Drive-response scenario of chaos synchronization in identical nonlinear systems

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We present an alternative way of achieving chaos synchronization in identical nonlinear systems with one-way coupling. We specifically realize this method of chaos synchronization in the van der Pol–Duffing oscillator and obtain a suitable Lyapunov function to establish the criterion for synchronization based on asymptotic stability. Synchronization of chaos is further demonstrated for the case of two identical Duffing oscillators. We also investigate the possibility of secure communication of analog signals in these systems.

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Recently, Pecora and Carroll have introduced the concept of synchronized chaos [1,2] and the method of cascading synchronization [3]. This idea has been in fact successfully tested in a variety of nonlinear dynamical systems, including Lorenz equations, the Rössler system, hysteresis circuits, Chua's circuit, and so on [1–5]. Also, synchronization of chaos has been investigated in two mutually coupled identical nonlinear systems [4]. Here, in this paper, we address the following question: Considering two identical chaotic systems, can one make a chaotic trajectory of one system to synchronize with a chaotic trajectory of the other system with a one-way coupling element alone without requiring that the system under study be divided into two subsystems? In such a model, one system undergoing chaos acts as a *master* or *drive* system, and drives the second identical system, called the *slave* or *response* system. Here, the drive system variables run independently of the response system variables but the response system variables identically follow their drive counterpart under the influence of drive variables as time elapses. In addition, the response system can have different sets of initial conditions than that of the drive system. As time progresses, the two systems achieve a perfect synchronization among their state variables and maintain it, depending upon the one-way coupling strength. We demonstrate this method of chaos synchronization with the aid of two physically interesting models, namely, the van der Pol–Duffing oscillator and the Duffing oscillator systems explicitly. (We have tested the efficacy of this method in a variety of other systems as well.)

As a first case, we consider the third-order autonomous van der Pol–Duffing oscillator system. The schematic representation of two identical circuits with a homogeneous coupling element is shown in Fig. 1. In this circuit, the two van der Pol–Duffing oscillator systems, namely, the *drive* and *response* (circuits within the broken line boxes), are coupled by a linear resistor (R_c) and a buffer. The buffer acts as a signal driving element that isolates the drive system variables from the response system variables, thereby providing one-way coupling. In the absence of the buffer the system represents two identical oscillators coupled by a common resistor (R_c). In this case, both the drive and response systems mutually affect each other. Also, each of the independent systems has a close resemblance with that of Chua's circuit [4,6] in that the piecewise linear element of the latter is presently replaced by a cubic nonlinear element of the form $i_N = aV_{c1} + bV_{c1}^3$ ($a<0$, $b>0$) [7].

By applying Kirchoff's laws to the various branches of the circuit of Fig. 1, and after appropriate rescaling [7], the following set of dynamical equations is obtained:

drive:

$$\dot{x} = -\nu[x^3 - \alpha x - y] , \quad (1a)$$

$$\dot{y} = x - y - z , \quad (1b)$$

$$\dot{z} = \beta y ; \quad (1c)$$

response:

$$\dot{x}' = -\nu[(x')^3 - \alpha x' - y'] + \nu\varepsilon(x - x') , \quad (1d)$$

$$\dot{y}' = x' - y' - z', \quad (1e)$$

$$\dot{z}' = \beta y' , \quad (1f)$$

where the overdot means d/dt. Here, x, y, z, x', y', and z' correspond to the rescaled form of the voltage across C_1, the voltage across C_2, the current through L, the voltage across C_1', the voltage across C_2', and the current through the inductor L', respectively. α, ν, and β are the rescaled circuit parameters. Here, $\varepsilon(=R'/R_c)$ is the coupling parameter. A numerical simulation of Eqs. (1a)–(1c) with the fixed values of ν and α exhibits period-

