Jeans instability of a dusty plasma

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We study the linear and the nonlinear stages of the Jeans instability of charged grains with mass m_d and charge q_d in the regime $Gm_d^2/q_d^2 \approx O(1)$. Various conditions for stable electrostatic levitation, condensation, and dispersion of grains in the plasma background are obtained. The nonlinear solutions show that there is a condensation of grains even when the effect of self-gravity is annulled by self-electrostatic repulsion. Astrophysical situations relevant to the results are briefly discussed.

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I. INTRODUCTION

A dusty plasma is a three component plasma with electrons, ions, and a dispersed (low number density) phase of very massive charged grains of solid matter. The size of the heavy grains is typically in the range of 1 μm to 1 cm. Dusty plasma is usually found in the interstellar clouds, circumstellar clouds, interplanetary medium, cometary tails, planetary rings, and the Earth's magnetosphere [1,2]. In the laboratory, dusty plasma occurs as a result of high Z impurities from the tokamak walls, during plasma etching and impurity generation in magnetohydrodynamics (MHD) power generators.

As stated earlier, the charged grains are very massive as compared to electrons and ions. For instance the micrometeorized flux observed by Helios probes furnishes a range of 10^{-2}–10^{-9} g for the grain mass within 1 A.U. from the Sun [3]. The charge of the grains is also large as compared to that of the electrons. In typical cases it could be as large as 10^4 times that of electronic charge. The presence of this very massive, charged, low density grains in the plasma introduces new time and space scales in the plasma behavior leading to new waves, instabilities, etc. This has been the subject of many recent investigations [4–14]. The process of charging of grains and grain charge fluctuations is also interesting and has been investigated recently [15–18].

In astrophysical scenarios, the dynamics of large bodies like planets, stars and satellites, etc. is controlled overwhelmingly by gravitation, while that of electrons and ions is influenced overwhelmingly by electromagnetic forces. The two forces operate on two widely different scales. It is now well established that for micron and submicron size grains these two forces become comparable, at least to within an order of magnitude [19]. If we consider only electrostatic forces then this would require $Gm_d^2/q_d^2 \approx O(1)$. The interplay between gravitational and electrostatic forces in the dynamics of such grains is responsible for many interesting phenomena in the terrestrial and solar environment, e.g., rings of Saturn and Jupiter, etc. For instance it is known that during the spoke formation in the B ring of Saturn, the grains are electrostatically levitated against self-gravity above the plane of the ring [20,21]. The critical thickness of Jovian rings has been attributed to electrostatic levitation against the gravity of the planet [22]. Similarly, the process of condensation or dispersion of charged grains under the action of self-gravity has important implications for the overall process of star formation [23,23]. With this in view, we carry out a detailed analysis of the process of electrostatic levitation, condensation, and dispersion of charged grains $[Gm_d^2/q_d^2 \approx O(1)]$ in a plasma background under the influence of self-gravity of grains. The problem of condensation of neutral grains due to self-gravity was first studied by Jeans—the so-called "Jeans instability" [25]. This is a process in which a slight rearrangement of uniform distribution of mass by the effect of self-gravitation leads to the further localized condensation of grains. In the case of charged grains, since the electrostatic forces are significant, unusual and interesting deviations from the corresponding processes of neutral grains are expected. Our approach to the problem is as follows. We first consider an infinite homogeneous dusty plasma with spatially uniform densities of electron, ion, and dust. As is well known, in gravitational systems, infinite homogeneous distribution of matter cannot occur. However, for the purpose of studying the linear stability, such a distribution of matter is artificially conceived by invoking the so-called "Jeans swindle" where effects of zero order gravitational fields are neglected [25]. Our approach is somewhat similar. We first study the linear stability of an infinite homogeneous plasma invoking Jeans swindle. We then construct a more realistic model of the equilibrium which takes into account the zero order fields. In this equilibrium, the grains are electrostatically levitated against the self-gravity. The linear stability of this equilibrium is studied. Finally, using the method of Lagrange variables [26–28], the dynamical equations are exactly integrated for a wide class of initial conditions. From these solutions various conditions for electrostatic levitation, condensation, and dispersion (where density everywhere approaches zero) of

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grains are delineated. The results are interesting. It is shown that in certain cases the results of linear stability analysis based on Jeans swindle may not be correct. For instance, for $Gm_d^2/q_d^2=1$, i.e., when the electrostatic repulsion between the grains balances the self-gravity, the linear stability analysis based on Jeans swindle implies marginal stability, while the nonlinear solution predicts condensation or dispersion of the grains.

In the context of a two component grain plasma, the two stream instability for grains with $Gm_d^2/q_d^2=O(1)$ has been studied by Gisler *et al.* [29]. They also find substantial modification of the usual two stream instability because of interplay of electrostatic and gravitational forces.

