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Time-dependent Beltrami fields in free space: Dyadic Green functions and radiation potentials

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The Beltrami-Maxwell equations for free space arise as specializations of the Beltrami-Maxwell equations for general material continua. Here, we investigate the novel concept of time-dependent Beltrami fields to solve the electromagnetic radiation problem in free space. We derive Beltrami field representations in terms of dyadic Green functions and vector potentials. Closed-form results for the Beltrami fields are presented for elementary point sources.

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I. INTRODUCTION

The concept of Beltrami fields and flows has a long and distinguished history in fluid mechanics. In electromagnetism, following early work by Silberstein [1], the Beltrami field concept has been repeatedly rediscovered throughout this century—see, e.g., [2–4]—though its antecedents have generally remained muddled. It is fair to state that these approaches mainly viewed the Beltrami field concept as a convenient tool to rearrange the time-harmonic electromagnetic field equations. Only in recent years, with considerable interest in the study of complex media, has there been a shift in emphasis: Beltrami fields are essential for the description of time-harmonic electromagnetic fields in chiral and biisotropic media [5].

While all these developments applied to fields that are either static or time-harmonic, a Beltrami field formulation for time-dependent fields has been lacking until recently. Such a description of time-dependent Beltrami fields has now been achieved, and the fundamental Beltrami-Maxwell equations have been formulated for a general material continuum [6]. It was shown that the time-dependent Beltrami-Maxwell equations in a material continuum are given as [6]

$$\nabla\cdot\mathbf{F}_{\pm}(\mathbf{r},t)=\mp icw_{\pm}(\mathbf{r},t)\ , \tag{1}$$

$$\nabla\times\mathbf{Q}_{\pm}(\mathbf{r},t)\mp(i/c)\partial_t\mathbf{F}_{\pm}(\mathbf{r},t)=\mathbf{W}_{\pm}(\mathbf{r},t)\ . \tag{2}$$

In Eqs. (1) and (2), $\mathbf{Q}_{\pm}(\mathbf{r},t)$ are the time-dependent Beltrami fields, and $\mathbf{F}_{\pm}(\mathbf{r},t)$ are the time-dependent Beltrami induction fields. The impressed Beltrami charge densities $w_{\pm}(\mathbf{r},t)$ and the Beltrami current densities $\mathbf{W}_{\pm}(\mathbf{r},t)$ are related through the continuity relation

$$\nabla\cdot\mathbf{W}_{\pm}(\mathbf{r},t)+\partial_t w_{\pm}(\mathbf{r},t)=0\ . \tag{3}$$

All fields and sources in Eqs. (1)–(3) are complex-valued. Here and hereafter, i is the imaginary unit, $c=1/\sqrt{\epsilon_0\mu_0}$ is the speed of light in free space (vacuum), and $\epsilon_0=8.854\times10^{-12}$ F/m and $\mu_0=4\pi\times10^{-7}$ H/m are the vacuum permittivity and permeability, respectively; ∂_t is the partial derivative with respect to time; $\mathbf{r}$ is the position vector of the observation point; vectors appear in boldface; dyadics are underlined.

Equations (1) and (2) can be shown to be Lorentz covariant [7]. However, they do not form a closed system of differential equations unless constitutive relations between $\mathbf{F}_{\pm}(\mathbf{r},t)$ and $\mathbf{Q}_{\pm}(\mathbf{r},t)$ are formulated. To set up appropriate constitutive relations, Beltrami polarization fields $\mathbf{P}_{\pm}(\mathbf{r},t)$ are conceptualized through

$$\mathbf{F}_{\pm}(\mathbf{r},t)=\mathbf{Q}_{\pm}(\mathbf{r},t)+\mathbf{P}_{\pm}(\mathbf{r},t)\ ; \tag{4}$$

thus a prescription for $\mathbf{P}_{\pm}(\mathbf{r},t)$ in terms of $\mathbf{Q}_{\pm}(\mathbf{r},t)$ is necessitated. Such a formulation, whereby the macroscopic properties of the material continuum can arise from a model of the behavior of the microscopic constituents of the medium, has been given for linear, spatially nonhomogeneous, spatially local, temporally causal, bianisotropic materials [6]. The constitutive relations are

$$\mathbf{P}_{+}(\mathbf{r},t)=\underline{a}_{11}(\mathbf{r},t)\circ\mathbf{Q}_{+}(\mathbf{r},t)+\underline{a}_{12}(\mathbf{r},t)\circ\mathbf{Q}_{-}(\mathbf{r},t)\ , \tag{5}$$

