Attractors of dissipative structure in three dissipative fluids

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A theory to find attractors of dissipative structure is developed by using autocorrelations for distributions. It is shown that the realization of coherent structures in dissipative dynamical systems is equivalent to that of self-organized states with the minimum change rate of autocorrelations for their instantaneous values, which usually represent the system's total energy. It is shown that attractors of dissipative structure are given by eigenfunctions for dissipative dynamic operators and they constitute the self-organized and self-similar decay phase. Three typical examples applied to incompressible viscous fluids, to incompressible viscous and resistive magnetohydrodynamic (MHD) fluids, and to compressible resistive MHD plasmas are presented in order to find attractors in the three dissipative fluids and to describe a common physical picture of self-organization and bifurcation of the dissipative structure.

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I. INTRODUCTION

Dissipative structures realized in dissipating nonlinear dynamical systems include various self-organized structures in thermodynamic systems [1,2], the force-free fields of cosmic magnetism [3], and the self-organized relaxed state of the magnetized fusion plasmas such as in the reversed field pinch experiment [4–6], in the spheromak experiment [7,8], and in the simple toroidal Z pinch experiment [9]. They also include the flow structures in incompressible viscous fluids such as the two-dimensional (2D) flow patterns after grid turbulence [10] and the helical flow patterns which follow turbulent puffs [11]. We can see some common mathematical structures among the self-organized relaxed states of the dissipative structure and also among the proposed theories themselves to explain those dissipative structures [3,12–19]. The study of the common universal mathematical structures embedded in dissipative nonlinear dynamic systems and leading to those dissipative structures is an area of deep interest. We have recently proposed a theory which clarifies that attractors of the dissipative structure are given by eigenfunctions for dissipative dynamic operators in dynamic systems of interest [20]. In this paper, we refine this theory [20] by introducing autocorrelations between two instants in time evolution in order to identify and/or to define the realization of coherent structures. We also present three typical applications of the refined theory in order to find attractors in three dissipative fluids and to describe a common physical picture of self-organization and bifurcation of the dissipative structure.

We present the refined theory in Sec. II, where we clarify that the realization of coherent structures in time evolution is equivalent to that of self-organized states with the minimum change rate of autocorrelations for their instantaneous values, which usually represent the system's total energy. We present three typical examples applied to incompressible viscous fluids in Sec. III, to incompressible viscous and resistive magnetohydrodynamic (MHD) fluids such as liquid metals in Sec. IV, and to compressible resistive MHD plasmas in Sec. V in order to find attractors of the dissipative structure in these dissipative fluids and to describe a common physical picture of self-organization and bifurcation of the dissipative structure.

II. GENERAL THEORY OF SELF-ORGANIZATION AND DISSIPATIVE STRUCTURE

We present here a more refined theory, which stands upon the concept of the coherent structure included in the self-organized dissipative structure, than the theory in a previous report [20]. In the previous report [20], we assumed *a priori* a quasisteady state with approximate equilibrium equations for the self-organized states. In the present refined theory, however, it will be shown that the self-organized states with a coherent structure have to be in equilibrium states.

Quantities with n elements in dynamic systems of interest shall be expressed as $\mathbf{q}(t,\mathbf{x})=\{q_1(t,\mathbf{x}), q_2(t,\mathbf{x}),\ldots,q_n(t,\mathbf{x})\}$. Here t is time, $\mathbf{x}$ denotes m-dimensional space variables, and $\mathbf{q}$ represents a set of physical quantities having n elements, some of which are vectors such as the velocity $\mathbf{u}$, the magnetic field $\mathbf{B}$, the current density $\mathbf{j}$, etc., and others are scalars such as the mass density, the energy density, the specific entropy, and so on. We consider a dissipative nonlinear dynamic system which may be generally described by

$$\frac{\partial q_i}{\partial t}=L_i^N[\mathbf{q}]+L_i^D[\mathbf{q}] , \tag{1}$$

where $L_i^N[\mathbf{q}]$ and $L_i^D[\mathbf{q}]$ denote, respectively, nondissipative and dissipative, linear or nonlinear dynamic operators, such as $q_i=\mathbf{u}$, $L_i^N[\mathbf{q}]=-\nabla p/\rho-\nabla u^2/2+\mathbf{u}\times\omega$, and $L_i^D[\mathbf{q}]=(\nu/\rho)\nabla^2\mathbf{u}$ in the Navier-Stokes equation for incompressible viscous fluid dynamics with the coefficient of viscosity ν. (In some cases, the operator $L_i^D[\mathbf{q}]$ may include negative dissipation terms such as energy input terms.) When the dynamic system has some unstable regions, the nondissipative dynamic operator $L_i^N[\mathbf{q}]$ may become dominant and lead to the rapid growth of perturbations there and further to turbulent phases. This may yield spectrum transfers or spectrum spreadings toward both the higher and the lower wave number regions in q_i distributions, as in the normal energy cascade and also the inverse cascade shown by 3D MHD simulations in [21–23] or in the turbulent region of the turbulent puffs in incompressible viscous fluids shown in Fig. 4 of [11]. When the higher wave number becomes a large fraction of the spectrum, the dissipative dynamic operator $L_i^D[\mathbf{q}]$ may become dominant to yield higher dissipations for the higher wave number components of W_{ii}. In this rapid dissipation phase, which is far from equilibrium, the unstable regions in the dynamic system are considered to vanish to produce a stable configuration again. Since this newly self-organized relaxed state is identified by the realization of its coherent structure from the standpoint of observation, we notice and find the following definition (i) for the configuration of the self-organized relaxed state, by using autocorrelations $q_i(t_R,\mathbf{x})q_i(t_R+\Delta t,\mathbf{x})$ between the time of relaxed state t_R and a slightly later time $t_R+\Delta t$ with a small Δt:

(i) The state with

$$\min\left|\frac{\int q_i(t_R,\mathbf{x})q_i(t_R+\Delta t,\mathbf{x})dV}{\int q_i(t_R,\mathbf{x})q_i(t_R,\mathbf{x})dV}-1\right| .$$

Substituting the Taylor expansion of $q_i(t_R+\Delta t,\mathbf{x}) = q_i(t_R,\mathbf{x})+[\partial q_i(t_R,\mathbf{x})/\partial t]\Delta t+\frac{1}{2}[\partial^2 q_i(t_R,\mathbf{x})/\partial t^2](\Delta t)^2 + \cdots$ into definition (i) and taking account of the arbitrary smallness of Δt, we obtain the following equivalent definition (ii) for the configuration of the self-organized relaxed state from the first order of Δt in definition (i):

(ii) The state with

$$\min\left|\frac{\int q_i(t_R,\mathbf{x})[\partial q_i(t_R,\mathbf{x})/\partial t]dV}{\int q_i(t_R,\mathbf{x})q_i(t_R,\mathbf{x})dV}\right| .$$

On the other hand, if this dynamic system has no dissipative term of $L_i^D[\mathbf{q}]$ and also has no external input through its boundary, then global autocorrelations $W_{ii}(t)=\int q_i(t,\mathbf{x})q_i(t,\mathbf{x})dV=\int [q_i(t,\mathbf{x})]^2 dV$ across the space volume of the system, which usually represent the system's total energy, are conserved because there is no dissipation by the nondissipative operator $L_i^N[\mathbf{q}]$. In this case, we accordingly obtain $\partial W_{ii}/\partial t=2\int q_i(\partial q_i/\partial t)dV =2\int q_i L_i^N[\mathbf{q}]dV=0$, and therefore the definition of the nondissipative operator $L_i^N[\mathbf{q}]$ (i.e., the conservative operator) is written

$$\int q_i L_i^N[\mathbf{q}]dV=0 . \tag{2}$$

Using Eq. (2), the dissipation rate or the change rate of $W_{ii}(t)$ in the dissipative dynamic system of Eq. (1) is written as follows:

$$\frac{\partial W_{ii}(t)}{\partial t}=2\int q_i(t,\mathbf{x})\frac{\partial q_i(t,\mathbf{x})}{\partial t}dV=2\int q_i L_i^D[\mathbf{q}]dV . \tag{3}$$

Using the term of $W_{ii}(t)$, the equivalent definition (ii) for the configuration of the self-organized relaxed state shown above is rewritten as follows:

(iii) The state with

$$\min\left|\frac{\partial W_{ii}(t_R)/\partial t}{W_{ii}(t_R)}\right| .$$

This definition (iii) leads to the following two equivalent definitions of (iv) and (v) for the configuration of the self-organized relaxed state:

(iv) The state with $\min|\partial W_{ii}/\partial t|$ for a given value of W_{ii} at $t=t_R$.