II. LINEAR INSTABILITY

We consider an infinite dusty plasma with spatially uniform density of electrons, ions, and grains. Let these densities be denoted by n_e, n_i, and n_d, respectively. The charge of these species will be denoted by $q_e=q_i$ and q_d ($q_d<0$) and their masses are denoted by m_e, m_i, and m_d. Since typically $m_e/m_d<m_i/m_d\sim 10^{-20}$ and $n_d/n_e\sim n_d/n_i\sim 10^{-2}$ to 10^{-3} we have $m_e n_e < m_i n_i \ll m_d n_d$ so that the gravitational potential is mainly due to grains. We further assume $Gm_d^2/q_d^2\approx O(1)$ so that self-gravitational field and electric field of grains are comparable. On the other hand, this ratio is too small for electrons and ions, hence self-gravitational field of electrons and ions will be neglected in our model. This, however, does not mean that there is no gravitational effect on electrons and ions. The effect of gravitational field of grains on electrons and ions will be taken into account. Further we consider a nonthermalized situation where grains are cold while electrons and ions are hot and are mutually thermalized, i.e., $T_d\ll T_e=T_i=T$. This assumption is reasonable because grains are very massive. As a result, the energy equilibration time between grains, electrons, and ions is much larger than the time scale of gravitational dynamics (Jeans time) which will be considered here. The dynamics of the plasma is governed by the following set of equations:

$$\frac{\partial n_\alpha}{\partial t}+\nabla(n_\alpha \mathbf{v}_\alpha)=0\ , \tag{1}$$

where α stands for electron, ion, and grains;

$$\nabla^2\phi=4\pi G m_d n_d\ , \tag{2}$$

$$\nabla^2\phi=-4\pi[q(n_i-n_e)-q_d n_d]\ , \tag{3}$$

$$m_\alpha n_\alpha\frac{d\mathbf{v}_\alpha}{dt}=-n_\alpha q_\alpha\nabla\phi-n_\alpha m_\alpha\nabla\psi-T\nabla n_\alpha\ , \tag{4}$$

where α = electrons and ions, while the equation of motion for the cold grain is given by

$$m_d n_d=\frac{d\mathbf{v}_d}{dt}=n_d q_d\nabla\phi-n_d m_d\nabla\psi\ , \tag{5}$$

where $\mathbf{v}_\alpha$ is the fluid velocity of the αth species. We consider a quasineutral equilibrium, i.e., $q_d n_d=q(n_i-n_e)$. In this equilibrium there is no electric field and the free energy is due to the gravitational field of the grains. Strictly speaking this equilibrium cannot be regarded as homogeneous. However, invoking Jeans swindle we will neglect the zero order gravitational field and regard the equilibrium as "homogeneous." Mathematically, Jeans swindle can be incorporated in our set of equations by modifying Poisson's equation of ψ as

$$\nabla^2\psi=4\pi G m_d(n_d-n_{d0})\ , \tag{6}$$

where n_{d0} is the equilibrium number density of grains. The set of equations (1), (3), (4), (5), and (6) form the basic set of equations for the linear analysis. To reiterate, we consider the linear gravitational stability of an infinite homogeneous dusty plasma characterized by

$$\begin{aligned}
&n_{e0}=\text{const}\ ,\quad n_{i0}=\text{const}\ ,\quad n_{d0}=\text{const}\ ,\\
&n_{e0}\ ,\quad n_{i0}\gg n_{d0}\ ,\\
&qn_i=(q_e n_e+q_d n_d)\ ,\\
&T_e=T_i=T=\text{const}\ ,\quad T_d=0\ ,\\
&\psi_0=0\ ,\quad \phi_0=0\ ,\quad v_{d0}=v_{e0}=v_{i0}=0\ .
\end{aligned}$$

The linearized set of equations are

$$\frac{\partial\delta n_\alpha}{\partial t}+n_{\alpha 0}\nabla\cdot\delta\mathbf{v}_\alpha=0\ , \tag{7}$$

with α equal to electrons, ions, and dust and

$$\nabla^2\delta\psi=4\pi G m_d\delta n_d\ , \tag{8}$$

$$\nabla^2\delta\phi=-4\pi[q(\delta n_i-\delta n_e)-q_d\delta n_d]\ , \tag{9}$$

$$m_e n_{e0}\frac{\partial\delta\mathbf{v}_e}{\partial t}=n_{e0}q\nabla\delta\phi-m_e n_{e0}\nabla\delta\phi-T\nabla\delta n_e\ , \tag{10}$$

$$m_i n_{i0}\frac{\partial\delta\mathbf{v}_i}{\partial t}=-n_{i0}q\nabla\delta\phi-m_i n_{i0}\nabla\delta\psi-T\nabla\delta n_i\ , \tag{11}$$

$$m_d n_{d0}\frac{\partial\delta\mathbf{v}_d}{\partial t}=n_{d0}q_d\nabla\delta\phi-m_d n_{d0}\nabla\delta\psi\ . \tag{12}$$

Since the equilibrium is homogeneous, the perturbations are proportional to $\propto \exp i(kx-\omega t)$. Using this in the continuity equation for the dust and also in Eq. (8) we have

$$\delta v_d=\frac{\omega\delta n_d}{k n_{d0}}\ , \tag{13}$$

$$\delta\psi=-\frac{4\pi G m_d\delta n_d}{k^2}\ , \tag{14}$$

which can be used to eliminate $\delta\psi$ and δv_d in terms of δn_d in Eq. (12). To eliminate $\delta\phi$ in terms of δn_d we use the continuity equation to express δv_e and δv_i in terms of δn_e and δn_i. This can be used in the equations of motion (10) and (11) to express δn_e and δn_i in terms of δn_d and $\delta\phi$ as

