Role of nonlinear dissipation in soft Duffing oscillators

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The effect of a strictly dissipative force (velocity to the pth power model) on the response and bifurcations of driven, soft Duffing oscillators is considered. The method of harmonic balance is used to obtain the steady state harmonic response. An anomalous jump in the harmonic response (signifying a break in the resonance curve), obtained in the case of linearly damped, soft Duffing oscillators, is shown to persist even in the presence of nonlinear damping. It is shown that the bifurcation structure and the structure of the chaotic attractor are quite insensitive to the damping exponent p. However, the threshold values of the parameters, at which bifurcations occur, depend both on the damping index and the damping coefficient. The Melinkov criterion and an analytical criterion for the period-doubling bifurcation have been obtained in the presence of combined linear and cubic damping.

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I. INTRODUCTION

Duffing oscillators are employed as models of various physical and engineering situations such as Josephson junctions, optical bistability, plasma oscillations, buckled beam, ship dynamics, vibration isolators, electrical circuits, etc. [1–6]. Since Ueda's work in 1979 [7], it is known that chaotic responses are commonplace in Duffing oscillators subjected to harmonic excitation. The existence of a symmetry-breaking precursor and a subsequent period-doubling route to chaos in these oscillators has been explained through the stability analysis using Mathieu and Hill's equations [8]. Huberman and Crutchfield [9,10] considered the role of fluctuations on the onset of the period-doubling route to chaos. They showed that the structure of the strange attractor is very stable even under the influence of large fluctuating forces and that the role of noise is to introduce a symmetric gap in the deterministic bifurcation sequence. Wiesenfeld [11] addressed the role of noisy precursors on the period-doubling and other instabilities encountered in Josephson junction circuits. Parlitz and Lauterborn [12] examined the connection between the bifurcation set and resonance structure of a forced Duffing oscillator. The bifurcation structures in various types of forced Duffing oscillators and the role of symmetry of the potential well have been considered in Ref. [13]. Several authors [14–18] obtained an extra jump in the harmonic response of soft Duffing oscillators indicating a break in the resonance curve. Miles [16,17] referred to this jump as an "anomalous jump" and conjectured that this break in the resonance curve for the oscillator is a necessary antecedent to symmetry breaking.

In all the literature cited so far, the phenomenological model of the dissipative force has been assumed to be linear. In this context it is appropriate to recall what Pippard [19] has said: "There is something of a tendency among physicists to try to reduce everything to linearity. . . , reality may not always conform to what we might wish, rather more so with the damping forces than with the restoring force in small-amplitude vibrations." In the context of his discussion of the experimental results obtained by Wraight [20], Pippard also suggested that the nonlinear damping curves may give a measure of the pinning and frictional forces involved when the magnetic fields penetrate and move within a superconductor. The role of dissipation in Josephson junctions has received special attention in recent years both from the experimental [21] and theoretical [22] points of view. Landauer [23] presented a nice discussion of ensuing philosophical debate on the origin of dissipation, fluctuations, and irreversibility. The modern geometrical theory of the dissipative dynamical systems introduced several new notions like "strange attractors," "basins of attraction," etc., [24]. These concepts, being essentially the facets of dissipation in dynamics, seem to inherit all the associated problems. Milnor [25] gave a detailed account of the conceptual difficulties for giving a precise mathematical definition of an attractor.

A good account of the role of damping in vibrations from the engineering point of view has been given by Crandall [26]. He mentioned that damping is of great relevance since it decides the border of stability and instability. Consideration of nonlinear damping models is necessary in several engineering applications such as the "roll damping" in ship dynamics [4], vibration isolators [27,28], and the drag forces encountered in flow induced vibration problems [3]. An interesting experimental work, on the modeling of nonlinearly damped vibration isolators used in space applications, can be found in Ref. [6]. The importance of nonlinear damping in engineering stems from the fact that it can be used as an effective pas-

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sive control strategy to suppress various instabilities [5,29]. With these applications in mind we study the role of nonlinear damping in the dynamics of soft Duffing oscillators.

Several phenomenological models of nonlinear damping are given in the literature [19,27,28]. In this paper, the nonlinear dissipative force is assumed to be proportional to the pth power of velocity. Furthermore, the damping is assumed to be strictly dissipative as opposed to a self-excited system like a Van der Pol oscillator [2]. The method of harmonic balance is used to obtain the steady state response and the existence of the so-called anomalous jump even in the presence of nonlinear damping has been demonstrated. Numerical simulations reveal that the structure of the chaotic attractor is insensitive to the value of p. This statement also holds good for the bifurcation structure. A parametric study is carried out to indicate the role of various types of damping on the onset of period doubling. The condition for the occurrence of fractal basin boundaries (in the presence of combined linear and cubic damping) is obtained via the Melinkov criterion.