$$\mathbf{P}_{-}(\mathbf{r},t)=\underline{a}_{21}(\mathbf{r},t)\circ\mathbf{Q}_{+}(\mathbf{r},t)+\underline{a}_{22}(\mathbf{r},t)\circ\mathbf{Q}_{-}(\mathbf{r},t)\ , \tag{6}$$

where $\underline{a}_{ij}(\mathbf{r},t)$ $(i,j=1,2)$ are the dyadic susceptibility operators, and $\circ$ denotes temporal convolution:

$$\underline{a}_{ij}(\mathbf{r},t)\circ\mathbf{Q}_{\pm}(\mathbf{r},t)=\int_{-\infty}^{t}dt'\underline{a}_{ij}(\mathbf{r},t-t')\cdot\mathbf{Q}(\mathbf{r},t')\ . \tag{7}$$

It is emphasized that the differential equations (1) and (2) and the constitutive relations (4)–(6) provide a fully self-consistent theoretical apparatus, and no recourse has yet been taken to the usual formulation of time-dependent problems in terms of the electromagnetic fields $\mathbf{E}(\mathbf{r},t)$, $\mathbf{H}(\mathbf{r},t)$, $\mathbf{D}(\mathbf{r},t)$, and $\mathbf{B}(\mathbf{r},t)$. It is important to remember that the four fields $\mathbf{Q}_{\pm}$ and $\mathbf{F}_{\pm}$ are complex valued and therefore possess twice as many degrees of freedom as the real-valued electromagnetic fields $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$. A connection between the Beltrami-Maxwell fields and the electromagnetic fields can be made to exist by using the dictionary:

$$\mathbf{Q}_{\pm}(\mathbf{r},t)=\mathbf{E}(\mathbf{r},t)\pm i\sqrt{\mu_0/\epsilon_0}\mathbf{H}(\mathbf{r},t)\ , \tag{8}$$

$$\mathbf{F}_{\pm}(\mathbf{r},t)=\mathbf{D}(\mathbf{r},t)/\epsilon_0\pm ic\mathbf{B}(\mathbf{r},t)\ , \tag{9}$$

$$\mathbf{W}_{\pm}(\mathbf{r},t)=-\mathbf{J}_m(\mathbf{r},t)\pm i\sqrt{\mu_0/\epsilon_0}\mathbf{J}_e(\mathbf{r},t)\ , \tag{10}$$

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$$w_{\pm}(\mathbf{r},t)=-\rho_m(\mathbf{r},t)\pm i\sqrt{\mu_0/\epsilon_0}\,\rho_e(\mathbf{r},t)\ , \tag{11}$$

relations which effectively amount to a reduction in the number of degrees of freedom of the complex Beltrami-Maxwell formalism. Here ρ_e and ρ_m are the electric and magnetic charge densities; $\mathbf{J}_e(\mathbf{r},t)$ and $\mathbf{J}_m(\mathbf{r},t)$ are the electric and magnetic current densities. Through relations (8)–(11) it is then a straightforward exercise to establish a mapping between the dyadic susceptibility operators $\underline{a}_{ij}$ in the Beltrami-Maxwell formulation and the constitutive dyadic operators (permittivity $\underline{\epsilon}$, etc.) of the conventional Maxwell formalism [8]. Boundary and/or initial conditions are usually formulated in terms of $\mathbf{E}$ and $\mathbf{H}$ in the conventional electromagnetic formalism. These conditions can be translated into those for the Beltrami fields $\mathbf{Q}_{\pm}$ by using Eq. (8). In the context of the present paper, boundary conditions are of lesser interest. As we are motivated to present closed-form results applicable to the radiation of confined sources in an unbounded region of free space, the only conditions on our solutions are a general necessity for them to be *causal,* thus implying an outgoing-wave condition [9].

The purpose of this paper is to explore the time-dependent Beltrami-Maxwell equations for free space, wherein the polarization fields $\mathbf{P}_{\pm}$ vanish identically. Therefore, from Eq. (4), the relation

$$\mathbf{F}_{\pm}(\mathbf{r},t)\equiv\mathbf{Q}_{\pm}(\mathbf{r},t)\ , \tag{12}$$

holds in free space, and Eqs. (1) and (2) reduce to

$$\nabla\cdot\mathbf{Q}_{\pm}(\mathbf{r},t)=\mp icw_{\pm}(\mathbf{r},t)\ , \tag{13}$$

$$\nabla\times\mathbf{Q}_{\pm}(\mathbf{r},t)\mp(i/c)\partial_t\mathbf{Q}_{\pm}(\mathbf{r},t)=\mathbf{W}_{\pm}(\mathbf{r},t)\ . \tag{14}$$