$$\delta n_i = \left[-\frac{n_{i0} q \delta\phi}{T} + \frac{\omega_{Jd}^2}{k^2 v_{th_i}^2} \frac{n_{i0}}{n_{d0}} \delta n_d \right] \left[1 - \frac{\omega^2}{k^2 v_{th_i}^2} \right]^{-1} , \tag{15}$$

$$\delta n_e = \left[\frac{n_{e0} q \delta\phi}{T} + \frac{\omega_{Jd}^2}{k^2 v_{th_e}^2} \frac{n_{e0}}{n_{d0}} \delta n_d \right] \left[1 - \frac{\omega^2}{k^2 v_{th_e}^2} \right]^{-1} , \tag{16}$$

where $v_{th_e}^2 = 2T/m_e$, $v_{th_i}^2 = 2T/m_i$, and $\omega_{Jd}^2 = 4\pi G m_d n_{d0}$ is the Jeans frequency for the grains. Using Eqs. (15) and (16) in the Poisson's equation for ϕ we obtain

$$\delta\phi = \frac{-4\pi q_d \lambda_D^2}{[A_e^{-1} + A_i^{-1} + k^2\lambda_D^2]} \left[1 - \frac{q}{q_d} \frac{\omega_{Jd}^2}{k^2 v_{th_i}^2} \frac{1}{A_i} \frac{n_0}{n_d} \right] \delta n_d , \tag{17}$$

with $A_e = [1 - \omega^2/(k^2 v_{th_e}^2)]$, $A_i = [1 - \omega^2/(k^2 v_{th_i}^2)]$, and $\lambda_D^2 = T/(4\pi n_0 q^2)$, where we have used $n_{e0} \approx n_{i0} \approx n_0$ and $v_{th_e}^2 \gg v_{th_i}^2$. Substituting this in the equation of motion for the dust we finally obtain the dispersion relation as

$$\omega^2 = -\omega_{Jd}^2 + \frac{k^2 \lambda_D^2 \omega_{pd}^2}{[A_e^{-1} + A_i^{-1} + k^2 \lambda_D^2]} \times \left[1 - \frac{q}{q_d} \frac{\omega_{Jd}^2}{k^2 v_{th_i}^2} \frac{1}{A_i} \frac{n_0}{n_d} \right] , \tag{18}$$

where $\omega_{pd}^2 = 4\pi q_d^2 n_{d0}/m_d$. This is the dispersion relation for the Jeans instability of the dusty plasma with massive grains, such that $G m_d^2 / q_d^2 \simeq 1$. It is a transcendental equation for ω which can be solved for a given k. If the grains are electrically neutral then $\omega_{pd}^2 = 0$ and Eq. (18) describes the usual Jeans instability of the neutral gas. On the other hand, for $\omega_{Jd}^2 = 0$, one gets a purely electrostatic oscillation of the dusty plasma which has been studied in [7].

We are interested in a very low frequency root of Eq. (18) such that $\omega^2 \sim \omega_{Jd}^2 \sim \omega_{pd}^2 \ll \omega_{pi}^2, \omega_{pe}^2$. In this regime $\omega^2/(k^2 v_{th_i}^2) \sim (\omega_{pd}^2/\omega_{pi}^2)(1/k^2\lambda_D^2)$ and $(\omega_{Jd}^2/\omega_{pi}^2) \sim (\omega_{pd}^2/\omega_{pi}^2) \sim (m_i/m_d)(q_d^2/q_i^2)(n_{d0}/n_0)$. Typically $m_i/m_d \approx 10^{-20}$, $q_d^2/q_i^2 \approx 10^6$ and $n_d/n_0 \approx 10^{-2}$. Hence $\omega^2/(k^2 v_{th_i}^2) \sim \omega_{Jd}^2 (k^2 v_{th_i}^2) \ll 1$, $A_i \approx 1$, $A_e \approx 1$, and the last term in Eq. (18) can be dropped. Clearly, on this time scale the inertial and the gravitational effects on electrons and ions are relatively weaker; they both follow Boltzmann's relation. The modified dispersion relation is

$$\omega^2 = \frac{k^2 \lambda_D^2 \omega_{pd}^2}{2 + k^2 \lambda_D^2} - \omega_{Jd}^2 . \tag{19}$$

Now if $k^2 \lambda_D^2 \geq 1$, i.e., the scale size of the perturbation is of the same order as the Debye length, then the unshielded electric field due to the charge separation affects the gravitational condensation, i.e.,

$$\omega^2 = \omega_{pd}^2 - \omega_{Jd}^2 . \tag{20}$$

In this regime, the background electrons and ions can be regarded as fixed ($\delta n_e = \delta n_i = 0$). If $G m_d^2 / q_d^2 < 1$ then the gravitational condensation of the grains is inhibited by the space charge electric field. The grains slosh around with a frequency given by $\omega = (\omega_{pd}^2 - \omega_{Jd}^2)^{1/2}$. A more interesting root of Eq. (20) is when gravitational condensation is stopped by the pressure of electrons and ions. To see this, consider the limit $k^2 \lambda_D^2 \ll 1$. In this case

$$\omega^2 = k^2 C_d^2 - \omega_{Jd}^2 . \tag{21}$$

Here, $C_d^2 = (T n_{d0} g_d^2 / m_d n_{e0} q^2)$. The first term in Eq. (21) represents the dust acoustic wave [10]. The critical length below which condensation of grains is stopped is given by $\lambda_e = 2\pi C_d^2/\omega_{Jd}^2 = T q_d^2/(2 G m_d^2 n_e q^2)$. The physical reason for stabilization is clear. As the grains begin to condense due to the self-gravitational field, an electrostatic field due to charge separation is created between electrons, ions, and the grains (recall that in this frequency range the gravitational effects on electrons and ions are unimportant). The electrons and ions rush to shield this field thereby creating density perturbations. If the speed of the wave is fast enough, so that it can travel one wavelength in the Jeans time ($k^2 C_d^2 \gg \omega_{Jd}^2$), the density perturbations will be smoothed and the condensation will be inhibited.

