D. F. Parker, and N. F. Smyth, Phys. Rev. E **48**, 1418 (1993).