Scaling and density of Lee-Yang zeros in the four-dimensional Ising model

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All of the information on the behavior of the four-dimensional Ising model is contained in the distribution and density of its partition function zeros. This model is believed to belong to the same universality class as the ϕ_4^4 model which plays a central role in relativistic quantum field theory. Here the scaling behavior of the edge of the distribution of zeros and the asymptotic form for the density of zeros are determined. The finite-size dependency of the density of zeros—or the distance between zeros—at the infinite volume critical point is found using both analytic and numerical approaches. As with a previous analysis of the lowest lying zero, emphasis is laid on the multiplicative logarithmic corrections to mean field scaling behavior which are related to the triviality of the Ising and ϕ^4 models in four dimensions.

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I. ZEROS OF THE PARTITION FUNCTION

The four-dimensional (4D) Ising model is believed to belong to the same universality class as the ϕ^4 model which plays a central role in relativistic quantum field theory. At a value κ of the intersite coupling and in the presence of an external field H, the grand canonical partition function may be written as a polynomial in the fugacity $z = \exp(-2\kappa H)$ [1]

$$Z_N(t,z) = z^{-\frac{N}{2}} \sum_{n=0}^{N} \rho_n(t) z^n \quad , \tag{1.1}$$

where N is the number of lattice sites, ρ_n an integrated density, and the reduced temperature $t = 1 - \kappa/\kappa_c$ is a measure of the distance away from the infinite volume critical value κ_c of κ.

That the partition function (1.1) is analytic for finite N establishes that no phase transition can occur in a finite-size system. However, as N is allowed to approach infinity, phase transitions which manifest themselves as points of nonanalyticity can and do occur. In 1952 Yang and Lee [1] showed that the study of the onset of criticality is equivalent to that of the scaling behavior of the zeros of the partition function. For a finite system, or in the thermodynamic limit but in the symmetric phase ($t > 0$), the zeros in H are strictly complex and the free energy is analytic in a nonvanishing neighborhood of the real axis. As criticality is approached ($N \to \infty$, $t \to 0$) the Lee-Yang zeros pinch the real H axis, precipitating a phase transition. The Lee-Yang theorem states that for the Ising model these zeros lie on the unit circle in the complex fugacity plane (the imaginary axis in the complex external field plane). This theorem holds independent of the size, dimension, and structure of the lattice.

Itzykson, Pearson, and Zuber [2] connected the concept of partition function zeros to the renormalization group and thereby formulated a finite-size scaling theory for these zeros. Their work applies to dimensions of three or less where scaling behavior is of a power-law nature. This was later extended to dimensions above the upper critical dimension $d = 4$ where the scaling behavior of the thermodynamic functions simplifies and the critical exponents are exactly those of the mean field theory [3]. At the upper critical dimension $d = 4$, perturbation theory and renormalization group considerations [4] imply that the mean field power-law scaling behavior is modified by multiplicative logarithmic corrections — a circumstance intimately related to the expected triviality of the theory [5–7]. These logarithmic corrections have recently been identified from a perturbative renormalization group analysis of finite-size scaling of the partition function zeros backed up by a high precision numerical study [8].

While a study of these lowest lying partition function zeros suffices as a numerical confirmation of the theoretically predicted existence of multiplicative logarithmic corrections, of further fundamental significance to the theory of critical phenomena is the *density* of zeros. Although the latter contains all of the information on the behavior of these Lee-Yang systems, its exact form is unknown for all but $d = 1$ dimensions.

It has, however, been shown rigorously that for isotropic nearest neighbor interactions, and for t sufficiently positive (the symmetric phase), there exists a region around $H = 0$ which is free from zeros [9]. This means there exists a gap $|\mathrm{Im}H| < H_1(t)$ where the density of zeros is zero. The free energy is analytic in H in the gap and no phase transition can occur (as a function of H). The point $H = iH_1(t)$, which is a branch point of the partition function, is called the Yang-Lee edge [10]. One expects that this property (the existence of a gap) holds in fact for all $t > 0$.

The scaling behavior of the Yang–Lee edge in the thermodynamic limit was studied by Abe [11] and by Suzuki [12] in 1967 for Ising models strictly below the upper critical dimension as well as for the mean field theory. They found asymptotic forms for the density of zeros

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and a power-law behavior for the scaling of the edge in the symmetric phase.