III. ASYMPTOTIC HOMOGENEOUS EQUILIBRIUM

In the preceding section we considered a "homogeneous" equilibrium which is quasineutral, so that there is no electric field and the zero order gravitational field was neglected invoking Jeans swindle. Strictly speaking, there is no rigorous justification for discarding the zero order gravitational field. In this section, therefore, we construct a legitimate equilibrium where zero order fields are retained. This equilibrium is shown to be homogeneous asymptotically. The linear analysis of this asymptotically homogeneous infinite dusty plasma is then studied. The homogeneous equilibrium can be constructed as follows. We combine the electric and the gravitational fields into a new field with potential $[\psi_0 - (q_d/m_d)\phi_0]$. The Poisson's equation for this field is given by

$$\nabla^2 \left[\psi_0 - \frac{q_d}{m_d} \phi_0 \right] = 4\pi \left\{ G m_d n_{d0} + \frac{q_d}{m_d} [q(n_{i0} - n_{e0}) - q_d n_{d0}] \right\} . \tag{22}$$

For dust equilibrium $[\psi_0 - (q_d/m_d)\phi_0] = 0$ which gives

$$n_{d0}(x) = -\frac{[q(n_{i0}(x) - n_{e0}(x))]}{q_d (G m_d^2 / q_d^2 - 1)} . \tag{23}$$

This is the condition for the electrostatic levitation of dust against the self-gravity. The conditions for the equilibrium with positive and finite n_{d0} are (i) if $G m_d^2 q_d^2 > 1$ then $n_{e0} > n_{i0}$, (ii) $G m_d^2 / q_d^2 < 1$ then $n_{i0} > n_{e0}$, and (iii)

$Gm_d^2/q_d^2=1$ then $n_{i0}=n_{e0}$. The reason for this is clear. The parameter Gm_d^2/q_d^2 measures the ratio between the electrostatic repulsion and the gravitational attraction of the dust. If $Gm_d^2/q_d^2>1$, then attraction dominates over the self-repulsion. In this case, a background of negative charges is required ($n_{e0}>n_{i0}$) to compensate for the extra attraction on the dust. The equilibrium of the electrons and ions is given by

$$0=-\frac{\partial\psi_0}{\partial x}-\frac{q_i}{m_i}\frac{\partial\phi_0}{\partial x}-\frac{T}{m_i}\frac{\nabla n_i}{n_i}\ , \tag{24}$$

$$0=-\frac{\partial\psi_0}{\partial x}+\frac{q_e}{m_e}\frac{\partial\phi_0}{\partial x}-\frac{T}{m_e}\frac{\nabla n_e}{n_e}\ . \tag{25}$$

Eliminating ψ_0 using $\psi_0=(q_d/m_d)\phi_0$ in Eqs. (24) and (25) it can be shown that the gravitational effects on electrons and ions can be neglected. Solving the remaining equations for n_{i0} and n_{e0} we have

$$n_{i0}=\hat{n}_{i0}\exp\left[-\frac{q\phi_0(x)}{T}\right]\ , \tag{26}$$

$$n_{e0}=\hat{n}_{e0}\exp\left[\frac{q\phi_0(x)}{T}\right]\ . \tag{27}$$

In the limit $|q\phi_0/T|\ll 1$, n_{i0} and n_{e0} tend to become spatially uniform. From Eq. (23) n_{d0} also becomes spatially uniform. The equilibrium of the dusty plasma thus becomes asymptotically homogeneous in the limit $|q\phi_0/T|\gg 1$. Astrophysical situations where $|q\phi_0/T|\ll 1$ are described at the end.

We next study the stability of this equilibrium to perturbations in the range $k^2\lambda_D^2\gg 1$. As shown earlier in this regime, electrons and ions can be regarded as stationary and the dispersion relation is again given by

$$\omega^2=-\omega_{Jd}^2+\omega_{pd}^2 \tag{28}$$

except that now the dust density n_{d0} is not fixed by the quasineutrality condition but by Eq. (23). Eliminating n_{d0} by Eq. (23) we have

$$\omega^2=\frac{4\pi q_d q(n_{i0}-n_{e0})}{m_d}\ . \tag{29}$$

If $n_{i0}>n_{e0}$ then $\omega^2>0$ and the plasma oscillates with the frequency given by Eq. (29). These are new kinds of hybrid electrostatic oscillations, which are governed by background charge density and the dust mass. The condition for instability is $n_{e0}>n_{i0}$. This is consistent with the instability criterion based on Jeans swindle, which is $Gm_d^2/q_d^2>1$. For $n_{e0}>n_{i0}$ Eq. (23) requires $Gm_d^2/q_d^2>1$ for positive n_{d0}. Similarly the analysis of the preceding section implies that if $Gm_d^2/q_d^2=1$ then $\omega^2=0$ and the equilibrium is marginally stable. This is consistent with the analysis of this section because for $Gm_d^2/q_d^2=1$ Eq. (23) implies $n_{e0}=n_{i0}$ (for finite n_{d0}) in which case $\omega^2=0$.