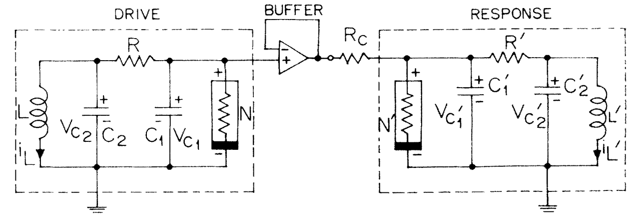

FIG. 1. Circuit realization of two identical van der Pol–Duffing oscillators with one-way coupling.

doubling bifurcations leading to chaos as the parameter β is decreased from a large value [7]. If the parameters are fixed as $\alpha=0.35$, $\nu=100$, and $\beta=300$, one observes a double-band chaotic attractor as shown in Fig. 2(a). Now let us denote the difference between the unprimed and primed quantities by "starring" them. Choosing a specific coupling $\varepsilon=(1+\alpha)$ and by assigning $(x-x')=x^*$, $(y-y')=y^*$, and $(z-z')=z^*$, the differential equations for the difference system of Eq. (1) are given as

$$\dot{x}^* = -\nu[(x^3-(x')^3)-\alpha x^*-y^*]-\nu\varepsilon x^* ,$$

$$= -\nu[(x^2+xx'+(x')^2)x^*-\alpha x^*-y^*]-\nu\varepsilon x^* ,$$

$$=[-a\nu x^*-\nu x^*+\nu y^*] , \quad (2a)$$

$$\dot{y}^* = x^*-y^*-z^* , \quad (2b)$$

$$\dot{z}^* = \beta y^* , \quad (2c)$$

where $a=(x^2+xx'+(x')^2)\geq 0$ and $\varepsilon=(1+\alpha)$.

Recently, a criterion based on the asymptotic stability has been developed as a necessary and sufficient condition for the synchronization of periodic and chaotic systems [2,8]. One of the practical ways to establish this asymptotic stability is to find an appropriate Lyapunov function [8]. Now by considering the Lyapunov function of the form $E=(\beta/2)x^{*2}+(\nu\beta/2)y^{*2}+(\nu/2)z^{*2}$, then $\dot{E}=-\nu\beta a x^{*2}-\nu\beta(x^*-y^*)^2\leq 0$ $(\beta,\nu>0,\ a\geq 0)$.

If $\dot{E}$ is to vanish identically for $t>t_1$, then x^* and y^* must be zero for all $t>t_1$. From (2b) this requires that $\dot{y}^*=0$ for all $t>t_1$ and so z^* must also be equal to zero [9] for all $t>t_1$. Thus $\dot{E}$ vanishes identically only at the origin. Therefore, Eqs. (1d)–(1f) are globally asymptotically stable for the specific value $\varepsilon=(1+\alpha)$. Thus both the drive [Eqs. (1a)–(1c)] and response [Eqs. (1d)–(1f)] systems eventually synchronize for all $t>t_1$ for the above choice of coupling parameter. A synchronized chaotic behavior of Eq. (1) between x and x' variables for $\varepsilon=1+\alpha$, $\alpha=0.35$, $\nu=100$, and $\beta=300$ is shown in Fig. 2(b). The initial values are fixed as $x(t=0)=x(0)=0.1$, $y(0)=0.1$, $z(0)=0.2$, $x'(0)=0.15$, $y'(0)=0.2$, and $z'(0)=0.3$. Furthermore, our detailed numerical simulation results seem to show that the system (1) eventually synchronizes for all values of $\varepsilon>0.8$. Presently, the

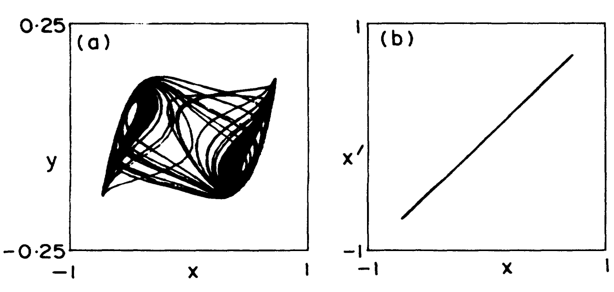

FIG. 2. (a) Chaotic attractor in the $(x\text{-}y)$ plane for $\alpha=0.35$, $\nu=100$, $\beta=300$. (b) Synchronization of chaos in the $(x\text{-}x')$ plane for $\varepsilon=1+\alpha$.