Indeed, Eqs. (13) and (14) follow naturally from the time-dependent Maxwell's equations for free space via the definition of the Beltrami fields and sources through Eqs. (8)–(11). Here the Beltrami-Maxwell equations for free space have been given as specializations of the Beltrami-Maxwell equations for general material continua to underscore the importance of the Beltrami-Maxwell field formalism, viewed not so much as a mathematical manipulation of the conventional electromagnetic formalism but as a theoretical apparatus in its own right.

We can observe an important difference at the level of the mathematical formalism between Beltrami fields in material continua and those in free space. It is apparent from the constitutive relations (5) and (6) that at the microscopic level both fields $\mathbf{Q}_{+}$ and $\mathbf{Q}_{-}$ are necessary for a consistent description. In free space, however, the polarization fields vanish. Therefore, there is no coupling between $\mathbf{Q}_{+}$ and $\mathbf{Q}_{-}$ through the differential equations (13) and (14) and it is sufficient to carry out the analysis for either $\mathbf{Q}_{+}$ or $\mathbf{Q}_{-}$; because by virtue of Eq. (8) we then have $\mathbf{Q}_{+}^{*}=\mathbf{Q}_{-}$ (the symbol $*$ indicates complex conjugation). Nevertheless, to remain in the spirit of the Beltrami-Maxwell formalism, we will continue to write subsequent equations and solutions in terms of $\mathbf{Q}_{\pm}$.

In the following, we will investigate the free-space equations (13) and (14) and present field representations in terms of dyadic Green functions and vector potentials (the concept of purely scalar Beltrami-Hertz potentials is explored elsewhere [10]). Closed-form expressions for simple radiating sources will be derived.

II. DYADIC GREEN FUNCTIONS

Due to the linearity of the Beltrami-Maxwell equations the (particular) solution to Eq. (14) can be represented as

$$\underline{Q}_{\pm}(\mathbf{r},t)=\int_{\Lambda'}d^3\mathbf{r}'dt'\underline{G}_{\pm}(\mathbf{r},t;\mathbf{r}',t')\cdot\mathbf{W}_{\pm}(\mathbf{r}',t')\ . \tag{15}$$

Here $\underline{G}_{\pm}(\mathbf{r},t;\mathbf{r}',t')$ are the dyadic Green functions, and the integration is over the space-time span Λ' in which the source current densities are nonzero. Incidentally, the divergence relation (13) is automatically satisfied if Eq. (14) and the continuity relation (3) hold.

The dyadic Green functions $\underline{G}_{\pm}(\mathbf{r},t;\mathbf{r}',t')$ satisfy the differential equations

$$[\nabla\times\underline{I}\mp(i/c)\underline{I}\partial_t]\cdot\underline{G}_{\pm}(\mathbf{r},t;\mathbf{r}',t')=\underline{I}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')\ , \tag{16}$$

$\underline{I}$ being the unit dyadic and $\delta(\cdot)$, the Dirac delta function. The solution of Eq. (16) can be written in the form

$$\underline{G}_{\pm}(\mathbf{r},t;\mathbf{r}',t')=[\nabla\times\underline{I}\pm(i/c)\underline{I}\partial_t]\cdot\underline{G}_{fs}(\mathbf{r},t;\mathbf{r}',t')\ , \tag{17}$$

where the free space dyadic Green function $\underline{G}_{\mathrm{fs}}(\mathbf{r},t;\mathbf{r}',t')$ is a solution of

$$[\nabla\times\nabla\times\underline{I}+(1/c^2)\partial_{tt}\underline{I}]\cdot\underline{G}_{\mathrm{fs}}(\mathbf{r},t;\mathbf{r}',t')=\underline{I}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')\ . \tag{18}$$

Equation (18) has the well-known solution (see, for example, [9])

$$\underline{G}_{\mathrm{fs}}(\mathbf{r},t;\mathbf{r}',t')=(1/\mu_0\partial_t)[\mu_0\partial_t\underline{I}-(1/\epsilon_0\partial_t)\nabla\nabla]g(\mathbf{r},t;\mathbf{r}',t')\ , \tag{19}$$

where $g(\mathbf{r},t;\mathbf{r}',t')$ is the scalar Green function of the time-dependent scalar wave equation

$$[\nabla^2-(1/c^2)\partial_{tt}]g(\mathbf{r},t;\mathbf{r}',t')=-\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')\ . \tag{20}$$

We note parenthetically that

$$(1/\partial_t)f(t)\equiv(\partial_t)^{-1}f(t)\equiv\int_{-\infty}^{t}f(t')dt'$$

is a unique operational definition of $1/\partial_t$ if a causality/initial condition is imposed.