(v) The state with $\max W_{ii}$ for a given value of $|\partial W_{ii}/\partial t|$ at $t=t_R$.

These two equivalent definitions of (iv) and (v) belong to typical problems of variational calculus with respect to the spatial variable $\mathbf{x}$ to find the spatial profiles of $q_i(t_R,\mathbf{x})$, and they are known to be equivalent to each other by the reciprocity of the variational calculus.

We find from the definitions (i)–(iv) shown above that the realization of coherent structures in dissipative dynamical systems is equivalent to that of self-organized states with the minimum change rate of autocorrelations for their instantaneous values.

If the dynamical system of Eq. (1) has both the negative and the positive dissipative terms in the dissipative operator $L_i^D[\mathbf{q}]$, and if it has a steady state such that it satisfies $\partial q_i/\partial t=0$ and therefore $\partial W_{ii}/\partial t=0$, then this steady state may constitute the self-organized state described by definitions (i), (ii), or (iii). In this case, if the system has come to the steady state, then the system will never deviate from the steady state because of $\partial q_i/\partial t=0$.

On the other hand, if the total dissipation in the dynamical system is always either positive (i.e., $\partial W_{ii}/\partial t<0$) or negative (i.e., $\partial W_{ii}/\partial t>0$), then the steady state of $\partial q_i/\partial t=0$ will never be realized in the system. Hereafter, we consider those dynamical systems with $\partial W_{ii}/\partial t<0$ or $\partial W_{ii}/\partial t>0$. We use the notation $\mathbf{q}^*(W_{ii},\mathbf{x})$ or simply q_i^* for the profiles of q_i that satisfy definition (iv). The mathematical expressions for definition (iv) are written as follows, defining a functional F with use of a Lagrange multiplier α:

$$F\equiv-\epsilon\frac{\partial W_{ii}}{\partial t}-\alpha W_{ii} , \tag{4}$$

$$\delta F=0 , \tag{5}$$

$$\delta^2 F>0 , \tag{6}$$

where $\epsilon=1$ for $\partial W_{ii}/\partial t<0, \epsilon=-1$ for $\partial W_{ii}/\partial t>0$, and δF and $\delta^2 F$ are the first and the second variations of F with respect to the variation $\delta\mathbf{q}(\mathbf{x})$ only for the spatial variable $\mathbf{x}$. Substituting Eq. (3) into Eqs. (4)–(6), we obtain

$$\delta F=-2\int\{\delta q_i(\epsilon L_i^D[\mathbf{q}]+\alpha q_i)+\epsilon q_i\delta L_i^D[\mathbf{q}]\}dV=0 . \tag{7}$$

$$\delta^2 F=-2\int\delta q_i\left[\epsilon\delta L_i^D[\mathbf{q}]+\frac{\alpha}{2}\delta q_i\right]dV>0 . \tag{8}$$

Using the same method as in [20], imposing and using the following self-adjoint property upon the operators $L_i^D[\mathbf{q}]$,

$$\int q_i \delta L_i^D[\mathbf{q}]dV = \int \delta q_i L_i^D[\mathbf{q}]dV + \oint \mathbf{P}\cdot d\mathbf{S} \ , \tag{9}$$

we obtain the following from Eq. (7):

$$\delta F = -2\int \delta q_i(2\epsilon L_i^D[\mathbf{q}] + \alpha q_i)dV - 2\oint \mathbf{P}\cdot d\mathbf{S} = 0 \ , \tag{10}$$

where $\oint \mathbf{P}\cdot d\mathbf{S}$ denotes the surface integral term which comes out as from the Gaussian theorem. We then obtain the Euler-Lagrange equations from the volume integral term in Eq. (10) for arbitrary variations of δq_i as follows:

$$L_i^D[\mathbf{q}^*] = -\frac{\epsilon\alpha}{2} q_i^* \ , \tag{11}$$

where $\epsilon^2 = 1$ is used. We find from Eq. (11) that the profiles of q_i^* are given by the eigenfunctions for the dissipative dynamic operators $L_i^D[\mathbf{q}^*]$ and therefore have a feature uniquely determined by the operators $L_i^D[\mathbf{q}^*]$ themselves. As a boundary value problem, we may assume that Eq. (11) can be solved for given boundary values of q_i. The value of the Lagrange multiplier α is determined by using the given value of W_{ii} for the global constraint, as is common practice in variational calculus. Substituting the eigenfunctions q_i^* into Eq. (3) and using Eq. (11), we obtain the following:

$$\frac{\partial W_{ii}^*}{\partial t} = -\epsilon\alpha \int (q_i^*)^2 dV = -\epsilon\alpha W_{ii}^* \ . \tag{12}$$

This equation leads to the following:

$$W_{ii}^* = e^{-\epsilon\alpha t} W_{iiR}^* = e^{-\epsilon\alpha t}\int [q_{iR}^*(\mathbf{x})]^2 dV = \int [q_{iR}^*(\mathbf{x}) e^{-(\epsilon\alpha/2)t}]^2 dV \ , \tag{13}$$

$$q_i^* = q_{iR}^*(\mathbf{x}) e^{-(\epsilon\alpha/2)t}, \tag{14}$$

where $q_{iR}^*(\mathbf{x})$ denotes the eigensolution for Eq. (11) which is supposed to be realized at the state with the minimum change rate during the time evolution of the dynamical system of interest. Using Eq. (14) at first and then substituting Eq. (11), we obtain the following:

$$\frac{\partial q_i^*}{\partial t} = -\frac{\epsilon\alpha}{2} q_i^* = L_i^D[\mathbf{q}^*] \ . \tag{15}$$

Substituting the eigenfunctions q_i^* into Eq. (1) and comparing with Eq. (15), we obtain the following equilibrium equations at $t = t_R$:

$$L_i^N[\mathbf{q}^*] = 0 \ . \tag{16}$$

This result indicates that the self-organized states with coherent structure have to be in equilibrium states of Eq. (16). We find from Eqs. (14)–(16) that the eigenfunctions q_i^* for the dissipative dynamic operators $L_i^D[\mathbf{q}^*]$ constitute the self-organized and self-similar change phase (decay for $\epsilon = 1$ and increase for $\epsilon = -1$) with the minimum change rate and with the equilibrium state of Eq. (16) in the time evolution of the present dynamic system. We see from Eq. (12) that the factor α of Eq. (11), the Lagrange multiplier, is equal to the time constant of change of W_{ii} at the self-organized and self-similar change phase. Since the present dynamic system evolves basically by Eq. (1), the dissipation by $L_i^D[\mathbf{q}^*]$ of Eq. (11) during the self-similar change couples with $L_i^N[\mathbf{q}]$ and the boundary wall conditions to cause gradual deviation from self-similar change. This gradual deviation may yield some new unstable region in the dynamic system. When some external positive input for $\epsilon = 1$ or negative input for $\epsilon = -1$ is applied through the boundary in order to recover the dissipation of W_{ii}, the present dynamic system is considered to return repeatedly close to the self-organized and self-similar change phase. The observation of the time evolution of the system of interest for long periods reveals a physical picture in which the system appears to be repeatedly attracted towards and trapped in the self-organized and self-similar change phase of Eq. (14). On the other hand, if the dynamic system has a steady state such that $\partial q_i/\partial t = 0$ is satisfied and therefore $\partial W_{ii}/\partial t = 0$, then this system will never deviate from the steady state after realization of the steady state without external input.