Lyapunov function E for the specific value of $\varepsilon=(1+\alpha)$ alone is given. However, it should be possible to choose an appropriate Lyapunov function E for other sets of ε values, also a question that we have not pursued further.

Interestingly, due to the simplicity of this method of chaos synchronization, namely, two identical nonlinear systems with one-way coupling, it can be easily implemented with minimal effort. In fact we have tested the applicability of the method numerically for a wide variety of third-order nonlinear systems including the familiar Chua's circuit, Fitzhugh-Nagumo oscillator, modified van der Pol oscillator, etc.

In the following, we focus on the use of a synchronizing chaotic signal, in which the above synchronized set of chaotic van der Pol–Duffing oscillators can be effectively utilized as a vehicle to transmit analog signals in the context of secure communications. A method of transmitting signals in a secure way through a cascaded chaos synchronization approach has been recently reported [10–13]. For the subsequent numerical analysis, we use the $x(t)$ signal of the drive system [Eqs. (1a)–(1c)] as a noiselike "masking signal" and $s(t)$ as an information signal to be transmitted in a potentially secure way. Now let us consider the actual transmitted signal $r(t)=x(t)+s(t)$. Then the response system [Eqs. (1d)–(1f)] is now modified as

$$\dot{x}' = -\nu[(x')^3-\alpha(x')-y']+\nu\varepsilon(r(t)-(x')) , \quad (1\text{d}')$$

$$\dot{y}' = x'-y'-z' , \quad (1\text{e}')$$

$$\dot{z}' = \beta y' . \quad (1\text{f}')$$

By assuming that the power level of the information bearing signal $s(t)$ is significantly lower than that of the $x(t)$ signal and the solution $x^*(t)=[x'(t)-x(t)]$ is significantly small with respect to $s(t)$, we see that $s(t)$ can be recovered from the response system as [10–13]

$$s(t)=r(t)-x'(t)=x(t)+s(t)-x'(t)\approx s^1(t) . \quad (3)$$

We have numerically solved the system of equations (1a)–(1c) and (1d′)–(1f′) simultaneously with parameters $\alpha=0.35$, $\nu=100$, $\beta=300$, and $\varepsilon=1.0$. The information bearing signal is assumed to be of a *single-tone, amplitude-modulated wave, or phase-modulated wave.* For example, with the form $s(t)=F\sin(\omega t)$ (single tone, $F=0.02$, $\omega=1.0$), the information signal $s^1(t)$ is recovered at the response system by adopting Eq. (3). Figure 3 depicts the power spectra of the signals $s(t)$, the actual transmitted signal $r(t)$ $(=s(t)+x(t))$, and the recovered signal $s^1(t)$. As noted in Ref. [12], the component of signal frequency $s(t)$ is not discernible or detectable in Fig. 3(b) because of the chaotic (broadband) nature of the actual transmitted signal $r(t)$. For the other signals we have also performed a similar analysis, and the detailed results will be reported separately. In view of the typical broadband spectra, the chaotic signal $x(t)$ becomes an ideal candidate for spread-spectrum communication applications [10–12].