With the exception of [13,14,2], there has been very little numerical work concerning the actual *density* of zeros. Early numerical work [13], which involved the exact calculation of the density of states $\rho_n(t)$, and was thus restricted to very small low-dimensional lattices, yielded some evidence for the existence of such a gap. High temperature and high field expansions (in the thermodynamic limit) were used in [14] to numerically approximate the density of zeros for two- and three-dimensional lattices and for the mean field theory. Again, power-law behavior was evident. The density of zeros was also studied for finite volume three-dimensional lattices in [2]. Recently Salmhofer [15] has proved the existence of a unique density of zeros in the thermodynamic ($N \to \infty$) limit.

Here we present analytical and numerical results on the *density* of Lee-Yang zeros in *four* dimensions. These include (i) the scaling behavior of the Yang-Lee edge itself in the thermodynamic limit, (ii) the asymptotic form for the density of zeros which is sufficient to recover the scaling form for the specific heat and magnetic susceptibility, and (iii) the finite-size behavior of the density of zeros. The numerical results are due to a more detailed analysis (on some more statistics) of our earlier study; results on the scaling of the closest Fisher and Lee-Yang zeros have been published there [8].

II. THE DENSITY OF LEE-YANG ZEROS AND THE YANG-LEE EDGE

According to the Lee-Yang theorem [1] the zeros of the partition function all lie on the unit circle in the complex fugacity plane. Denoting the (t-dependent) position of these zeros by

$$z_j(t) = e^{i\theta_j(t)} \quad , \quad \theta_j \in \mathbb{R} \quad , \quad j = 1, \ldots, N \qquad (2.1)$$

the partition function may be written as

$$Z_N(t,z) = z^{-\frac{N}{2}} \rho_N(t) \prod_{i=1}^{N} \left(z - e^{i\theta_j(t)} \right) \quad . \qquad (2.2)$$

The largest coefficient $\rho_N(t)$ plays no role in the following and we henceforth set it to unity. The free energy density,

$$f_N(t,z) = \frac{1}{N} \ln Z_N(t,z) \quad , \qquad (2.3)$$

can be written as

$$f_N(t,z) = -\frac{1}{2} \ln z + \frac{1}{N} \sum_{j=1}^{N} \ln \left(z - e^{i\theta_j(t)} \right) \quad . \qquad (2.4)$$

The discrete measure dG_N is formally given by

$$g_N(\theta,t) = \frac{dG_N(\theta,t)}{d\theta} = \frac{1}{N} \sum_{j=1}^{N} \delta(\theta - \theta_j(t)) \quad . \qquad (2.5)$$

The t-dependent density of Lee-Yang zeros on the unit circle in the complex z plane is given by g_N and the cumulative density of zeros G_N is a function monotonically increasing in θ from $G(0,t) = 0$ to $G(2\pi,t) = 1$. The free energy is

$$f_N(t,z) = -\frac{1}{2} \ln z + \int_{\theta=0}^{\theta=2\pi} \ln \left(z - e^{i\theta} \right) dG_N(\theta,t). \qquad (2.6)$$

The thermodynamic limit is

$$g(\theta,t) = \lim_{N\to\infty} g_N(\theta,t) \quad , \qquad (2.7)$$

$$G(\theta,t) = \lim_{N\to\infty} G_N(\theta,t) \quad , \qquad (2.8)$$

$$f(t,z) = \lim_{N\to\infty} f_N(t,z) \quad . \qquad (2.9)$$

The coefficients $\rho_k(t)$ of the polynomial (1.1) are real and hence $g(-\theta,t) = g(\theta,t)$. Therefore it is sufficient to consider only the interval $0 \le \theta \le \pi$ in the integrals. The Yang-Lee edge $\theta_c(t)$ is defined by

$$g(\theta,t) = 0 \quad \text{for} \quad -\theta_c(t) < \theta < \theta_c(t) \quad . \qquad (2.10)$$

Integrating (2.6) by parts gives for the free energy

$$f(t,z) = \frac{1}{2} \ln [2\cosh(2h) + 2] - \int_{\theta_c(t)}^{\pi} \frac{\sin\theta}{\cosh(2h) - \cos\theta} G(\theta,t) d\theta \quad . \qquad (2.11)$$