The retarded solution of Eq. (20) is

$$g(\mathbf{r},t;\mathbf{r}',t')=\delta(\tau)/4\pi R\ , \tag{21}$$

where $R=|\mathbf{R}|$, $\mathbf{R}=\mathbf{r}-\mathbf{r}'$, and $\tau=t-t'-R/c$ is the retarded time.

Using Eqs. (19)–(21) in Eq. (17), we obtain the causal solution for $\underline{G}_{\pm}$ for which

$$\underline{G}_{\pm}(\mathbf{r},t;\mathbf{r}',t')\equiv0\quad\text{for }\tau=t-t'-R/c<0\ , \tag{22}$$

in the form

$$\underline{G}_{\pm}(\mathbf{r},t;\mathbf{r}',t') = [\nabla\times\underline{I}\pm(i/c)\partial_t\underline{I}\mp(ic/\partial_t)\nabla\nabla](\delta(\tau)/4\pi R) \ . \tag{23}$$

Explicitly, solution (23) can be given as

$$\pm 4\pi\underline{G}_{\pm}(\mathbf{r},t;\mathbf{r}',t') = (i/Rc)(\underline{I}-\hat{\mathbf{R}}\hat{\mathbf{R}}\pm\hat{\mathbf{R}}\times\underline{I})\delta'(\tau) + (i/R^2)(\underline{I}-3\hat{\mathbf{R}}\hat{\mathbf{R}}\pm i\hat{\mathbf{R}}\times\underline{I})\delta(\tau) + (ic/R^3)(\underline{I}-3\hat{\mathbf{R}}\hat{\mathbf{R}})U(\tau) \ , \tag{24}$$

where we have introduced the unit vector $\hat{\mathbf{R}}=\mathbf{R}/R$; $\delta'(\cdot)$ is the derivative of $\delta(\cdot)$ with respect to the argument, and $U(\cdot)$ is the unit step function defined via

$$U(\tau)=\begin{cases}0 & \tau<0\\ 1 & \tau\geq 0\ ,\end{cases} \quad \delta(\tau)=U'(\tau) \ . \tag{25}$$

Elementary point sources

As an application of the above, we will now evaluate the fields radiated by elementary point sources. First, we consider an elementary point source located at the origin $\mathbf{r}=0$ that flashes on and off at time $t=0$. Such a source can be modeled through Beltrami current densities containing an impulse doublet $\delta'(t)$ at $t=0$ [11] as per

$$\mathbf{W}_{\pm}(\mathbf{r},t)=\mathbf{W}_{1\pm}\delta(\mathbf{r})\delta'(t) \ , \tag{26}$$

where $\mathbf{W}_{1\pm}$ are some constant vectors. Substituting Eq. (26) into Eq. (15) and using Eq. (24), we obtain

$$\pm 4\pi\mathbf{Q}_{\pm}(\mathbf{r},t) = (i/rc)(\underline{I}-\mathbf{r}\mathbf{r}/r^2\pm\mathbf{r}\times\underline{I}/r)\cdot\mathbf{W}_{1\pm}\delta''(t-r/c) + (i/r^2)(\underline{I}-3\mathbf{r}\mathbf{r}/r^2\pm i\mathbf{r}\times\underline{I}/r)\cdot\mathbf{W}_{1\pm}\delta'(t-r/c) + (ic/r^3)(\underline{I}-3\mathbf{r}\mathbf{r}/r^2)\cdot\mathbf{W}_{1\pm}\delta(t-r/c) \ . \tag{27}$$

Here, $r=|\mathbf{r}|$, and $\delta''(\cdot)$ is the impulse triplet [11]. It is clear from Eq. (27) that the radiated field is precisely null for any $t<r/c$ or $t>r/c$.