Using the same method as in [20] for the discussion on the mode transition point or bifurcation point of the dissipative structure, we consider the following associated eigenvalue problem for critical perturbations δq_i that make $\delta^2 F$ in Eq. (8) vanish:

$$\epsilon(\delta L_i^D[\mathbf{q}])_k + \frac{\alpha_k}{2}\delta q_{ik} = 0, \tag{17}$$

with boundary conditions given for δq_i, for example, $\delta q_i = 0$ at the boundary wall. Here α_k is the eigenvalue and $(\delta L_i^D[\mathbf{q}])_k$ and δq_{ik} denote the eigensolutions. Substituting the eigensolution δq_{ik} into Eq. (8) and using Eq. (17), we obtain the following:

$$\delta^2 F = (\alpha_k - \alpha)\int \delta q_{ik}^2 dV > 0 \ . \tag{18}$$

Since Eq. (18) is required for all eigenvalues, we obtain the following condition for the state with the minimum change rate:

$$0 < \alpha < \alpha_1 \ , \tag{19}$$

where α_1 is the smallest positive eigenvalue and α is assumed to be positive. When the value of α goes beyond the condition of Eq. (19), as when $\alpha_1 < \alpha$, then the mixed mode, which consists of the basic mode by the solution of Eq. (11) where $\alpha = \alpha_1$, and the lowest eigenmode of Eq. (17) become the self-organized dissipative structure with the minimum change rate. The bifurcation point of the dissipative structure is given by $\alpha = \alpha_1$.

III. ATTRACTORS IN INCOMPRESSIBLE VISCOUS FLUIDS

We apply here the theory from the preceding section, which is based on the realization of the coherent structure and does not start with an assumption on the equilibrium equation, to incompressible viscous fluids described by the Navier-Stokes equation:

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \nu \nabla^2 \mathbf{u} \ , \tag{20}$$

where ρ, $\mathbf{u}$, and p are the fluid mass density, the fluid velocity, and the pressure, respectively, and $\nabla \cdot \mathbf{u} = 0$. For simplicity, we assume ν to be spatially uniform. Using vector formulas, Eq. (20) is rewritten as

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{\nabla p}{\rho} - \frac{1}{2}\nabla u^2 + \mathbf{u} \times \boldsymbol{\omega} - \frac{\nu}{\rho} \nabla \times \nabla \times \mathbf{u} \ , \tag{21}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity. We find from Eq. (21) that $L_i^N[\mathbf{q}] = -\nabla p / \rho - \nabla u^2 / 2 + \mathbf{u} \times \boldsymbol{\omega}$ and $L_i^D[\mathbf{q}] = -(\nu/\rho)\nabla \times \nabla \times \mathbf{u}$, where $q_i \equiv \mathbf{u}$. Substituting these two operators of $L_i^N[\mathbf{q}]$ and $L_i^D[\mathbf{q}]$ into Eqs. (7)–(10) and using $\delta\boldsymbol{\omega} = \nabla \times \delta\mathbf{u}$, $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}$, and the Gaussian theorem, we obtain the following:

$$\begin{aligned} \delta F &= 4 \int \delta\mathbf{u} \cdot \left[\frac{\nu}{\rho} \nabla \times \nabla \times \mathbf{u} - \frac{\alpha}{2} \mathbf{u} \right] dV \\ &\quad + \frac{2\nu}{\rho} \oint [\delta\mathbf{u} \times (\nabla \times \mathbf{u}) + (\nabla \times \delta\mathbf{u}) \times \mathbf{u}] \cdot d\mathbf{S} \\ &= 0 \ , \end{aligned} \tag{22}$$

$$\delta^2 F = 2 \int \delta\mathbf{u} \cdot \left[\frac{\nu}{\rho} \nabla \times \nabla \times \delta\mathbf{u} - \frac{\alpha}{2} \delta\mathbf{u} \right] dV > 0 \ . \tag{23}$$

Here the present operator $L_i^D[\mathbf{q}]$ satisfies the self-adjoint property of Eq. (9) as follows:

$$\begin{aligned} \int \mathbf{u} \cdot \left[\frac{\nu}{\rho} \nabla \times \nabla \times \delta\mathbf{u} \right] dV &= \int \delta\mathbf{u} \cdot \left[\frac{\nu}{\rho} \nabla \times \nabla \times \mathbf{u} \right] dV \\ &\quad + \frac{\nu}{\rho} \oint [\delta\mathbf{u} \times (\nabla \times \mathbf{u}) + (\nabla \times \delta\mathbf{u}) \times \mathbf{u}] \cdot d\mathbf{S} \ . \end{aligned} \tag{24}$$

We obtain the Euler-Lagrange equation from the volume integral term in Eq. (22) for the arbitrary variation $\delta\mathbf{u}$, corresponding to Eq. (11), as follows:

$$\nabla \times \nabla \times \mathbf{u}^* = \frac{\alpha\rho}{2\nu} \mathbf{u}^* \ . \tag{25}$$

The eigenfunction of Eq. (25) can be obtained for a given boundary value of $\mathbf{u}$, as a boundary value problem. Using the eigenfunction of Eq. (25) and referring to Eqs. (12)–(15), we obtain the following:

$$\frac{\partial W_{ii}^*}{\partial t} = -\alpha \int (\mathbf{u}^*)^2 dV = -\alpha W_{ii}^* \ , \tag{26}$$

$$\begin{aligned} W_{ii}^* = e^{-\alpha t} W_{iiR}^* &= e^{-\alpha t} \int [\mathbf{u}_R^*(\mathbf{x})]^2 dV \\ &= \int [\mathbf{u}_R^*(\mathbf{x}) e^{-(\alpha/2)t}]^2 dV \ , \end{aligned} \tag{27}$$

$$\mathbf{u}^* = \mathbf{u}_R^*(\mathbf{x}) e^{-(\alpha/2)t} \ , \tag{28}$$

$$\frac{\partial \mathbf{u}^*}{\partial t} = -\frac{\alpha}{2} \mathbf{u}^* = -\frac{\nu}{\rho} \nabla \times \nabla \times \mathbf{u}^* \ , \tag{29}$$

where $\mathbf{u}_R^*(\mathbf{x})$ denotes the eigensolution of Eq. (25) for the given boundary value of $\mathbf{u}$, which is supposed to be realized at the state with the minimum dissipation rate during the time evolution of the dynamical system of interest. Substituting the eigenfunction $\mathbf{u}^*$ into Eq. (21) and using Eq. (29), we obtain the equilibrium equation at $t = t_R$:

$$\nabla p^* + \frac{\rho}{2} \nabla (u^*)^2 = \rho (\mathbf{u}^* \times \boldsymbol{\omega}^*) \ . \tag{30}$$

We find from Eq. (28) that the eigenfunction $\mathbf{u}^*$ for the present dissipative dynamic operator $-(\nu/\rho)\nabla \times \nabla \times \mathbf{u}$ constitutes the self-organized and self-similar decay phase with the equilibrium equation (30) during the time evolution of the present dynamic system. We see from Eq. (26) that the factor α of Eq. (25), which is the Lagrange multiplier, is equal to the decay constant of the flow energy W_{ii} at the self-organized and self-similar decay phase.