The magnetization is then

$$\frac{\partial f}{\partial h} = \tanh(h) + 2\sinh(2h) \times \int_{\theta_c(t)}^{\pi} \frac{\sin\theta}{[\cosh(2h) - \cos\theta]^2} G(\theta,t) d\theta \quad , \qquad (2.12)$$

and the zero field susceptibility

$$\chi(t) = \left(\frac{\partial^2 f}{\partial h^2} \right)_{h=0} = 1 + 4 \int_{\theta_c(t)}^{\pi} \frac{\sin\theta}{(1-\cos\theta)^2} G(\theta,t) d\theta \quad . \qquad (2.13)$$

One expects the contribution of small θ to be dominant [11,12]. In particular we want to study its contribution singular in t. Expanding the trigonometric functions in (2.13) (and dropping the constant term),

$$\chi(t) = 16 \int_{\theta_c(t)}^{\pi} \frac{G(\theta,t)}{\theta^3} \{1 + O(\theta^2)\} d\theta \quad . \qquad (2.14)$$

In four dimensions and in the symmetric phase the perturbative renormalization group gives [4]

$$\chi(t) \sim t^{-1} (-\ln t)^{\frac{1}{3}} \quad . \qquad (2.15)$$

A change of variables is introduced via $\theta = \theta_c x$. Then in the critical region where $t > 0$ is sufficiently small

$$t^{-1}(-\ln t)^{1/3} \sim \theta_c(t)^{-2} \int_1^{\pi/\theta_c(t)} \frac{G(x\theta_c,t)}{x^2} dx \quad . \qquad (2.16)$$

Following [11,12], the upper integral limit can be replaced by infinity near criticality. This leads to the requirement that

$$\frac{t(-\ln t)^{-\frac{1}{3}}}{\theta_c(t)^2}\int_1^\infty \frac{G(x\theta_c,t)}{x^2}dx \sim \text{const.} \tag{2.17}$$

For fixed t the integral is bounded due to the boundedness of G. The constancy leads to a differential equation [11,12] for G with the general solution

$$G(\theta,t) = t^{-1}(-\ln t)^{\frac{1}{3}}\,\theta_c(t)^2\Phi\left(\frac{\theta}{\theta_c(t)}\right) \quad , \tag{2.18}$$

$\Phi(x)$ being an arbitrary function of x with $\Phi(|\,x\,|\leq 1) = 0$. Then

$$g(\theta,t) = \frac{dG(\theta,t)}{d\theta} = t^{-1}(-\ln t)^{\frac{1}{3}}\,\theta_c(t)\Phi'\left(\frac{\theta}{\theta_c(t)}\right), \tag{2.19}$$

where $\Phi'(x) = \frac{d\Phi(x)}{dx}$.

From (2.11) [and using the fact that $G(\theta_c,t) = 0$], one gets the specific heat

$$\begin{aligned} C_V(t) &= \left.\frac{\partial^2 f(t,z)}{\partial t^2}\right|_{h=0} \\ &= -2\int_{\theta_c(t)}^{\pi}\theta^{-1}\frac{d^2G(\theta,t)}{dt^2}\{1+O(\theta^2)\}d\theta \quad . \end{aligned} \tag{2.20}$$

Now the cumulative density of zeros in four dimensions may be found from (2.18). In four dimensions one expects the power-law scaling behavior characteristic of dimensions below the upper critical one to be modified by multiplicative logarithmic corrections. Assume therefore that the Yang-Lee edge has the scaling behavior

$$\theta_c(t) = At^p(-\ln t)^{-\lambda} \tag{2.21}$$

for small $t > 0$ and with $0 < p < 1$. This gives

$$\begin{aligned} \frac{d^2G(\theta,t)}{dt^2} &= A^2t^{2p-3}(-\ln t)^{\frac{1}{3}-2\lambda}\left[1+O\left(\frac{1}{\ln t}\right)\right] \\ &\quad\times\{2(1-3p+2p^2)\Phi(x) \\ &\quad+p(3-4p)x\Phi'(x)+p^2x^2\Phi''(x)\}, \end{aligned} \tag{2.22}$$

where $x = \theta/\theta_c$ and a prime indicates derivative with respect to x. The specific heat is then

$$C_V \propto t^{2p-3}(-\ln t)^{\frac{1}{3}-2\lambda}\left[1+O\left(\frac{1}{\ln t}\right)\right]\int_1^{\frac{\pi}{\theta_c}}\frac{I(x)}{x}dx \quad , \tag{2.23}$$