As a second example we consider an elementary point source at $\mathbf{r}=0$ which is switched on at $t=0$ and remains turned on; the temporal behavior of the corresponding current densities is then modeled through a δ function according to

$$\mathbf{W}_{\pm}(\mathbf{r},t)=\mathbf{W}_{2\pm}\delta(\mathbf{r})\delta(t) \ , \tag{28}$$

where now $\mathbf{W}_{2\pm}$ are constant vectors. It follows from Eqs. (28), (15), and (24) that

$$\pm 4\pi\mathbf{Q}_{\pm}(\mathbf{r},t) = (i/rc)(\underline{I}-\mathbf{r}\mathbf{r}/r^2\pm\mathbf{r}\times\underline{I}/r)\cdot\mathbf{W}_{2\pm}\delta'(t-r/c) + (i/r^2)(\underline{I}-3\mathbf{r}\mathbf{r}/r^2\pm i\mathbf{r}\times\underline{I}/r)\cdot\mathbf{W}_{2\pm}\delta(t-r/c) + (ic/r^3)(\underline{I}-3\mathbf{r}\mathbf{r}/r^2)\cdot\mathbf{W}_{2\pm}U(t-r/c) \ . \tag{29}$$

Therefore, at large times $t>r/c$ we find

$$\pm 4\pi\mathbf{Q}_{\pm}(\mathbf{r},t)=(ic/r^3)(\underline{I}-3\mathbf{r}\mathbf{r}/r^2)\cdot\mathbf{W}_{2\pm} \ , \tag{30}$$

in agreement with the behavior expected of a static dipolar source. We note that both elementary point sources which we have chosen as examples are canonical radiators; i.e., $\mathbf{W}_+$ ($\mathbf{W}_-$) generates the radiated Beltrami field $\mathbf{Q}_+$ ($\mathbf{Q}_-$).

III. RADIATION POTENTIALS

In order to solve the time-dependent free space Beltrami-Maxwell equations (13) and (14) in terms of vector (and scalar) potentials, we first eliminate the Beltrami charge densities $w_{\pm}(\mathbf{r},t)$ by virtue of the continuity relation (3). Then

$$w_{\pm}=-(1/\partial_t)\nabla\cdot\mathbf{W}_{\pm} \ . \tag{31}$$

Upon substitution of Eq. (31) into Eq. (13), we obtain

$$\nabla\cdot[\mathbf{Q}_{\pm}\mp(ic/\partial_t)\mathbf{W}_{\pm}]=0 \ . \tag{32}$$

Consequently,

$$\mathbf{Q}_{\pm}=\pm(ic/\partial_t)\mathbf{W}_{\pm}+\nabla\times\mathbf{A}_{\pm} \ , \tag{33}$$

where $\mathbf{A}_{\pm}(\mathbf{r},t)$ are as yet undetermined vector functions. Inserting Eq. (33) into Eq. (14), we find

$$\nabla\times[\nabla\times\mathbf{A}_{\pm}\mp(i\partial_t/c)\mathbf{A}_{\pm}\pm(ic/\partial_t)\mathbf{W}_{\pm}]=0 \ . \tag{34}$$

This equation suggests that $\mathbf{A}_{\pm}(\mathbf{r},t)$ must satisfy the differential equation

$$\nabla\times\mathbf{A}_{\pm}\mp(i\partial_t/c)\mathbf{A}_{\pm}\pm(ic/\partial_t)\mathbf{W}_{\pm}=\nabla V_{\pm} \ , \tag{35}$$

containing the arbitrary scalar functions $V_{\pm}(\mathbf{r},t)$.

A solution of Eq. (35) can be derived by making the ansatz

$$\mathbf{A}_{\pm}=\alpha\partial_t\mathbf{p}_{\pm}+\beta(\nabla\times\mathbf{p}_{\pm})+\nabla\Lambda_{\pm} \ , \tag{36}$$

$$V_{\pm}=\gamma\nabla\cdot\mathbf{p}_{\pm}+\kappa\partial_t\Lambda_{\pm} \tag{37}$$

for the unknown vector potentials $\mathbf{p}_{\pm}(\mathbf{r},t)$ and scalar potentials $\Lambda_{\pm}(\mathbf{r},t)$. Four constants—$\alpha$, β, γ, and κ—are involved here, but one of the four will remain arbitrary as it amounts to a redefinition of $\mathbf{p}_{\pm}$ only. The motivation for the ansatz (36) and (37) is to use first-order derivatives in space and time only when expressing $\mathbf{A}_{\pm}$ and $V_{\pm}$. With the choices

$$\beta=1, \quad \gamma=\beta, \quad \alpha=\pm(i\beta/c), \quad \kappa=\pm(i/c) \ , \tag{38}$$

we find Eqs. (36) and (37) turn out to be solutions of the differential equations (35), provided that $\mathbf{p}_{\pm}(\mathbf{r},t)$ are solutions of the inhomogeneous vector wave equations

$$[\nabla^2-(1/c^2)\partial_{tt}]\mathbf{p}_{\pm}(\mathbf{r},t)=\pm(ic/\partial_t)\mathbf{W}_{\pm}(\mathbf{r},t) \ . \qquad (39)$$