Referring to Eqs. (17)–(19) for the discussion on the bifurcation point of the self-organized dissipative structure, we obtain the associated eigenvalue problem from Eq. (23) for critical perturbations $\delta\mathbf{u}$ that make $\delta^2 F$ vanish and the condition for the state with the minimum dissipation rate that corresponds to Eq. (19) as follows:

$$\nabla \times \nabla \times \delta\mathbf{u}_k - \frac{\alpha_k \rho}{2\nu} \delta\mathbf{u}_k = \mathbf{0} \ , \tag{31}$$

$$0 < \alpha < \alpha_1 \ . \tag{32}$$

Here α_k is the eigenvalue, $\delta\mathbf{u}_k$ denotes the eigensolution, α_1 is the smallest positive eigenvalue, the boundary conditions are $\delta\mathbf{u}_w \cdot d\mathbf{S} = 0$ and $[\delta\mathbf{u}_w \times (\nabla \times \delta\mathbf{u}_w)] \cdot d\mathbf{S} = 0$, and the subscript w denotes the value at the boundary wall. Owing to the self-adjoint property of Eq. (24), the eigenfunctions $\mathbf{a}_k$ for the associated eigenvalue problem of Eq. (31) form a complete orthogonal set and the appropriate normalization is written as

$$\begin{aligned} \int \mathbf{a}_k \cdot (\nabla \times \nabla \times \mathbf{a}_j) dV &= \int \mathbf{a}_j \cdot (\nabla \times \nabla \times \mathbf{a}_k) dV \\ &= \frac{\alpha_k \rho}{2\nu} \int \mathbf{a}_j \cdot \mathbf{a}_k dV \\ &= \frac{\alpha_k \rho}{2\nu} \delta_{jk} \ , \end{aligned} \tag{33}$$

where $\nabla \times \nabla \times \mathbf{a}_k - (\alpha_k \rho / 2\nu)\mathbf{a}_k = \mathbf{0}$ is used. The flow $\mathbf{u}$ distribution at each instant can then be expanded by using the eigensolution $\mathbf{u}^*$ for the boundary value problem of Eq. (25) for the given boundary value and also by using orthogonal eigenfunctions $\mathbf{a}_k$ for the eigenvalue problem of Eq. (31) as follows:

$$\mathbf{u} = \mathbf{u}^* + \sum_{k=1}^{\infty} c_k \mathbf{a}_k \ . \tag{34}$$

Here the spectrum component of c_0 by this eigenfunction expansion corresponds to the basic component $\mathbf{u}^*$ and the spectrum of $c_k (k = 0, 1, 2, \ldots)$ depends now on time t. Substituting Eqs. (34), (25), and (31) into Eq. (21), we obtain the following:

$$\frac{\partial \mathbf{u}^*}{\partial t}+\sum_{k=1}^{\infty}\frac{\partial (c_k\mathbf{a}_k)}{\partial t}=L_i^N[\mathbf{q}]-\frac{\alpha}{2}\mathbf{u}^*-\sum_{k=1}^{\infty}\frac{\alpha_k}{2}c_k\mathbf{a}_k \ , \tag{35}$$

where the eigenvalues α_k are positive and α_1 is the smallest positive eigenvalue. In the same way as that shown by Eqs. (28) and (29), we see from Eq. (35) that the flow components of $\mathbf{u}^*$ and $c_k\mathbf{a}_k$ decay approximately by the decay constants of $\alpha/2$ and $\alpha_k/2$, respectively. When the flow-dynamics system has some unstable regions, the nondissipative and nonlinear operator $L_i^N[\mathbf{q}]=-\nabla p/\rho-(1/2)\nabla u^2+\mathbf{u}\times\boldsymbol{\omega}$ in Eq. (35) may become dominant, leading to the raid growth of perturbations and to turbulent phases. This nonlinear process may yield spectrum transfers in the spectrum of c_k toward both the higher and the lower mode number regions. At the same time, since the components with the larger eigenvalue α_k decay faster, the selective dissipation for the higher mode number components may take place. Due to these two key processes of the spectrum transfer and the selective dissipation in the spectrum of c_k, the lowest mode with the smallest decay constant will remain last after unstable regions have vanished. If $\alpha<\alpha_1$, the basic component $\mathbf{u}^*$ remains last. If the value of α becomes greater than α_1, then the basic component $\mathbf{u}^*$ decays faster than the eigenmode $\mathbf{a}_1$. This faster decay of $\mathbf{u}^*$ continues to yield a further decrease of W_{ii}, resulting in the decrease of α itself, until α becomes equal to α_1, i.e., the same decay rate state with the lowest eigenmode $\mathbf{a}_1$. Consequently, the mixed mode which consists of both $\mathbf{u}^*$, having $\alpha=\alpha_1$, and the lowest eigenmode $\mathbf{a}_1$ remains last. In other words, the bifurcation of the self-organized dissipative structure from the basic mode $\mathbf{u}^*$ to the mixed mode with $\mathbf{u}^*$ and $\mathbf{a}_1$ takes place at $\alpha=\alpha_1$.

If $\mathbf{g}(\mathbf{x})$ is a solution of Eq. (25), then another solution $\mathbf{h}(\mathbf{x})$ defined by $\mathbf{h}(\mathbf{x})\equiv\nabla\times\mathbf{g}(\mathbf{x})$ satisfies again Eq. (25) and has the same decay constant α as that of the component $\mathbf{g}(\mathbf{x})$, as is easily shown by taking the rotation of Eq. (25). Linear combinations of $\mathbf{u}^*=e_1\mathbf{g}(\mathbf{x})+e_2\mathbf{h}(\mathbf{x})$ also satisfy Eq. (25) and have the same decay constant α. In a special case of $e_2=\sqrt{2\nu/\alpha\rho}\,e_1$, the linear combinations of $\mathbf{u}^*$ can be shown straightforwardly to satisfy the following:

$$\nabla\times\mathbf{u}^*=\kappa\mathbf{u}^* \quad \left[|\kappa|\equiv\sqrt{\alpha\rho/2\nu}\right] . \tag{36}$$

In this special case, $\mathbf{u}^*\times\boldsymbol{\omega}^*=\mathbf{0}$, and then the equilibrium equation, Eq. (30), becomes

$$\nabla p^*+\frac{\rho}{2}\nabla(u^*)^2=\mathbf{0} . \tag{37}$$

In the more general case with $e_2\neq\sqrt{2\nu/\alpha\rho}\,e_1$, $\mathbf{u}^*$ contains another component so that $\mathbf{u}^*\times\boldsymbol{\omega}^*\neq\mathbf{0}$.

When self-organized relaxed states of interest have some kind of symmetry along one coordinate x_s in $\mathbf{x}$, i.e., $\partial/\partial x_s=0$ (for example, translational, axial, toroidal, or helical symmetry, or two-dimensional flow systems perpendicular to x_s), then Eq. (25) can be separated into two mutually independent equations by using two components of $\mathbf{u}_s^*$ along x_s and $\mathbf{u}_{s\perp}^*$ perpendicular to x_s, as follows:

$$\nabla\times\nabla\times\mathbf{u}_s^*=\frac{\alpha\rho}{2\nu}\mathbf{u}_s^* \ , \tag{38}$$

$$\nabla\times\nabla\times\mathbf{u}_{s\perp}^*=\frac{\alpha\rho}{2\nu}\mathbf{u}_{s\perp}^* \ . \tag{39}$$

The time evolution of self-organized and coherent surface flow structures after grid turbulence, shown in Figs. 1 and 4 in [10], is considered to be represented by Eq. (35) with the use of Eq. (39). In three-dimensional flow systems, when self-organized states have a feature of $\sqrt{\alpha\rho/2\nu}\,\mathbf{u}_s^*=\nabla\times\mathbf{u}_{s\perp}^*$, then the total flow of $\mathbf{u}^*=\mathbf{u}_s^*+\mathbf{u}_{s\perp}^*$ can be shown straightforwardly to constitute solutions of the helical flow of Eq. (36), by using Eq. (39). This type of helical flow solution for Eq. (36) is considered to represent approximately the helical flow pattern after the turbulent puffs shown in Fig. 4 of [11] with the use of the NMR imaging observation.