where I is some function of x. As $t \to 0$ [$\theta_c(t) \to 0$], one has

$$C_V \propto t^{2p-3}(-\ln t)^{\frac{1}{3}-2\lambda}\left[1+O\left(\frac{1}{\ln t}\right)\right] \tag{2.24}$$

in the symmetric phase ($t > 0$, $H = 0$) and near criticality. From perturbation renormalization group analyses it is known [4] that the zero field specific heat scales as

$$C_V(t) \sim (-\ln t)^{\frac{1}{3}} \tag{2.25}$$

in four dimensions. Therefore $p = \frac{3}{2}$ and $\lambda = 0$. From (2.21) the Yang-Lee edge in four dimensions scales as

$$\theta_c(t) \sim t^{\frac{3}{2}} \quad . \tag{2.26}$$

It should be clear that the derivation of (2.26) does not involve mean field approximation but is derived in what is genuinely four dimensions. That the power of the multiplicative logarithmic corrections happens to be zero means that the scaling behavior of the Yang-Lee edge in four dimensions coincides with that yielded by mean field theory [2,11,12] in the thermodynamic limit.

The density of Lee-Yang zeros is given by (2.19) as

$$g(\theta,t) = t^{\frac{1}{2}}(-\ln t)^{\frac{1}{3}}\,\Phi'\left(\frac{\theta}{\theta_c}\right) \quad , \tag{2.27}$$

in which Φ' is an unknown function. This form is sufficient to recover the singular behavior of the susceptibility and of the specific heat.

The behavior of the zero at $\theta = x\theta_c$ (for fixed x) as a function of t ($t > 0$) is given by (2.27). At fixed t, $g(\theta,t)$ is an unknown function of θ/θ_c. Kortman and Griffiths emphasized the study of the density of zeros close to the Yang-Lee edge [14]. Using high temperature and high field series they concluded that below the upper critical dimension and for a fixed (strictly positive) t, the density of zeros near the edge exhibits a power-law behavior

$$g(\theta,t) \sim [\theta-\theta_c(t)]^\sigma \quad . \tag{2.28}$$

In zero dimensions (a single site) σ is known to be -1 [16]. For the exactly solvable one-dimensional Ising model $\sigma = -1/2$ for all $t > 0$ [1], i.e., the density of zeros diverges as the edge is approached. The Ising model in the presence of an external field has not been solved in more than one dimension. Nonetheless the value of σ in two dimensions has been found to be $-1/6$ by Dhar [17] by mapping the two-dimensional Ising ferromagnet into a solvable model of three-dimensional directed animals. Cardy [18] found the same result by using the conformal invariance of two-dimensional systems at the critical point. Using high temperature numerical methods, Kurtze and Fisher [10,19] found $\sigma = 0.086(15)$ in three dimensions. It is believed that these values hold independent of the lattice parametrization used [10]. For the mean field theory $\sigma = 1/2$ [14]. Thus there seems to be a systematic increase of σ with dimensionality.

At criticality $t = 0$, however, the Yang-Lee gap vanishes and one may expect the critical exponent σ to take on a value different than that in the symmetric phase. Now, the density of zeros is proportional to the discontinuity in the magnetization M crossing the locus of zeros [1],

$$\lim_{r\to1^+} M(t,z=re^{i\theta}) - \lim_{r\to1^-} M(t,z=re^{i\theta}) \propto g(\theta,t). \tag{2.29}$$

The infinite volume behavior of the magnetization below the upper critical dimension

$$M(t=0,H) \sim H^{\frac{1}{\delta}} \tag{2.30}$$

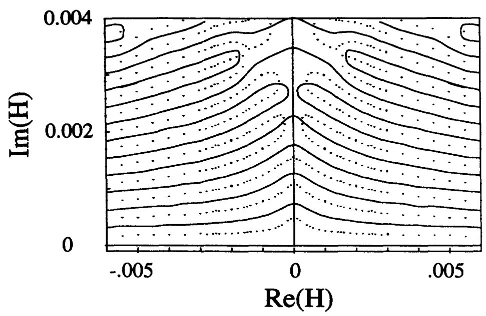

FIG. 1. Contours along which $\mathrm{Re}Z = 0$ (dotted lines) and $\mathrm{Im}Z = 0$ (full lines) (for $L = 24$ and $\kappa = 0.149\,703$).