Finally, the Beltrami fields $\mathbf{Q}_{\pm}(\mathbf{r},t)$ are represented in the form

$$\mathbf{Q}_{\pm}=\pm(ic/\partial_i)\mathbf{W}_{\pm}\pm(i\partial_t/c)(\nabla\times\mathbf{p}_{\pm})+\nabla\times(\nabla\times\mathbf{p}_{\pm}) \ . \qquad (40)$$

An alternative representation for $\mathbf{Q}_{\pm}$ can be obtained by using Eq. (39) in Eq. (40). Then we have

$$\mathbf{Q}_{\pm}=\pm(i\partial_t/c)(\nabla\times\mathbf{p}_{\pm})+\nabla\nabla\cdot\mathbf{p}_{\pm}-(1/c^2)\partial_{tt}\mathbf{p}_{\pm} \ . \qquad (41)$$

Interestingly, the scalar potentials $V_{\pm}$ play no role in the Beltrami field representation. Furthermore, if we define new vector potentials $\mathbf{q}_{\pm}(\mathbf{r},t)$ through

$$\mathbf{q}_{\pm}(\mathbf{r},t)=\nabla\times\mathbf{p}_{\pm}(\mathbf{r},t) \ , \qquad (42)$$

then Eq. (40) becomes

$$\mathbf{Q}_{\pm}=\pm(ic/\partial_t)\mathbf{W}_{\pm}\pm(i\partial_t/c)\mathbf{q}_{\pm}+\nabla\times\mathbf{q}_{\pm} \ , \qquad (43)$$

and the potentials $\mathbf{q}_{\pm}$ can be thought of as Lorentz potentials in view of Eq. (8).

The particular solution to Eq. (39) can be written by using the scalar Green function $g(\mathbf{r},t;\mathbf{r}',t')$ defined earlier [see Eqs. (20) and (21)]. We obtain

$$\mathbf{p}_{\pm}(\mathbf{r},t)=\mp(ic)\int_{\Lambda'}d^3\mathbf{r}'dt'(\delta(\tau)/4\pi R) \times[(1/\partial t')\mathbf{W}_{\pm}(\mathbf{r},t')] \ . \qquad (44)$$

Performing the integration with respect to t', we obtain

$$\mathbf{p}_{\pm}(\mathbf{r},t) =\mp(ic)\int_{V'}d^3\mathbf{r}'[(1/\partial t')\mathbf{W}_{\pm}(\mathbf{r}',t')]_{t'=t-R/c}/4\pi R \ , \qquad (45)$$

where V' is the volume wherein $\mathbf{W}_{\pm}(\mathbf{r}',t')$ is nonzero; the Lorentz potentials $\mathbf{q}_{\pm}$ then follow simply by using Eq. (42).

Elementary point sources

To exemplify the general results for the vector potentials $\mathbf{p}_{\pm}$ we specialize the Beltrami current densities to those in Sec. II for elementary point sources. In the first example, with $\mathbf{W}_{\pm}(\mathbf{r},t)$ given by Eq. (26), the general expression (45) yields

$$\mathbf{p}_{\pm}(\mathbf{r},t)=\mp ic\mathbf{W}_{1\pm}\delta(t-r/c)/4\pi r \ . \qquad (46)$$

The second elementary point source, given by the Beltrami current density (38), provides us with

$$\mathbf{p}_{\pm}(\mathbf{r},t)=\mp ic\mathbf{W}_{2\pm}U(t-r/c)/4\pi r \ . \qquad (47)$$

On substituting the vector potentials from Eqs. (46) and (47) into Eq. (40), we obtain the Beltrami fields $\mathbf{Q}_{\pm}$ in identical form as derived previously in Eqs. (27) and (29), respectively, by using the Green function technique.

IV. SUMMARY

We have shown that the Beltrami-Maxwell equations for free space arise as specializations of the Beltrami-Maxwell equations for general material continua. Subsequently, we delineated the general problem of radiation of confined sources in an unbounded region of free space. We found closed-form results for the time-dependent dyadic Green function associated with the Beltrami fields and, alternatively, derived a representation of the Beltrami fields in terms of vector potentials. Detailed, closed-form results for the Beltrami fields were then given for two types of elementary point sources.

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