IV. ATTRACTORS IN INCOMPRESSIBLE VISCOUS AND RESISTIVE MHD FLUIDS

We show here another application of the theory in Sec. II to incompressible viscous and resistive MHD fluids such as liquid metals which are described by the following extended Navier-Stokes equation and the equation for the magnetic field:

$$\rho\frac{\partial \mathbf{u}}{\partial t}=\mathbf{j}\times\mathbf{B}-\nabla p-\frac{\rho}{2}\nabla u^2+\rho\mathbf{u}\times\boldsymbol{\omega}-\nu\nabla\times\nabla\times\mathbf{u} \ , \tag{40}$$

$$\frac{\partial \mathbf{B}}{\partial t}=\nabla\times(\mathbf{u}\times\mathbf{B})-\nabla\times(\eta\mathbf{j}) \ , \tag{41}$$

where Ohm's law is used and $d\mathbf{u}/dt$ is rewritten by $\partial\mathbf{u}/\partial t$ in Eq. (40) in the same way as that used in Eq. (20) for Eq. (21). In this system, the flow energy $\rho u^2/2$ and the magnetic energy $B^2/2\mu_0$ interchange with each other through the terms of $\mathbf{j}\times\mathbf{B}$ and $\nabla\times(\mathbf{u}\times\mathbf{B})$ in Eqs. (40) and (41). The global autocorrelation W_{ii}, corresponding to the total energy, and its dissipation rate $\partial W_{ii}/\partial t$ are written, respectively, as $W_{ii}=2\int[(\rho u^2/2)+(B^2/2\mu_0)]dV$ and $\partial W_{ii}/\partial t=-2\int[\nu\mathbf{u}\cdot\nabla\times\nabla\times\mathbf{u}+\mathbf{B}\cdot\nabla\times(\eta\mathbf{j})/\mu_0]dV$. Using the vector formula $\nabla\cdot(\mathbf{a}\times\mathbf{b})$ $\mathbf{b}\cdot\nabla\times\mathbf{a}-\mathbf{a}\cdot\nabla\times\mathbf{b}$ and the Gauss theorem, $\partial W_{ii}/\partial t$ is known to be rewritten by volume integrals of $(\nu\omega^2+\eta j^2)$. We assume here, for simplicity, that the resistivity η at the relaxed state has a fixed spatial dependence like $\eta(\mathbf{x})$. In the same way as that used in Eqs. (22) and (23), substituting those equations of W_{ii} and $\partial W_{ii}/\partial t$ into Eqs. (4)–(6), we obtain the following:

$$\delta F=4\int\left\{\delta\mathbf{u}\cdot\left[\nu\nabla\times\nabla\times\mathbf{u}-\frac{\alpha}{2}\rho\mathbf{u}\right]+\frac{1}{\mu_0}\delta\mathbf{B}\cdot\left[\nabla\times(\eta\mathbf{j})-\frac{\alpha}{2}\mathbf{B}\right]\right\}dV+2\oint\left[\nu(\delta\mathbf{u}\times\omega+\delta\omega\times\mathbf{u})+\frac{\eta}{\mu_0}(\delta\mathbf{B}\times\mathbf{j}+\delta\mathbf{j}\times\mathbf{B})\right]\cdot d\mathbf{S}=0\ , \tag{42}$$

$$\delta^2F=2\int\left\{\delta\mathbf{u}\cdot\left[\nu\nabla\times\nabla\times\delta\mathbf{u}-\frac{\alpha}{2}\rho\delta\mathbf{u}\right]+\frac{1}{\mu_0}\delta\mathbf{B}\cdot\left[\nabla\times(\eta\delta\mathbf{j})-\frac{\alpha}{2}\delta\mathbf{B}\right]\right\}dV>0\ , \tag{43}$$

where $\mu_0\delta\mathbf{j}=\nabla\times\delta\mathbf{B}$ is used. Here we notice again that the dissipative operator $-\nabla\times(\eta\mathbf{j})$ [i.e., $-\nabla\times(\eta\nabla\times\mathbf{B}/\mu_0)$] satisfies the self-adjoint property of Eq. (9) as follows:

$$\int\mathbf{b}_k\cdot[\nabla\times(\eta\nabla\times\mathbf{b}_j)]dV=\int\mathbf{b}_j\cdot[\nabla\times(\eta\nabla\times\mathbf{b}_k)]dV+\oint[\eta(\nabla\times\mathbf{b}_j)\times\mathbf{b}_k-\eta(\nabla\times\mathbf{b}_k)\times\mathbf{b}_j]\cdot d\mathbf{S}\ . \tag{44}$$

We then obtain the Euler-Lagrange equations for arbitrary variations of $\delta\mathbf{u}$ and $\delta\mathbf{B}$ from the volume integral terms of Eq. (42) as follows:

$$\nabla\times\nabla\times\mathbf{u}^*=\frac{\alpha\rho}{2\nu}\mathbf{u}^*\ , \tag{45}$$

$$\nabla\times(\eta\mathbf{j}^*)=\frac{\alpha}{2}\mathbf{B}^*\ , \tag{46}$$

$$\nabla\times\nabla\times\mathbf{B}^*=\frac{\alpha\mu_0}{2\eta}\mathbf{B}^*\quad\text{for } \eta=\text{const}\ , \tag{47}$$

where $\mu_0\mathbf{j}=\nabla\times\mathbf{B}$ is used. Using the eigenfunctions of Eqs. (45) and (46) and referring to Eqs. (12)–(15), we obtain the following:

$$\frac{\partial W_{ii}^*}{\partial t}=-\alpha\int\left[\rho(\mathbf{u}^*)^2+\frac{(\mathbf{B}^*)^2}{\mu_0}\right]dV=-\alpha W_{ii}^*\ , \tag{48}$$

$$W_{ii}^*=e^{-\alpha t}W_{iiR}^*=\int\left\{\rho[\mathbf{u}_R^*(\mathbf{x})e^{-(\alpha/2)t}]^2+\frac{[\mathbf{B}_R^*(\mathbf{x})e^{-(\alpha/2)t}]^2}{\mu_0}\right\}dV\ , \tag{49}$$

$$\mathbf{u}^*=\mathbf{u}_R^*(\mathbf{x})e^{-(\alpha/2)t}\ , \tag{50}$$

$$\mathbf{B}^*=\mathbf{B}_R^*(\mathbf{x})e^{-(\alpha/2)t}\ , \tag{51}$$

$$\rho\frac{\partial\mathbf{u}^*}{\partial t}=-\frac{\alpha}{2}\rho\mathbf{u}^*=-\nu\nabla\times\nabla\times\mathbf{u}^*\ , \tag{52}$$

$$\frac{\partial\mathbf{B}^*}{\partial t}=-\frac{\alpha}{2}\mathbf{B}^*=-\nabla\times(\eta\mathbf{j}^*)\ , \tag{53}$$

where $\mathbf{u}_R^*(\mathbf{x})$ and $\mathbf{B}_R^*(\mathbf{x})$ denote again the eigensolutions of Eqs. (45) and (46) for given boundary values of $\mathbf{u}$ and $\mathbf{B}$, which are supposed to be realized at the state with the minimum dissipation rate during the time evolution of the dynamical system of interest. Substituting the eigenfunctions $\mathbf{u}^*$ and $\mathbf{B}^*$ into Eqs. (40) and (41) and using Eqs. (52) and (53), we obtain the equilibrium equation at $t=t_R$:

$$\nabla p^*+\frac{\rho}{2}\nabla u^{*2}=\mathbf{j}^*\times\mathbf{B}^*+\rho(\mathbf{u}^*\times\omega^*)\ . \tag{54}$$

$$\nabla\times(\mathbf{u}^*\times\mathbf{B}^*)=\mathbf{0}\ . \tag{55}$$

We find from Eqs. (50) and (51) that the eigenfunctions $\mathbf{u}^*$ and $\mathbf{B}^*$ for the present two dissipative dynamic operators $-\nu\nabla\times\nabla\times\mathbf{u}$ and $-\nabla\times(\eta\mathbf{j})$ constitute the self-organized and self-similar decay phase with the minimum dissipation rate and with equilibrium equations of (54) and (55) during the time evolution of the present dynamic system. We see from Eq. (48) that the factor α of Eqs. (45) and (46), which is the Lagrange multiplier, is equal to the decay constant of energy W_{ii} at the self-organized and self-similar decay phase.