should be recovered from (2.28) at $t = 0$ and therefore, for $d < 4$,

$$g(\theta, t = 0) \sim \theta^{\frac{1}{\delta}} \quad . \tag{2.31}$$

In four dimensions where $\delta = 3$, one expects the above formulas to be modified by multiplicative logarithmic corrections. There, (2.30) becomes [4]

$$M(t = 0, H) \sim H^{\frac{1}{3}}(-\ln H)^{\frac{1}{3}} \quad . \tag{2.32}$$

Therefore in 4D in the thermodynamic limit

$$g(\theta, t = 0) \sim \theta^{\frac{1}{3}}(-\ln \theta)^{\frac{1}{3}} \quad . \tag{2.33}$$

III. FINITE-SIZE ANALYSIS

Nonperturbative means of calculating thermodynamic functions in spin models are provided by stochastic techniques such as Monte Carlo integration. These numerical methods yield exact results subject only to statistical error. They are, however, limited to finite lattices. One has to rely on finite-size scaling (FSS) extrapolation methods to gain information on the corresponding thermodynamic limit.

Let $P_L(t)$ represent the value of some thermodynamic quantity P at reduced temperature t on a lattice characterized by a linear extent L. Then, if ξ is the correlation length, the FSS hypothesis is that [8,20]

$$\frac{P_L(t)}{P_\infty(t)} = f\left(\frac{\xi_L(t)}{\xi_\infty(t)}\right) \quad . \tag{3.1}$$

In four dimensions the scaling behavior of the correlation length is [21]

$$\xi_\infty(t) \sim t^{-\frac{1}{2}}(-\ln t)^{\frac{1}{6}} \quad . \tag{3.2}$$

Its scaling with L is [22]

$$\xi_l(0) \sim L(\ln L)^{\frac{1}{4}} \quad . \tag{3.3}$$

Therefore in $d = 4$ the scaling variable should include logarithmic terms [8]

$$x = \frac{L(\ln L)^{\frac{1}{4}}}{t^{-\frac{1}{2}}(-\ln t)^{\frac{1}{6}}} \quad . \tag{3.4}$$

Let H_1 be the position of the Yang-Lee edge in the complex external magnetic field plane in the thermodynamic limit and let $H_1(L)$ be its finite-size counterpart (i.e., the position of the lowest lying zero for a system of finite linear extent L). From (2.26), the FSS hypothesis applied to the Yang-Lee edge gives

$$H_1(L) \sim t^{\frac{3}{2}} f(x) \quad . \tag{3.5}$$

Fixing x (so that when rescaling L, the temperature is also rescaled in such a way as to keep x constant), we find

TABLE I. The positions of the first Lee-Yang zeros as obtained from the multihistograms for all five lattices and near κ_c. The real part of the zeros is always zero.

	$L=8$	$L=12$	$L=16$	$L=20$	$L=24$
κ	$\mathrm{Im}H_1$	$\mathrm{Im}H_1$	$\mathrm{Im}H_1$	$\mathrm{Im}H_1$	$\mathrm{Im}H_1$
0.149600	0.015281	0.004511	0.001958	0.001047	0.000637
0.149650	0.015091	0.004384	0.001860	0.000966	0.000567
0.149703	0.014892	0.004253	0.001761	0.000886	0.000500
0.149750	0.014718	0.004140	0.001677	0.000820	0.000447
0.149800	0.014535	0.004023	0.001593	0.000756	0.000398
0.149850	0.014355	0.003910	0.001512	0.000697	0.000355
0.149900	0.014177	0.003800	0.001437	0.000644	0.000319
0.149950	0.014001	0.003693	0.001366	0.000597	0.000289
0.150000	0.013828	0.003590	0.001299	0.000555	0.000264
0.150050	0.013657	0.003491	0.001237	0.000518	0.000244
0.150100	0.013488	0.003395	0.001180	0.000486	0.000228
0.150150	0.013322	0.003302	0.001127	0.000458	0.000215
0.150200	0.013159	0.003213	0.001077	0.000433	0.000204
0.150250	0.012997	0.003127	0.001032	0.000413	0.000195
0.150300	0.012838	0.003044	0.009909	0.000395	0.000187
0.150350	0.012682	0.002965	0.009530	0.000379	0.000180
0.150400	0.012527	0.002889	0.009185	0.000366	0.000174