From this section, we find that the condition for the stable electrostatic levitation of negatively charged grains (against self-gravity) is $n_i>n_e$ and $Gm_d^2/q_d^2<1$. The condition for the marginal levitation is $n_i=n_e$ and $GM_d^2/q_d^2=1$, while the condition for condensation of the grains is $n_e>n_i$ and $Gm_d^2/q_d^2>1$. The process of dispersion of grain density everywhere approaches zero as $t\to\infty$ is not shown by the linear analysis. In the next section, we integrate the dynamical equations to obtain time dependent solutions. These solutions describe the process of dispersion of grains.

IV. NONLINEAR STABILITY

In this section, we investigate the nonlinear stage of the Jeans instability. As stated earlier, in the $|q\phi_0 T|\ll 1$ limit the electron and ion densities are uniform. They can thus form a fixed background while the nonlinear equations corresponding to the dust dynamics can be integrated analytically. The case pertaining to the limit $|q\phi_0/T|>1$ is discussed at the end. The equations to be solved in the $|q\phi_0/T|\ll 1$ limit are the equation of motion and continuity equation for the grains and the Poisson's equation for the electric and the gravitational fields. Following the method of Shu [26] and Sturrock [27], which is based on the use of Lagrangian variables these equations can be integrated exactly in one dimension for the interesting class of initial conditions. The set of equations to be solved are

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{E}_1-\frac{q_d}{m_d}\mathbf{E}_2\ , \tag{30}$$

$$\frac{\partial n_d}{\partial t}+\nabla\cdot(n_d\mathbf{v}_d)=0\ , \tag{31}$$

$$\nabla\cdot\mathbf{E}_1=-4\pi G m_d n_d\ , \tag{32}$$

$$\nabla\cdot\mathbf{E}_2=4\pi[q(n_i-n_e)-q_d n_d]\ , \tag{33}$$

where $\mathbf{E}_1=-\nabla\psi$ is the gravitational field, $\mathbf{E}_2=-\nabla\phi$ is the electrostatic field, and $\mathbf{v}$ is the grain velocity (the subscript is dropped). To integrate these equations in one dimension we transform to Lagrangian variables (x_0,τ_0) given by [30]

$$\tau=t\ , \tag{34}$$

$$x_0=x-\int_0^\tau v(x_0,\tau')d\tau'\ . \tag{35}$$

The partial derivative with respect to x transforms as

$$\frac{\partial}{\partial x}=\left[1+\int_0^\tau d\tau'\frac{\partial v(x_0,\tau')}{\partial x_0}\right]^{-1}\frac{\partial}{\partial x_0}\ , \tag{36}$$

while the partial derivative with respect to time transforms as

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial\tau}-v(x_0,\tau)\left[1+\int_0^\tau d\tau'\frac{\partial v(x_0,\tau)}{\partial x_0}\right]^{-1}\frac{\partial}{\partial x_0}\ . \tag{37}$$

In the new variables the convective derivative transforms as $\partial/\partial t+v(\partial/\partial x)=\partial/\partial\tau$ and Eq. (30) can be written as

$$\frac{\partial v}{\partial\tau}=E_1-\frac{q_d}{m_d}E_2\ . \tag{38}$$

The continuity equation can be integrated to give

$$n(x_0,\tau)=\frac{n(x_0,0)}{\left[1+\int_0^{\tau}\frac{\partial v(x_0,\tau')}{\partial x_0}d\tau'\right]} . \tag{39}$$

Next eliminating n_d from Eq. (32) in the first term in Eq. (31) and extracting the divergence gives

$$\frac{\partial E_1}{\partial t}-4\pi G m_d n_d v=0 . \tag{40}$$

[It should be noted that, in general, extracting a divergence will leave a curl of an arbitrary function on the right side of Eq. (40). However, in one dimension it can be chosen to be zero.] Using Eq. (32) again we have

$$\frac{\partial E_1}{\partial t}+v(\nabla\cdot E_1)=\frac{\partial E_1}{\partial t}+v\frac{\partial E_1}{\partial x}$$
$$=\frac{\partial E_1}{\partial \tau}=0 . \tag{41}$$

This is an interesting result, which simplifies the nonlinear problem. It implies that gravitational fluid is convected with grain fluid. By similar manipulations, the convection of the electric field can be written as

$$\frac{\partial E_2}{\partial \tau}=4\pi q(n_i-n_e)v . \tag{42}$$

Now using Eqs. (41) and (42), Eq. (38) can be integrated to obtain v as a function of τ which then can be used to obtain E_1 and E_2 as functions of τ. Thus,

$$v=A(x_0)(e^{\gamma\tau}-e^{-\gamma\tau}) ,$$
$$v(x_0,0)=0 , \tag{43}$$

$$E_2(x_0,\tau)=-\frac{2\gamma m_d}{q_d}A(x_0)\cosh(\gamma\tau)+\frac{m_d}{q_d}C(x_0) , \tag{44}$$

$$E_1=C(x_0) , \tag{45}$$

$$n_d(x_0,\tau)=\frac{n_d(x_0,0)}{\left[1+\frac{2}{\gamma}\frac{\partial A}{\partial x_0}[\cosh(\gamma\tau)-1]\right]} , \tag{46}$$