Referring to Eqs. (8) and (17)–(19) for the discussion of the bifurcation point of dissipative structure, we obtain two associated eigenvalue problems from Eq. (43) for critical perturbations $\delta\mathbf{u}$ and $\delta\mathbf{B}$ that make δ^2F vanish and the condition for the state with the minimum dissipation rate that corresponds to Eq. (19) as follows:

$$\nabla\times\nabla\times\delta\mathbf{u}_k-\frac{\alpha_k\rho}{2\nu}\delta\mathbf{u}_k=\mathbf{0}\ , \tag{56}$$

$$\nabla\times(\eta\nabla\times\delta\mathbf{B}_k)-\frac{\mu_0\beta_k}{2}\delta\mathbf{B}_k=\mathbf{0}\ , \tag{57}$$

$$0<\alpha<\alpha_1,\beta_1\ . \tag{58}$$

Here α_k and β_k are eigenvalues, $\delta\mathbf{u}_k$ and $\delta\mathbf{B}_k$ denote the eigensolutions, α_1 and β_1 are the smallest positive eigenvalue of α_k and β_k, respectively, and the boundary conditions are $\delta\mathbf{u}_w\cdot d\mathbf{S}=0,[\delta\mathbf{u}_w\times(\nabla\times\delta\mathbf{u}_w)]\cdot d\mathbf{S}=0$, $\delta\mathbf{B}_w\cdot d\mathbf{S}=0$, and $[\eta(\nabla\times\delta\mathbf{B}_w)\times\delta\mathbf{B}_w]\cdot d\mathbf{S}=0$. Owing to the self-adjoint property of Eq. (44), the eigenfunctions $\mathbf{b}_k$ for the associated eigenvalue problem of Eq. (57) for the magnetic field form a complete orthogonal set and the appropriate normalization is written as

$$\int\mathbf{b}_k\cdot[\nabla\times(\eta\nabla\times\mathbf{b}_j)]dV=\int\mathbf{b}_j\cdot[\nabla\times(\eta\nabla\times\mathbf{b}_k)]dV=\frac{\mu_0\beta_k}{2}\int\mathbf{b}_j\cdot\mathbf{b}_k dV=\frac{\mu_0\beta_k}{2}\delta_{jk}\ , \tag{59}$$

where $\nabla\times(\eta\nabla\times\mathbf{b}_k)-(\mu_0\beta_k/2)\mathbf{b}_k=\mathbf{0}$ is used. Distributions of $\mathbf{u}$ and $\mathbf{B}$ at each instant can be expanded by using eigensolutions $\mathbf{u}^*$ and $\mathbf{B}^*$ for the boundary value problem and orthogonal eigenfunctions $\mathbf{a}_k$ and $\mathbf{b}_k$ for eigenvalue problems as follows:

$$\mathbf{u}=\mathbf{u}^*+\sum_{k=1}^{\infty}c_k\mathbf{a}_k\ , \tag{60}$$

$$\mathbf{B}=\mathbf{B}^*+\sum_{k=1}^{\infty} C_k \mathbf{b}_k \ . \tag{61}$$

We note here that the spectrum components of c_0 and C_0 by this eigenfunction expansion correspond, respectively, to the basic components $\mathbf{u}^*$ and $\mathbf{B}^*$, and the spectra of c_k and C_k $(k=0,1,2,\dots)$ depend on time t. Substituting Eqs. (60), (61), (45), (46), (56), and (57) into Eqs. (40) and (41), we obtain the following:

$$\frac{\partial \mathbf{u}^*}{\partial t}+\sum_{k=1}^{\infty}\frac{\partial (c_k \mathbf{a}_k)}{\partial t}=L_1^N[\mathbf{q}]-\frac{\alpha}{2}\mathbf{u}^*-\sum_{k=1}^{\infty}\frac{\alpha_k}{2}c_k \mathbf{a}_k \ , \tag{62}$$

$$\frac{\partial \mathbf{B}^*}{\partial t}+\sum_{k=1}^{\infty}\frac{\partial (C_k \mathbf{b}_k)}{\partial t}=\nabla\times(\mathbf{u}\times\mathbf{B})-\frac{\alpha}{2}\mathbf{B}^*-\sum_{k=1}^{\infty}\frac{\beta_k}{2}C_k \mathbf{b}_k \ . \tag{63}$$

When the present system has some unstable regions, the nondissipative and nonlinear operators $L_1^N[\mathbf{q}]$ $[\equiv \mathbf{j}\times\mathbf{B}-\nabla p-(\rho/2)\nabla u^2+\rho\mathbf{u}\times\boldsymbol{\omega}]$ and $\nabla\times(\mathbf{u}\times\mathbf{B})$ in Eqs. (62) and (63) may become dominant and yield spectrum transfers in the spectra of c_k and C_k toward both the higher and the lower mode number regions. (Field reconnections have features to induce spectrum transfers toward both the lower and the higher mode number regions.) We find again from Eqs. (62) and (63) that the two key processes of the spectrum transfer and the selective dissipation in the spectra of c_k and C_k give us a detailed physical picture of the self-organization and the bifurcation of the dissipative structure, in the same way as that shown after Eq. (33) in Sec. III. If $\alpha<\alpha_1,\beta_1$, then through an interchange between two of the flow and the magnetic energies by the two terms of $\mathbf{j}\times\mathbf{B}$ and $\nabla\times(\mathbf{u}\times\mathbf{B})$ in Eqs. (40) and (41), and after the faster decay component catches the slower decay component of the two energies, the basic components of $\mathbf{u}^*$ and $\mathbf{B}^*$ with the same value of α remain last. The bifurcation of the self-organized dissipative structure takes place when the value of α becomes equal to the lower one of α_1 and β_1, where the mixed mode with ($\mathbf{u}^*$ and $\mathbf{B}^*$) and the corresponding lowest eigenmode ($\mathbf{a}_1$ or $\mathbf{b}_1$) remain last.

In the same way as that used in Eq. (36), Eqs. (45) and (47) can be shown to have the following helical solutions:

$$\nabla\times\mathbf{u}^*=\kappa\mathbf{u}^* \quad (|\kappa|\equiv\sqrt{\alpha\rho/2\nu}) \ , \tag{64}$$

$$\nabla\times\mathbf{B}^*=\lambda\mathbf{B}^* \quad (|\lambda|\equiv\sqrt{\alpha\mu_0/2\eta}) \ , \tag{65}$$

where spatially uniform η has to be assumed. In this special case, $\mathbf{u}^*\times\omega^*=\mathbf{0}$ and $\mathbf{j}^*\times\mathbf{B}^*=\mathbf{0}$, and then the equilibrium equation, Eq. (54), becomes

$$\nabla p^*+\frac{\rho}{2}\nabla(u^*)^2=\mathbf{0} \ . \tag{66}$$

In more general cases, $\mathbf{u}^*$ and $\mathbf{B}^*$ contain other components so that $\mathbf{u}^*\times\boldsymbol{\omega}^*\neq\mathbf{0}$ and $\mathbf{j}^*\times\mathbf{B}^*\neq\mathbf{0}$.