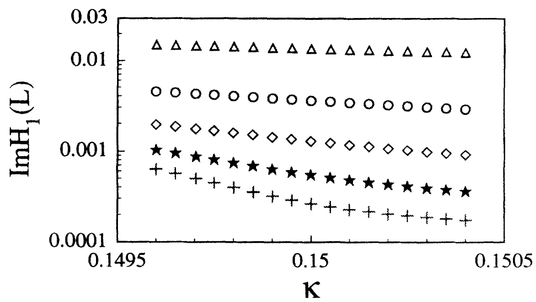

FIG. 2. The zeros approach the real axis as κ increases; at κ_c they should scale with the lattice size L according to (3.6). Here the triangles, circles, diamonds, stars, and crosses correspond to lattice sizes 8, 12, 16, 20, and 24, respectively.

$$H_1(L) \sim [x^{-1}L^{-2}(\ln L)^{-\frac{1}{6}}]^{\frac{3}{2}} f(x) \sim L^{-3}(\ln L)^{-\frac{1}{4}} \quad . \tag{3.6}$$

This FSS formula agrees with that derived recently by perturbative renormalization group methods [8].

The perturbative renormalization group analysis of the finite-size ϕ_4^4 model [8] gives the relationship between the magnetization $M_L(t, H)$ and external field H at reduced temperature t,

$$H = c_1 t M_L (\ln L)^{-\frac{1}{3}} + c_2 M_L^3 (\ln L)^{-1} \quad , \tag{3.7}$$

where c_1 and c_2 are constants. At $t = 0$, therefore,

$$M_L(t = 0, H) \sim H^{\frac{1}{3}} (\ln L)^{\frac{1}{3}} \quad . \tag{3.8}$$

For a finite-size system the position of the Yang-Lee edge is not zero at $t = 0$ and the origin of nonvanishing density of zeros has to be correspondingly shifted as in (2.28). One therefore expects the density of zeros to be

$$g_L[H_j(L)] \sim [H_j(L) - H_1(L)]^{\frac{1}{3}} (\ln L)^{\frac{1}{3}} \quad , \tag{3.9}$$

where $H_j(L)$ is the position of the jth Lee-Yang zero. Defining the cumulative density of zeros at the jth zero by the fractional total of zeros up to $H_j(L)$,

$$G_L[H_j(L)] = \frac{j-1}{L^4} \quad , \tag{3.10}$$

we find [integrating g_L in (3.9) to G_L]

$$\frac{j-1}{L^4} \sim [H_j(L) - H_1(L)]^{\frac{4}{3}} (\ln L)^{\frac{1}{3}} \quad . \tag{3.11}$$

TABLE II. The positions of the first three Lee-Yang zeros as obtained from the jackknifed multihistograms at $\kappa = 0.149\,703$. The real part of the zeros is always zero.

L	Im(H_1)	Im(H_2)	Im(H_3)
8	0.014892(22)	0.033057(48)	0.047357(174)
12	0.004253(16)	0.009426(15)	0.013349(71)
16	0.001761(6)	0.003905(22)	0.005388(24)
20	0.000886(5)	0.001970(12)	0.002743(29)
24	0.000500(4)	0.001106(5)	0.001541(12)

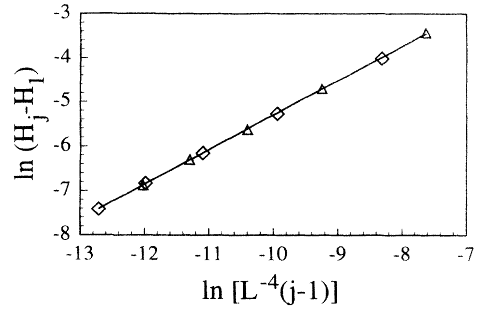

FIG. 3. The FSS of the density of zeros is given by (3.12). The leading power-law behavior is revealed by a log-log plot. Here the open diamonds and triangles correspond to $j = 2$ and $j = 3$, respectively. This gives a slope of 0.778(2), the deviation away from 0.75 being due to the presence of logarithmic corrections.

Therefore

$$H_j(L) - H_1(L) \sim \left(\frac{j-1}{L^4}\right)^{\frac{3}{4}} (\ln L)^{-\frac{1}{4}} \quad . \tag{3.12}$$

Equation (3.12) gives the FSS behavior of the distance between lowest lying zeros, i.e., of the density of zeros.

We now compare these FSS results with data obtained for the 4D Ising model in a high statistics Monte Carlo calculation. The simulation was done with the Swendsen-Wang cluster updating algorithm [23] applied to lattices of sizes L^4 with linear extension $L = 8, 12, 16, 20, 24$ (details of the numerics can be found in [8]).