where $\gamma^2=[4\pi q_d q(n_e-n_i)]/m_d$. The functions $A(x_0)$ and $C(x_0)$ are to be evaluated from Poisson's equation at $t=0$ as follows. At $\tau=0$, the Poisson's equation for the two fields can be written as

$$\frac{\partial E_1(x,0)}{\partial x}=\frac{\partial E_1(x_0,0)}{\partial x_0}=\frac{dC}{dx_0}$$
$$=-4\pi G m_d n_d(x_0,0) , \tag{47}$$

$$\frac{dE_2}{dx}=\frac{dE_2}{dx_0}=-4\pi[q_d n_d-q(n_i-n_e)] . \tag{48}$$

Substituting for $E_2(x_0,0)$ from Eq. (44) gives

$$\frac{dA}{dx_0}=\frac{\gamma}{2}\left\{1-\left[\left[1-\frac{Gm_d^2}{q_d^2}\right]\frac{q_d n_d(x_0,0)}{q(n_i-n_e)}\right]\right\} . \tag{49}$$

This completes the solution in terms of x_0 and τ. Transformation to Eulerean variables (x,t) is achieved through

$$\tau=t , \tag{50}$$

$$x=x_0+\frac{2A(x_0)}{\gamma}[\cosh(\gamma\tau)-1] . \tag{51}$$

For a given profile of initial dust density $n_d(x_0,0)$ [at $t=\tau=0$, $x=x_0$, which is at rest $v(x,0)=0$], these equations can be integrated to obtain n_d, v, E_1, and E_2 as functions of (x,t). Generalization to $v(x_0,0)\neq 0$ is straightforward. It should be noted that, while obtaining the solution of the nonlinear equations (43)–(46), we have not made any assumption about whether or not there is a dust equilibrium at $t=0$. Accordingly, we will examine two classes of initial conditions (i) when there is an equilibrium of grains at $t=0$ and n_{d0} is given by Eq. (23) and (ii) when there is no equilibrium of the grains at $t=0$ and n_{d0} is arbitrary. However, before we proceed to do this, we would like to examine the nonlinear solutions in the following limiting cases (1) $Gm_d^2/q_d^2=0$. In this case, there is a quasineutral equilibrium of the grains given by

$$q_d n_{d0}=q(n_i-n_e) , \tag{52}$$

for $n_{d0}>0$ and $n_i>n_e$. According to Eq. (29) for $n_i>n_c$, $\gamma^2=-\omega^2=\omega_{pd}^2$. Let the initial perturbation around this equilibrium be given by

$$n_d(x,0)=n_{d0}+\Delta\cos(kx) . \tag{53}$$

For this case $n_d(x_0,\tau)$ is given by $[\cosh(\gamma t)\to\cos(\omega_{pd}t)]$ for $\gamma^2<0$,

$$n_d(x_0,\tau)=\frac{n_{d0}+\Delta\cos(kx_0)}{1-(\Delta/n_{d0})[\cos(\omega_p t)-1]\cos(kx_0)} , \tag{54}$$

where x_0 is related to x through

$$kx=kx_0+\frac{\Delta}{n_{d0}}[\cos(\omega_p t)-1]\sin(kx_0) . \tag{55}$$

These are large amplitude dust plasma oscillations. In order that dust density $n_d(x_0,t)$ is always positive $\Delta/n_{d0}<\frac{1}{2}$. (2) $Gm_d^2/q_d^2\neq 0$, $n_e=n_i$. In this limit, the plasma background is charge neutral and there is no dust equilibrium for finite $n_{d0}(x_0,0)$. In this limit, $\gamma\to 0$ and we expand $\cosh(\gamma t)-1=\gamma^2t^2/2$. Using this and Eq. (49) in Eq. (46) we obtain

$$n_d(x_0,t)=\frac{n_d(x_0,0)}{1+(\omega_{pd}^2-\omega_{Jd}^2)t^2/2} . \tag{56}$$

If the dust is charge neutral, then $\omega_{pd}^2=0$ and we obtain the nonlinear Jeans instability in one dimension. It is explosive in time, i.e., $n_d\propto 1/(t-t_0)$. On the other hand, if gravitation is weak then $\omega_{Jd}^2=0$ and $n_d\to 0$ as $t\to\infty$. This is because of continuous expansion of dust under self-electrostatic repulsion (note $n_i=n_e$). This is the process of grain dispersion. We now proceed to examine initial conditions with and without the dust equilibrium.

A. Initial condition with dust equilibrium

To begin with, we recapitulate the results of linear instability analysis based on Jeans swindle and Sec. II. If $Gm_d^2/q_d^2>1$, then $n_e>n_i$ for $n_d>0$ and $\omega^2>0$ indicating instability. Similarly if $Gm_d^2/q_d^2<1$, then $n_i>n_e$ for $n_d>0$ and $\omega^2>0$ indicating stable oscillation. If $Gm_d^2/q_d^2=1$, then $n_e=n_i$ for finite n_d and $\omega^2=0$ indicating marginal stability. To examine the nonlinear stability, consider a perturbation of the following form:

$$n_d(x,0)=n_{d0}+\Delta\cos(kx)\ , \tag{57}$$

where n_{d0} is given by Eq. (23). Using this and Eq. (49) in Eq. (46) we have

$$n_d(x_0,\tau)=\frac{n_d(x_0,0)}{1-(\Delta_1/n_{d0})[\cosh(\gamma t)-1]\cos(kx_0)(Gm_d^2/q_d^2-1)}\ , \tag{58}$$

where $\Delta_1=q_d\Delta/q[n_e(x_0)-n_i(x_0)]$.