II. SINGLE-WELL POTENTIAL OSCILLATOR

Consider the governing equation of motion in the nondimensional form as

$$\ddot{x}+2\zeta\dot{x}+2\zeta_p\dot{x}|\dot{x}|^{p-1}+x+\varepsilon x^3=F\cos\omega\tau\ , \quad \varepsilon<0, p>0\ , \tag{1}$$

where p is the damping exponent, ζ is the coefficient of viscous damping, ζ_p is the coefficient of pth power damping, ε is the nonlinearity parameter, F is the amplitude of excitation, and ω is the frequency of excitation. The dot denotes differentiation with respect to time τ. In this section we illustrate the effects of nonlinear damping on the harmonic response and bifurcation set.

Assuming the harmonic solution of Eq. (1) in the form

$$x=A\cos(\omega\tau-\phi)\ , \tag{2}$$

and using the method of harmonic balance, one gets the following equation for A:

$$A^2(1-\omega^2)^2+(\tfrac{9}{16})\varepsilon^2A^6+(\tfrac{3}{2})(1-\omega^2)\varepsilon A^4+(2\zeta A\omega+2\zeta_pA^p\omega^p\gamma_p)^2-F^2=0\ , \tag{3}$$

where γ_p is given by

$$\gamma_p=(2/\sqrt{\pi})\Gamma[(p+2)/2]/\Gamma[(p+3)/2]\ , \tag{4}$$

and Γ is the standard gamma function. It should be noted that Eq. (3) can also be obtained using the notion of the equivalent viscous damping coefficient [19,27,28].

Typical curves of A vs ω with $p=2$ are shown in Figs. 1(a)–1(d) for different values of F. One can see from Figs. 1(a)–1(c) that apart from the usual jump phenomenon, there exists another branch at low frequencies referred to as the anomalous jump. When the forcing amplitude F is increased beyond a critical value, the anomalous jump merges with the main jump and the response curve opens up (similar to the case of an undamped system) as shown in Fig. 1(d). This phenomenon was observed with $p=1$ and was related to the symmetry-breaking bifurcation [16–18].

To obtain the bifurcation set depicting the unstable zone in the F-ω plane, one can define

$$\rho=(1-\omega^2),\quad \sigma=2\zeta\omega,\quad \sigma_p=2\zeta_p\omega^p\gamma_p\ , \tag{5}$$

and rewrite Eq. (3) as

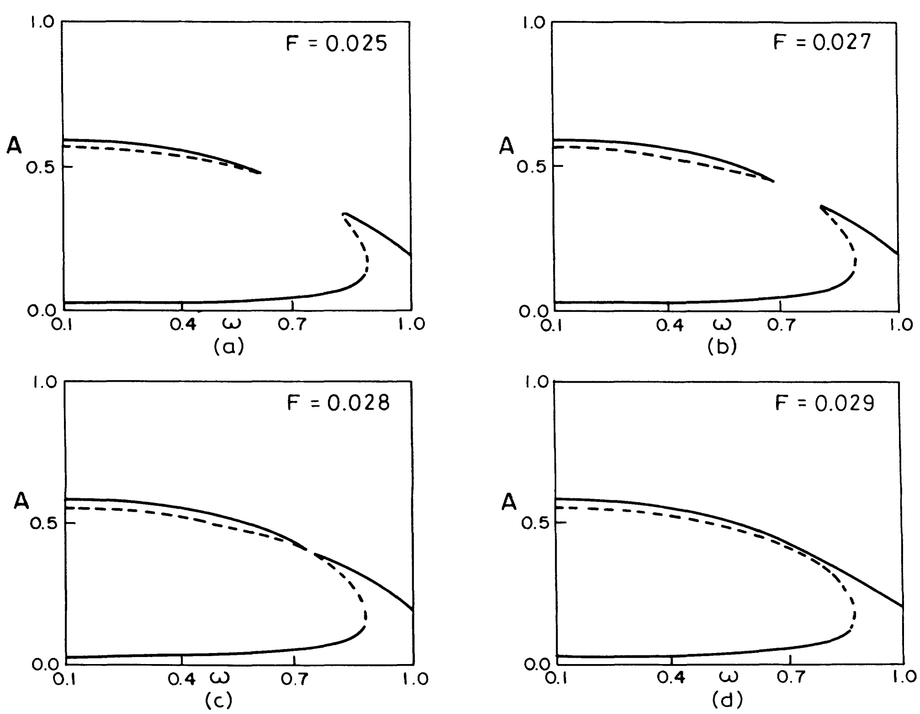

FIG. 1. Response curve A vs ω with $\zeta=0$, $\zeta_2=0.2$, and $\varepsilon=-4.0$.