The critical value of κ in four dimensions has been determined to $\kappa_c = 0.149\,703(15)$ [8]. Our data yield only three reliable Lee-Yang zeros for each lattice size. The reason for this is demonstrated in Fig. 1 where the contours along which Re$Z = 0$ and Im$Z = 0$ (for $L = 24$ and $\kappa = 0.149\,703$) are plotted. Because of the magnification of statistical errors far away from the simulation point $H = 0$ these contours fail to cross the imaginary Z axis when ImH is large. Thus the zeros move off the ImH axis and their positions are unreliable. The remaining lattices give qualitatively similar pictures.

Table I lists the positions of the first Lee-Yang zeros (the Yang-Lee edge) as obtained from the multihis-

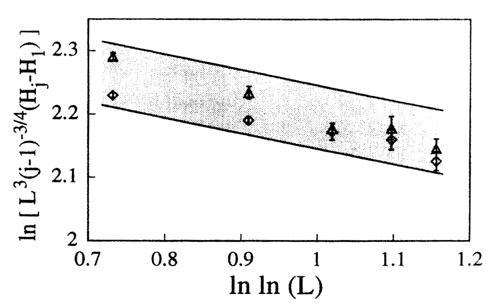

FIG. 4. Data as in Fig. 3, but with the leading power-law behavior removed; we clearly identify the negative exponent in the $\log L$ behavior. The shaded band indicates the result of a fit giving a slope value of $-0.248(17)$.

tograms for various κ values near κ_c and for all five lattices analyzed. As κ increases one expects the zeros to approach the real axis in the thermodynamic limit according to (2.26). Figure 2 shows the corresponding behavior for the finite-size systems considered. At κ_c they should scale according to the FSS formula (3.6).

Table II lists the positions of the first three Lee-Yang zeros as obtained from the multihistograms at our estimated value for the critical coupling in the infinite volume limit, $\kappa_c = 0.149\,703$. The errors in the quantities calculated from the multihistograms were estimated by the jackknife method as in [8].

The density of zeros should behave according to (3.9) or (equivalently) (3.12). The log-log plot of Fig. 3 gives a slope of 0.778(2). The deviation from the exponent 0.75 in (3.12) is presumably due to the presence of logarithmic corrections. This may be seen in Fig. 4 where we remove the expected leading behavior: A negative slope is clearly identified. In fact a best fit to all ten points gives a slope of −0.248(17). The shaded area is bordered by lines of this slope.

We find that both leading power-law scaling behavior and multiplicative logarithmic corrections for the density of zeros (or equivalently for the distance between zeros) are identified in Figs. 3 and 4. This is complementary to our previous analysis in which the scaling behavior of the actual positions of these zeros was analyzed [8]. Both approaches yield quantitative agreement with the (perturbative) theoretical predictions.

IV. CONCLUSIONS

The scaling behavior of the Lee-Yang zeros and in particular of the Yang-Lee edge in four dimensions and in the thermodynamic limit has been determined. The asymptotic form for the density of zeros in the infinite volume limit is sufficient to recover the scaling formulas for the specific heat, the magnetization, and the magnetic susceptibility. This extends the work of Abe and Suzuki to the case of four dimensions where mean field power-law scaling behavior is modified by multiplicative logarithmic corrections which are linked to the triviality of the theory. An analytical FSS study of the edge and the density of (i.e., distance between) zeros is in good quantitative agreement with a numerical analysis in the form of Monte Carlo simulations on finite-size lattices.

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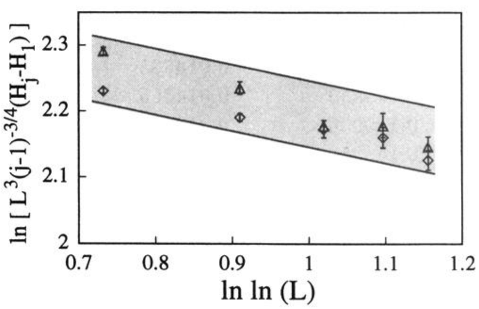

FIG. 4. Data as in Fig. 3, but with the leading power-law behavior removed; we clearly identify the negative exponent in the $\log L$ behavior. The shaded band indicates the result of a fit giving a slope value of $-0.248(17)$.