In the same way as that used in Eqs. (38) and (39), when self-organized relaxed states of interest have some kind of symmetry along one coordinate x_s in $\mathbf{x}$, i.e., $\partial/\partial x_s=0$ (two-dimensional systems are included in this case), Eqs. (45) and (46) can be separated into two mutually independent equations, by using two components of $\mathbf{u}_s^*$ and $\mathbf{B}_s^*$ along x_s, and $\mathbf{u}_{s\perp}^*$ and $\mathbf{B}_{s\perp}^*$ perpendicular to x_s, as follows:

$$\nabla\times\nabla\times\mathbf{u}_s^*=\frac{\alpha\rho}{2\nu}\mathbf{u}_s^* \ , \tag{67}$$

$$\nabla\times\nabla\times\mathbf{u}_{s\perp}^*=\frac{\alpha\rho}{2\nu}\mathbf{u}_{s\perp}^* \ . \tag{68}$$

$$\nabla\times(\eta\nabla\times\mathbf{B}_s^*)=\frac{\alpha\mu_0}{2}\mathbf{B}_s^* \ , \tag{69}$$

$$\nabla\times(\eta\nabla\times\mathbf{B}_{s\perp}^*)=\frac{\alpha\mu_0}{2}\mathbf{B}_{s\perp}^* \ , \tag{70}$$

where $\mu_0\mathbf{j}=\nabla\times\mathbf{B}$ is used. In three-dimensional systems, when self-organized states with uniform η have a feature of $\sqrt{\alpha\mu_0/2\eta}\mathbf{B}_s^*=\nabla\times\mathbf{B}_{s\perp}^*$, it can be shown by the straightforward use of Eq. (70) that the total field of $\mathbf{B}^*=\mathbf{B}_s^*+\mathbf{B}_{s\perp}^*$ constitutes solutions of the helical force-free field of Eq. (65) in the same way as that used for $\mathbf{u}^*$ after Eq. (39).

V. ATTRACTORS IN COMPRESSIBLE RESISTIVE MHD PLASMAS

We show here the third application of the theory in Sec. II to compressible resistive MHD plasmas described by the following simplified equations:

$$\rho\frac{d\mathbf{u}}{dt}=\mathbf{j}\times\mathbf{B}-\nabla p \ , \tag{71}$$

$$\frac{\partial\mathbf{B}}{\partial t}=\nabla\times(\mathbf{u}\times\mathbf{B})-\nabla\times(\eta\mathbf{j}) \ , \tag{72}$$

where the viscosity is assumed to be negligibly small. In this system, W_{ii} and its dissipation rate $\partial W_{ii}/\partial t$ are written, respectively, as $W_{ii}=2\int[B^2/2\mu_0+\rho u^2/2]dV$ and $\partial W_{ii}/\partial t=-(2/\mu_0)\int[\mathbf{B}\cdot\nabla\times(\eta\mathbf{j})]dV$. We assume here, for simplicity, that the resistivity η at the relaxed state has a fixed spatial dependence such as $\eta(\mathbf{x})$, as is indeed the case in all experimental plasmas where η goes to infinity near the boundary wall. Substituting those equations of W_{ii} and $\partial W_{ii}/\partial t$ into Eqs. (4)–(6) in the same way as that used in Eqs. (42) and (43), and taking account of compressible ρ, we obtain the following:

$$\delta F=\frac{2}{\mu_0}\int\left\{2\delta\mathbf{B}\cdot\left[\nabla\times(\eta\mathbf{j})-\frac{\alpha}{2}\mathbf{B}\right]-\alpha\mu_0\left[\delta\rho\frac{u^2}{2}+\rho\delta\mathbf{u}\cdot\mathbf{u}\right]\right\}dV+\frac{2}{\mu_0}\oint(\eta\delta\mathbf{B}\times\mathbf{j}+\eta\delta\mathbf{j}\times\mathbf{B})\cdot d\mathbf{S}=0 \ , \tag{73}$$

$$\delta^2F=\frac{2}{\mu_0}\int\left\{\delta\mathbf{B}\cdot\left[\nabla\times(\eta\delta\mathbf{j})-\frac{\alpha}{2}\delta\mathbf{B}\right]-\alpha\mu_0\left[\delta\rho\delta\mathbf{u}\cdot\mathbf{u}+\rho\frac{\delta u^2}{2}\right]\right\}dV>0 \ . \tag{74}$$

We obtain the Euler-Lagrange equation from the volume integral term in Eq. (73) for arbitrary variations of $\delta\mathbf{B}$, $\delta\rho$, and $\delta\mathbf{u}$ as Eq. (46) and the following:

$$u^*=0,\quad \rho^*\mathbf{u}^*=\mathbf{0}\ , \tag{75}$$

Using Eqs. (46) and (75), and referring to Eqs. (12)–(15), we obtain the following that lead to Eqs. (51) and (53):

$$\frac{\partial W_{ii}^*}{\partial t}=-\alpha\int\frac{(\mathbf{B}^*)^2}{\mu_0}dV=-\alpha W_{ii}^*\ , \tag{76}$$

$$W_{ii}^*=e^{-\alpha t}W_{iiR}^*=\int\frac{[\mathbf{B}_R^*(\mathbf{x})e^{-(\alpha/2)t}]^2}{\mu_0}dV\ . \tag{77}$$

Substituting $\mathbf{u}^*$ and $\mathbf{B}^*$ into Eqs. (71) and (72), and using Eqs. (46), (75), and (53), we obtain the equilibrium equation at $t=t_R$:

$$\nabla p^*=\mathbf{j}^*\times\mathbf{B}^*\ , \tag{78}$$

$$\nabla\times(\mathbf{u}^*\times\mathbf{B}^*)=\mathbf{0}\ . \tag{79}$$

We find again from Eqs. (76) and (51) that the eigenfunction $\mathbf{B}^*$ for the present dissipative dynamic operator $-\nabla\times(\eta\mathbf{j})$ constitutes the self-organized and self-similar decay phase with the minimum dissipation rate and with equilibrium equations of Eqs. (78) and (79) during the time evolution of the present dynamic system. We also see from Eq. (76) that the factor α of Eq. (46), which is the Lagrange multiplier, is equal to the decay constant of energy W_{ii} at the self-organized and self-similar decay phase, as was shown in Eqs. (11)–(16) in the general self-organization theory.