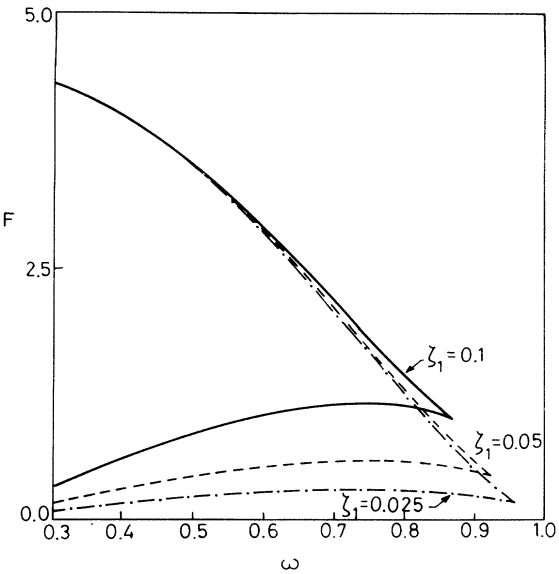

FIG. 2. Bifurcation set of the harmonic solution with $\zeta=0$, $p=1$, and $\varepsilon=-0.01$.

$$9\varepsilon^2 A^6+24p\varepsilon A^4+16p^2 A^2+16(\sigma A+\sigma_p A^p)^2-16F^2=0 . \tag{6}$$

So long as $\zeta=0$ the above equation is bicubic in A, for $p=1$, 2, and 3. In these cases one can employ the condition for repeated roots of a cubic equation to get the loci of the turning points (the points of vertical tangencies in the response curve) in the F-ω plane [15]. The bifurcation sets so obtained for $p=1$ and 2 are shown in Figs. 2 and 3, respectively. It may be seen that the nature of the bifurcation set is similar in both the cases. It may be noted that in a soft spring, there is a limiting amplitude beyond which the spring loses its stiffness and the above results are not valid beyond that amplitude.

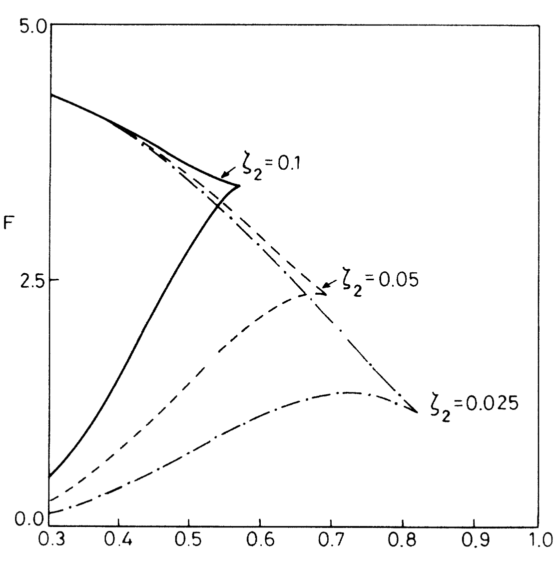

FIG. 3. Bifurcation set of the harmonic solution with $\zeta=0$, $p=2$, and $\varepsilon=-0.01$.

III. DOUBLE-WELL POTENTIAL OSCILLATOR

Now we consider the following nondimensional equation:

$$\ddot{x}+2\zeta\dot{x}+2\zeta_p\dot{x}\,|\dot{x}|^{p-1}+\varepsilon(-x+x^3)=F\cos\omega\tau ,$$

$$\varepsilon>0, p>0 . \tag{7}$$

In this section we study the effects of ζ_p on the period-doubling bifurcation, for $p=1$, 2, and 3, using numerical simulation. Also an analytical criterion for the period doubling bifurcation, based on the instability of the harmonic solution, is derived for combined linear and cubic damping. Further, the Melinkov criterion has also been obtained for such combined damping.