In Eq. (58) we consider the following cases:

(i) $Gm_d^2/q_d^2>1$. In this case whenever $\Delta_1\cos(kx_0)>0$, the density $n_d\to\infty$ at time $t=t_c$ defined by $\Delta_1\cos(kx_0)[\cosh(\gamma t_c)-1]=1$. In places where $\Delta_1\cos(kx_0)<0$, the density $n_{d0}\to 0$ as $t\to\infty$. This is the process of condensation of grains. This is the modified Jeans instability. The modification is due to the equilibrium at $t=0$. In Fig. 1, we plot the normalized density $[n_d(x_0,t)/n_d(x_0,0)]$ against $r=kx$ at different times to show the process of condensation.

(ii) $Gm_d^2/q_d^2<1$. In this case $\gamma^2<0$ and $\cosh(\gamma t)\to\cos(\gamma t)$. These are nonlinear dust plasma oscillations modified due to gravity. The modification is due to the fact that n_{d0} is given by Eq. (23) rather than by the quasineutrality condition.

(iii) $Gm_d^2/q_d^2=1$. In this case $n_e=n_i$ and $\gamma\to 0$ in which case $n_d(x_0,t)=n_d(x_0,0)$, i.e., there is no evolution and the equilibrium is marginally stable.

B. Initial condition without the dust equilibrium

In this case, there is no dust equilibrium at $t=0$ and $n_d(x_0,0)$ is arbitrary. As in the previous case, we consider the following cases.

(i) $Gm_d^2q_d^2=1$, $n_e\neq n_i$. Recall that in this case the analysis based on Jeans swindle implies marginal stability $\omega^2=0$ [Eq. (20)] and there is no evolution. Now if $n_e>n_i$, $\gamma^2>0$ and

$$n_d(x_0,t)=\frac{n_d(x_0,0)}{\cosh(\gamma t)}\ . \tag{59}$$

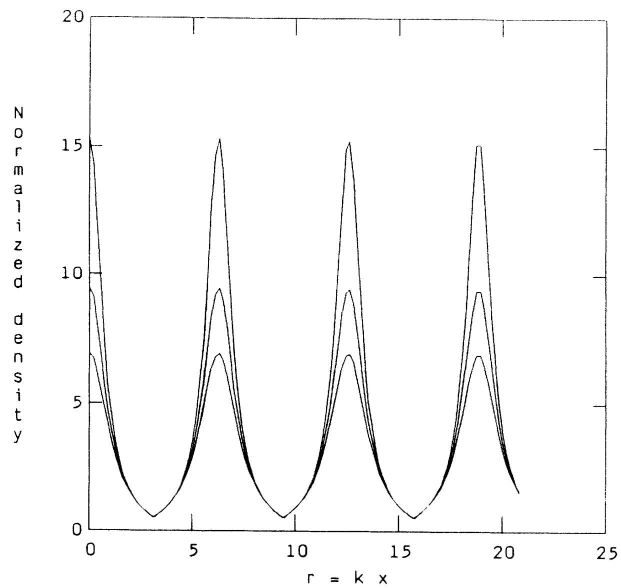

FIG. 1. Spatial variation of normalized dust density at different times given by (i) $t_c=0.55\pi/\gamma$, (ii) $t_c=0.56\pi/\gamma$, and (iii) $t_c=0.57\pi/\gamma$. Case (i) has the lowest amplitude and case (iii) has the highest amplitude. On the x axis $r=kx$ and on the y axis normalized dust density $=n_d(x,t)/n_d(x,0)$.

As $t\to\infty$, $n_d(x_0,t)\to 0$. The reason for this is clear. In the case $Gm_d^2/q_d^2=1$, the effect of self-gravity of dust is completely annulled by self-electrostatic repulsion between dust particles. If now $n_e>n_i$, then the background is uniformly negatively charged and repels negatively charged dust particles everywhere to reduce the dust density. On the other hand, if $n_i>n_e$ then $\gamma^2<0$ and

$$n_d(x_0,t)=\frac{n_d(x_0,0)}{\cos(\omega t)}\ . \tag{60}$$

In this case $n_d(x_0,t)=\infty$ at $t=t_c$ such that $\omega t_c=\pi/2$. This is a new result. We obtain a condensation of grains even when the effect of self-gravity (which was originally responsible for condensation) is completely annulled by self-electrostatic repulsion between dust particles. This condensation of negatively charged dust is electrostatic in nature and is caused by a positively charged background. From this analysis, we see that in cases where there is no equilibrium at $t=0$, linear stability based on Jeans swindle gives incorrect results. For this case, it predicted marginal stability and no temporal evolution while the nonlinear theory predicts dust condensation.