Referring to Eqs. (8) and (17)–(19) for the discussion of the bifurcation point of dissipative structure, we obtain the associated eigenvalue problem from Eq. (74) for critical perturbations $\delta\mathbf{B}$ that make δ^2F vanish and the condition for the state with the minimum dissipation rate that corresponds to Eq. (19) as Eq. (57) and the following:

$$0<\alpha<\beta_1\ , \tag{80}$$

with the boundary conditions of $\delta\mathbf{B}_w\cdot d\mathbf{S}=0$ and $[\eta(\nabla\times\delta\mathbf{B}_w)\times\delta\mathbf{B}_w]\cdot d\mathbf{S}=0$. In the same way as that used in Eqs. (59), (61), and (63), we obtain the same eigenfunction expansion of $\mathbf{B}$ by the eigensolution $\mathbf{B}^*$ for the boundary value problem and the orthogonal eigenfunction $\mathbf{b}_k$ for eigenvalue problems, and also the same field equation with Eqs. (61) and (63), as follows:

$$\mathbf{B}=\mathbf{B}^*+\sum_{k=1}^{\infty}C_k\mathbf{b}_k\ , \tag{81}$$

$$\frac{\partial\mathbf{B}^*}{\partial t}+\sum_{k=1}^{\infty}\frac{\partial(C_k\mathbf{b}_k)}{\partial t}=\nabla\times(\mathbf{u}\times\mathbf{B})-\frac{\alpha}{2}\mathbf{B}^*-\sum_{k=1}^{\infty}\frac{\beta_k}{2}C_k\mathbf{b}_k\ . \tag{82}$$

We find again from Eq. (82) that the two key processes of the spectrum transfer and the selective dissipation in the spectrum of C_k give us a detailed physical picture for the self-organization process and the bifurcation of the dissipative structure at $\alpha=\beta_1$, in the same way as that shown after Eq. (63) in Sec. IV. The flow energy in the present system dissipates to vanish by the dissipation term of $-\nabla\times(\eta\mathbf{j})$ in Eq. (72) through the interchange between the flow and the magnetic energies by the two terms of $\mathbf{j}\times\mathbf{B}$ and $\nabla\times(\mathbf{u}\times\mathbf{B})$ in Eqs. (71) and (72).

In the same way as that used in Eq. (36), it can be shown that Eq. (46) has the same force-free field solution with Eq. (65) for the case with spatially uniform η:

$$\nabla\times\mathbf{B}^*=\lambda\mathbf{B}^*\quad(|\lambda|\equiv\sqrt{\alpha\mu_0/2\eta})\ . \tag{83}$$

In this special case, $\mathbf{j}^*\times\mathbf{B}^*=\mathbf{0}$, and then the equilibrium equation, Eq. (78), becomes

$$\nabla p^*=0\ . \tag{84}$$

In more general cases, $\mathbf{B}^*$ contains other components so that $\mathbf{j}^*\times\mathbf{B}^*\neq\mathbf{0}$.

In the same way as that used in Eqs. (38) and (39), when self-organized relaxed states of interest have some kind of symmetry along one coordinate x_s in $\mathbf{x}$, i.e., $\partial/\partial x_s=0$ (two-dimensional systems are included in this case), Eq. (46) can be separated again into the two mutually independent equations, Eqs. (69) and (70). Conventional notations of $\mathbf{B}_s^*\equiv\mathbf{B}_t^*$ (toroidal component) and $\mathbf{B}_{s\perp}^*\equiv\mathbf{B}_p^*$ (poloidal component) are used for the case of toroidal symmetric relaxed states. As one branch of the toroidal relaxed states, the field reversal configuration (FRC) plasma without $\mathbf{B}_t^*$ has been observed recently in merging experiments of two spheromak plasmas, as shown in Fig. 2 in [8]. This FRC branch of the relaxed states can be represented by Eq. (70).

When self-organized states with uniform η in a three-dimensional system have a feature of $\sqrt{\alpha\mu_0/2\mu}\mathbf{B}_s^*=\nabla\times\mathbf{B}_{s\perp}^*$, it can be shown straightforwardly with use of Eq. (70) that the total field of $\mathbf{B}^*=\mathbf{B}_s^*+\mathbf{B}_{s\perp}^*$ constitutes a solution of the helical force-free field of Eq. (83), as shown after Eq. (70). This force-free field is realized approximately in experimental low β plasmas (i.e., negligible pressure gradient of $\nabla p^*\simeq 0$) when spatially uniform resistivity η is assumed. In more general cases with nonuniform η, substituting $\mathbf{j}^*=\mathbf{j}_\parallel^*+\mathbf{j}_\perp^*$ and $\mu_0\mathbf{j}_\parallel^*=f(\mathbf{x})\mathbf{B}^*$ into $\nabla\times(\eta\mathbf{j}^*)=(\alpha/2)\mathbf{B}^*$ of Eq. (46), using $\mu_0\mathbf{j}=\nabla\times\mathbf{B}$, and comparing the factor of $\mathbf{B}^*$, we obtain the following approximate solution for $\mathbf{j}_\parallel^*$ at the self-organized relaxed state:

$$\mu_0\mathbf{j}_\parallel^*\simeq\sqrt{\mu_0\alpha/2\eta}\mathbf{B}^*\ , \tag{85}$$

where the subscripts $\parallel$ and $\perp$ denote, respectively, the parallel and the perpendicular components to the field $\mathbf{B}^*$. As reported in [24], a comparison between this theoretical result of Eq. (85) and the results of 3D MHD simulations with both "nonuniform η" and "uniform η" supports this dependence of $\mathbf{j}_\parallel^*$ on η profiles.

VI. SUMMARY

We have presented a theory, more refined than that in [20], which stands upon the concept of the coherent structure included in the self-organized dissipative structure, and its application, which is in further detail than in [20]. As one of the universal mathematical structures embedded in dissipative dynamic systems of Eq. (1), we have clarified in Sec. II that the realization of coherent structures in time evolution, which is expressed by definition (i) with use of autocorrelations, is equivalent to that of self-organized states with the minimum value of $|\partial W_{ii}/\partial t|$ for instantaneously contained W_{ii}, expressed by the equivalent definition (iv). It is seen from a comparison of definitions (i)–(iv) and Eqs. (1)–(3) that this coherent structure of the self-organized state with the minimum change rate is determined essentially by the equations of the dynamic system themselves, which rule the time evolution of the system, and key terms are dissipative dynamic operators $L_i^D[\mathbf{q}]$ in the system. We find the following features (a)–(e) from the variational calculus of Eqs. (4)–(11) and from Eqs. (12)–(16): (a) The attractors of the dissipative structure are given by eigenfunctions q_i^* of Eq. (11) for dissipative operators $L_i^D[\mathbf{q}]$. (b) The attractors constitute the self-organized and self-similar change phase with the minimum change rate of the autocorrelation W_{ii}. (c) The Lagrange multiplier α becomes equal to the time constant of change of W_{ii} in the self-similar change phase. (d) The self-organized states with coherent structure have to satisfy the equilibrium equation Eq. (16). (e) The bifurcation point of the dissipative structure is generally given by $\alpha=\alpha_1$ with use of the smallest positive eigenvalue α_1 for the associated eigenvalue problem of Eq. (17).

We have presented three typical examples of detailed applications of the present refined theory to incompressible viscous fluids (Sec. III), to incompressible viscous and resistive MHD fluids such as liquid metals (Sec. IV), and to compressible resistive MHD plasmas (Sec. V) and have derived attractors of the dissipative structure in these dissipative fluids. We have clarified that all of the attractors in the three dissipative fluids have the same features as those of attractors in the general theory mentioned above. Using eigensolutions of basic modes for boundary value problems and complete orthogonal sets by eigenfunctions for associated eigenvalue problems for the three dissipative fluids, we have presented detailed physical pictures of the self-organization of these dynamic systems approaching basic modes and also of the bifurcation of the dissipative structures from basic modes to mixed modes. Those physical pictures consist of the following two common key processes: The first is the spectrum transfer toward both the higher and the lower eigenmode regions for dissipative dynamic operators, caused by such as instabilities and field reconnections. The second is the selective dissipation for higher eigenmode components associated inevitably with dissipative operators. Results of 3D MHD simulations show that these two key processes take place almost simultaneously during fast relaxation phase, as reported in [25].

Corresponding to the Fourier spectrum analysis shown in [21–23], Eqs. (35), (62), (63), and (82), with the use of the eigenfunction expansion, suggest that an eigenfunction spectrum analysis associated with dissipative dynamical operators $L_i^D[\mathbf{q}]$ will be useful to understand self-organization processes. This type of eigenfunction spectrum analysis for our computer simulations of self-organization processes in resistive MHD plasmas [24] and in incompressible viscous fluids is under investigation and the results will be reported elsewhere [25].

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