A. Numerical simulation

Equation (7) is numerically integrated using the Runge-Kutta-Merson method to investigate the role of the damping exponent on the structure of the chaotic attractor. The Poincaré maps of the chaotic attractors are shown in Fig. 4 for various values of p. It can be seen from Fig. 4 that the structure of the chaotic attractor is insensitive to the value of p. Numerical results (not presented here) suggest that both the period-doubling and intermittency routes to chaos observed with linear damping [13] persist even in the presence of nonlinear damping. Though the bifurcation structure is qualitatively similar to various values of p, the threshold values of the parameters at which the period-doubling bifurcation occurs depend on both p and ζ_p. It can be observed from Table I that, for the same values of $\zeta_p(>0.05)$, the critical values of F required to generate a period-2 orbit decrease with increasing p. This feature is just opposite to what has been observed in the case of a hard, Duffing oscillator [29] where the critical values were seen to increase with increasing p.

B. Period-doubling criterion

A criterion is derived for the period-doubling bifurcation based on the instability of the harmonic solution predicted by the classical stability analysis. Towards this end, we take $\varepsilon=\frac{1}{2}$ and $p=3.0$ in Eq. (7). Assuming the harmonic solution of Eq. (7) in the form

$$x=A_0+A\cos(\omega\tau-\phi) , \tag{8}$$

TABLE I. The critical values of F required to generate a period-2 orbit with $\zeta=0$, $\varepsilon=1.0$, and $\omega=1.5$.

ζ_p \ p	1	2	3
0.05	0.38	0.38	0.38
0.1	0.41	0.39	0.38
0.15	0.44	0.40	0.39
0.2	0.49	0.42	0.40
0.25	0.54	0.45	0.41

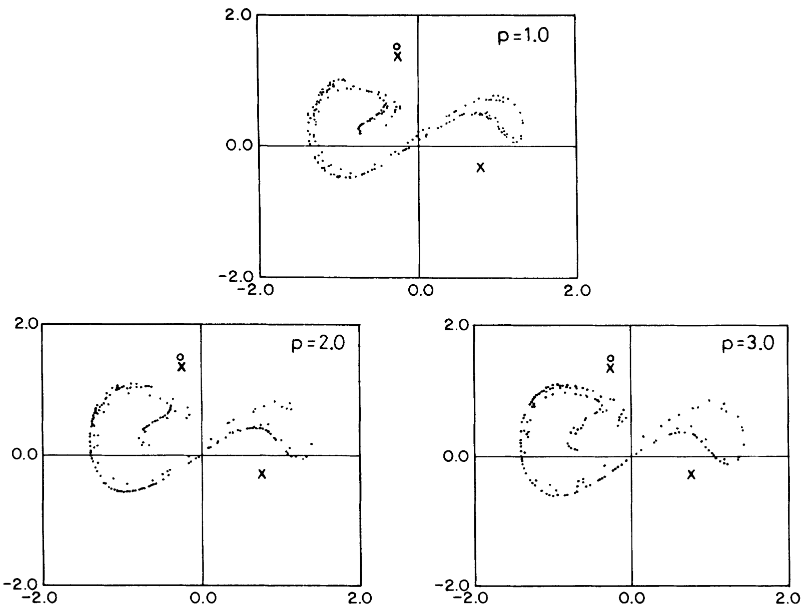

FIG. 4. Stroboscopic maps of chaotic attractors with $\zeta=0$, $\zeta_p=0.125$, $\varepsilon=1.0$, $F=0.4$, and $\omega=1.0$.

and using the method of harmonic balance [30], we get the following expressions for A_0 and A:

$$A_0=(2-3A^2)/2 \tag{9}$$

and

$$(\tfrac{15}{8})^2A^6-(\tfrac{15}{4})(1-\omega^2)A^4+(1-\omega^2)^2A^2+(2\zeta A\omega+3\zeta_3A^3\omega^3/2)^2-F^2=0\ . \tag{10}$$