(ii) $Gm_d^2/q_d^2<1$ and $n_e>n_i$. In this case $n_d(x_0,t)$ is given by

$$n_d(x_0,t)=\frac{n_d(x_0,0)}{\left\{1+\left[1+\frac{q_dn_d(x_0,0)}{q(n_e-n_i)}\left[1-\frac{Gm_d^2}{q_d^2}\right]\right][\cosh(\gamma t)-1]\right\}}\ . \tag{61}$$

As $t\to\infty$, $n_d\to 0$. This is again because negatively charged dust particles disperse in the negatively charged background and self-electrostatic repulsion, while gravity is too weak to overcome this.

(iii) $Gm_d^2/q_d^2 > 1$ and $n_i > n_e$. Here $n_d(x_0,t)$ is given by

$$n_d(x_0,t)=\frac{n_d(x_0,0)}{\left\{1+\left[1+\frac{q_d n_d(x_0,0)}{q(n_e-n_i)}\left[1-\frac{Gm_d^2}{q_d^2}\right]\right][\cosh(\gamma t)-1]\right\}} . \tag{62}$$

In this case, we again encounter grain condensation, i.e., $n_d(x_0,t_c)=\infty$ where

$$\cos(\omega t_c)=\frac{\left[\frac{q_d n_d(x_0,0)}{q(n_e-n_i)}\left[1-\frac{Gm_d^2}{q_d^2}\right]\right]}{1+\left[\frac{g_d n_d(x_0,0)}{q(n_e-n_i)}\left[1-\frac{Gm_d^2}{q_d^2}\right]\right]} . \tag{63}$$

This condensation is jointly triggered by a positive background (for $q_d<0$) and self-gravity which is stronger than the electrostatic repulsion in the limit $Gm_d^2/q_d^2>1$. These results are systematically presented in Table I.

These conditions are interesting as they show that to obtain levitation it is not enough to balance the self-gravitation with self-repulsion, i.e., $Gm_d^2/q_d^2=1$. Also if $Gm_d^2/q_d^2>1$ we obtain condensation but no dispersion or levitation while if $Gm_d^2/q_d^2<1$ we never obtain condensation. The properties of the plasma background are equally important. These conditions for levitation, condensation, and dispersion in the background of the plasma will have important implications in the dynamics of comets, planetary rings, and the formation of stars and planets.

The limit $|q\phi_0/T|>1$ is not amenable to analytic treatment. In this limit, background electrons and ions move in response to fluctuations in electrostatic potential ϕ through Boltzmann's relation given by Eqs. (26) and (27). As recently shown by Schamel and Bujarbarua [31], the Lagrangian formulation in this case gives rise to a complicated set of integro-differential equations. Such a formulation can be given for the present problem also. However, for initial condition of interest, the equations will have to be solved numerically. We postpone this problem to a future investigation.

TABLE I. Conditions on the plasma background and Gm_d^2/q_d^2 for levitation, dispersion, and condensation.

	Levitation	Dispersion	Condensation
$Gm_d^2/q_d^2>1$			$n_e>n_i$ $n_e=n_i$ $n_e<n_i$
$Gm_d^2/q_d^2=1$	$n_e=n_i$	$n_e>n_i$	$n_i>n_e$
$Gm_d^2/q_d^2<1$	$n_i>n_e$	$n_e=n_i$ $n_e>n_i$	

We now turn to astrophysical situations where $|q\phi_0/T|\ll 1$ and $|q\phi_0/T|\gg 1$. As shown by Goertz and Shan [22], typically in E and G rings of Saturn, Jovian rings, and ring halos $|q\phi_0/T|$ is small while the charge on the dust is significant. Our nonlinear calculations which assume a large charge on the dust so that $Gm_d^2/q_d^2\approx O(1)$ and $|q\phi_0/T\ll 1$ are applicable to these situations. On the other hand, for A, B, and F rings of Saturn and Uranus rings $|q\phi_0/T|\geq 1$ while the charge on the dust is small. This may imply $Gm_d^2/q_d^2\gg O(1)$ and thus both the assumptions of the present calculations are not valid.

In our calculations, we have not included any damping process. Inclusion of such processes will lead to a damping of nonlinear oscillations and a static equilibrium where the unshielded electric field balances the gravitational field. The compression of the dust density in the process of condensation can be enormous, i.e., $n_d(x_0,t)/n_d(x_0,0)\gg 1$. At these extreme compressions, the thermal energy of the dust, which has been neglected here, becomes important. This may have important consequences for the process of star formation.

To summarize, we have studied the Jeans instability of a dusty plasma. The charge and mass of the dust are in the range where $Gm_d^2/q_d^2\approx O(1)$ and electric and gravitational forces operate on the same scale. The linear instability is studied in two ways. First, linear instability of an infinite, homogeneous dusty plasma is studied using Jeans swindle. This analysis shows that if $k^2\lambda_D^2\gg 1$, the unshielded electric field inhibits condensation. If $k^2\lambda_D^2\ll 1$, then the pressure of the background electrons and ions can inhibit the gravitational condensation of the dust. Next, an equilibrium of dusty plasma is constructed, where a finite space charge field balances the gravitational field. The equilibrium is homogeneous only asymptotically. In the limit $k^2\lambda_D^2\gg 1$, the stability of the asymptotically homogeneous equilibrium is analyzed. Finally, using the method of the Lagrange variable due to Shu and Sturrock, the time dependent nonlinear solutions are studied in the limit $|q\phi_0/T|\ll 1$. From these solutions, various conditions on the plasma background for stable levitation against self-gravity, condensation, and dispersion of charged grains are delineated.

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