To analyze the stability of the solution given by Eq. (8), consider $x_0=x+\delta$, where δ is a small perturbation. Employing the usual variational procedure [19] one gets the following linear differential equation with coefficients varying periodically in time:

$$\ddot{\delta}+(\gamma_0+\gamma_1\cos 2\omega\tau)\dot{\delta}+(\lambda_0+\lambda_1\cos\omega\tau+\lambda_2\cos 2\omega\tau)\delta=0\ , \tag{11}$$

where

$$\begin{aligned}\gamma_0&=2\zeta+2\zeta_3A^2\omega^2\ ,\\ \gamma_1&=-3\zeta_3A^2\omega^2\ ,\\ \lambda_0&=3A_0^2/2+3A^2/4-\tfrac{1}{2}\ ,\\ \lambda_1&=3A_0A\ ,\\ \lambda_2&=3A^2/4\ .\end{aligned} \tag{12}$$

Let us assume

$$\delta=\delta_{1/2}\cos[(\omega\tau/2)+\vartheta] \tag{13}$$

to get the first approximate solution for a $\frac{1}{2}$ subharmonic instability [30]. Substituting Eq. (13) in Eq. (11) and using the method of harmonic balance, one finds that for a nontrivial solution, the following condition must be satisfied:

$$(\lambda_0-\omega^2/4)^2+(\zeta+3\zeta_3A^2\omega^2/2)^2\omega^2-\lambda_1^2/4=0\ . \tag{14}$$

Equation (14) in conjunction with Eq. (10) determines the boundary of the period-doubling bifurcation in the F-ω plane for combined linear and cubic damping. Figure 5 shows the results so obtained, separately for linear ($p=1$) and cubic ($p=3$) damping. It may be pointed out that the results for $p=1$ are in perfect agreement with those reported in Ref. [30]. It can be seen from Fig. 5 that the excitation amplitude needed to generate the period-doubling bifurcation for $p=3$ is lower than that for $p=1$. So this trend is in conformity with the numerical results presented in Table I.

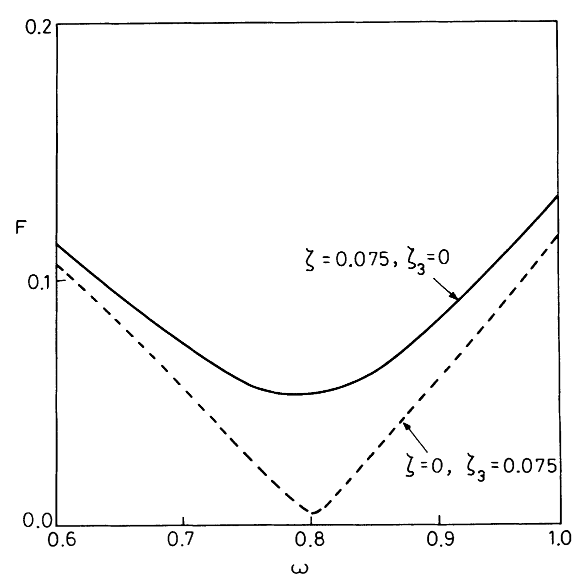

FIG. 5. Period-doubling criterion with $\varepsilon=\frac{1}{2}$.

C. Melinkov criterion

The Melinkov criterion for the double-well potential oscillator has been obtained in the case of linear damping [2]. It has been pointed out that the Melinkov criterion depicts the occurrence of fractal basin boundaries rather than the onset of chaos [3]. Following Ref. [2], the Melinkov criterion for combined linear and cubic damping has been obtained for $\varepsilon=0.5$ as

$$F=\left[\tfrac{8}{3}\zeta+8\zeta_3\int_{-\infty}^{\infty}\operatorname{sec}^4ht\,\tan^4ht\,dt\right]\frac{\cosh(\pi\omega/2)}{\sqrt{2}\pi\omega}\;. \tag{15}$$

Evaluating the integral in Eq. (15), one gets

$$F=(\tfrac{8}{3}\zeta+\tfrac{32}{35}\zeta_3)\frac{\cosh(\pi\omega/2)}{\sqrt{2}\pi\omega}\;. \tag{16}$$

The coefficient of ζ_3 in the above equation is smaller than that of ζ. One can thus conclude that the fractal basin boundaries, just like the period-doubling bifurcation, occur for a lower value of F in the case of cubic damping as compared to linear damping (of course, with $\zeta=\zeta_3$).

It should be noted that the present analysis can be easily extended to the case of a harmonically driven pendulum (with periodic potential) [31]. Following Ref. [31] it can be shown that for a driven pendulum the Melinkov criterion is obtained as

$$F=(8\zeta+\tfrac{128}{3}\zeta_3)\frac{\cos h\,(\pi\omega/2)}{\pi}\;. \tag{17}$$

It is worthwhile to note the difference in the coefficients of ζ and ζ_3 in Eqs. (16) and (17). For the driven pendulum (which is also a soft Duffing oscillator), the critical value of F, predicted by the Melinkov criterion, is more for $p=3$ than that for $p=1$ with $\zeta=\zeta_3$. This is in contradiction to what has been just observed for the double-well potential oscillator.

IV. CONCLUSIONS

A detailed investigation of the effects of strictly dissipative nonlinear damping on the harmonic response, bifurcation set, and chaotic motion of harmonically driven soft Duffing oscillators is carried out. Numerical results indicate that the structure of the chaotic attractor and the routes to chaos are quite insensitive to the value of the damping exponent p. A parametric study is reported to show the influence of the nonlinear damping on the onset of the period-doubling route to chaos. An analytical criterion has been derived, in the presence of combined linear and cubic damping, to estimate the critical values of the excitation at which the period-doubling bifurcation takes place. The Melinkov criterion for such systems has also been obtained. These results indicate that the correct model of damping nonlinearity is important for accurate prediction of the onset of chaos.

[1] P. Holmes, Proc. R. Soc. London, Ser. A **292**, 419 (1979).
[2] J. Guckenheimer and P. J. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields* (Springer, New York, 1983).
[3] F. C. Moon, *Chaotic and Fractal Dynamics* (Wiley, New York, 1992).
[4] J. M. T. Thompson *et al.*, Philos. Trans. R. Soc. London Ser. A **332**, 149 (1990).
[5] B. Ravindra and A. K. Mallik, Int. J. Non-Linear Mech. **28**, 427 (1993).
[6] M. L. Tinker and M. A. Cutchins, J. Sound Vib. **157**, 7 (1992).
[7] Y. Ueda, J. Stat. Phys. **20**, 181 (1979).
[8] S. Novak and R. G. Frehlich, Phys. Rev. A **26**, 3660 (1982).
[9] B. A. Huberman and J. P. Crutchfield, Phys. Rev. Lett. **42**, 1743 (1979).
[10] J. P. Crutchfield and B. A. Huberman, Phys. Lett. **77A**, 407 (1980).
[11] K. Wiesenfeld, J. Stat. Phys. **38**, 1071 (1985).
[12] U. Parlitz and W. Lauterborn, Phys. Lett. **107A**, 351 (1985).
[13] C. S. Wang *et al.*, Phys. Rev. A **45**, 3471 (1992).
[14] K. Magnus, *Vibrations* (Blakie, London, 1965).
[15] P. J. Holmes and D. A. Rand, J. Sound Vib. **44**, 237 (1976).
[16] J. Miles, Phys. Lett. A **130**, 276 (1988).
[17] J. Miles, Physica D **31**, 252 (1988).
[18] W. Szemplinska-Stupnicka, Int. J. Non-Linear Mech. **23**, 257 (1988).
[19] A. B. Pippard, *The Physics of Vibration* (Cambridge University Press, Cambridge, 1989).
[20] P. C. Wraight, Philos. Mag. **23**, 1261 (1971).
[21] P. H. Kes *et al.*, Phys. Rev. Lett. **64**, 1063 (1990).
[22] A. O. Caldeira and A. J. Leggett, Ann. Phys. (N.Y.) **149**, 374 (1983).
[23] R. Landauer, in *Selected Topics in Signal Processing*, edited by S. Haykin (Prentice Hall, Englewood Cliffs, NJ, 1989).
[24] S. Smale, *The Mathematics of Time: Essays on Dynamical Systems, Economic Processes and Related Topics* (Springer-Verlag, Berlin, 1980).
[25] J. Milnor, Commun. Math. Phys. **99**, 177 (1985).
[26] S. H. Crandall, J. Sound Vib. **11**, 3 (1970).
[27] A. K. Mallik, *Principles of Vibration Control* (Affiliated, New Delhi, 1990).
[28] J. E. Ruzicka and T. F. Derby, *Influence of Damping in Vibration Isolation* SVM-7 (Shock and Vibration Information Center, U.S. Department of Defense, Washington, DC, 1971).
[29] B. Ravinda and A. K. Mallik (unpublished).
[30] K. R. Asfar and K. K. Masoud, Trans. ASME J. Vib. Acoustics **114**, 489 (1992).
[31] P. Holmes, in Proceedings of Symposia in Applied Mathematics, Providence, RI, 1989 [Am. Math. Soc. **39**, 25 (1989)].