The applicability of this method of chaos synchronization is not restricted to third-order autonomous systems alone, but can be equally well used for second-order nonautonomous nonlinear systems as well. To illustrate this, we consider here the familiar Duffing oscillator. The chaotic dynamics of this system has been extensively investigated [14]. Following the above method, we consider the drive and response systems, for the Duffing oscillator as

drive:

$$\dot{x} = y \ , \tag{4a}$$

$$\dot{y} = -py - x^3 + F\cos(\omega t) \ ; \tag{4b}$$

response:

$$\dot{x}' = y' + \varepsilon(x - x') \ , \tag{4c}$$

$$\dot{y}' = -py' - (x')^3 + F\cos(\omega t) \ , \tag{4d}$$

where p, F, and ω are parameters. ε is the one-way coupling parameter. When $\varepsilon = 0$, Eq. (4) represents two independent identical Duffing oscillators. When $\varepsilon > 0$ Eqs. (4a) and (4b) act as the drive system and Eqs. (4c) and (4d) act as the response system. Figure (4a) shows the chaotic attractor for $p = 0.05$, $F = 7.5$, $\omega = 1.0$, and $\varepsilon = 1.0$. The initial conditions are fixed as $x(0) = 0.2$, $y(0) = 0.1$, $x'(0) = 0.1$, and $y'(0) = 0.2$. Figure 4(b) shows the synchronization of chaotic behavior that exists between x and x' for $\varepsilon = 1.0$. By following the approach of transmitting analog signals as in the case of the van der Pol–Duffing (VPD) oscillator, we can utilize the Duffing oscillator system (4) also to transmit analog signals. Then the response system Eqs. (4c) and (4d) is now modified as

$$\dot{x}' = y' + \varepsilon(r(t) - x') \ , \tag{4c'}$$

$$\dot{y}' = -py' - (x')^3 + F\cos(\omega t) \ . \tag{4d'}$$

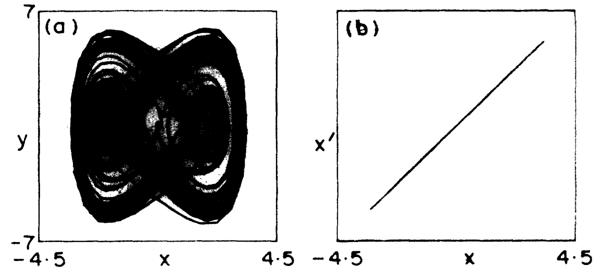

FIG. 4. (a) Chaotic attractor in $(x\text{-}y)$ plane for $p = 0.05$, $F = 7.5$, and $\omega = 1.0$. (b) Synchronization of chaos in the $(x\text{-}x')$ plane for $\varepsilon = 1.0$.

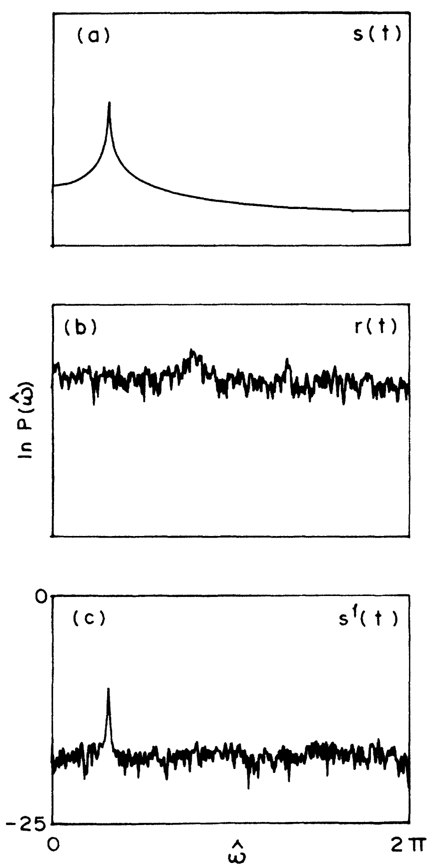

FIG. 3. Power spectra of the signals (a) $s(t) = F\sin(\omega t)$ ($F = 0.02$, $\omega = 1.0$, $\varepsilon = 1.0$); (b) $r(t)$; (c) $s^1(t)$.

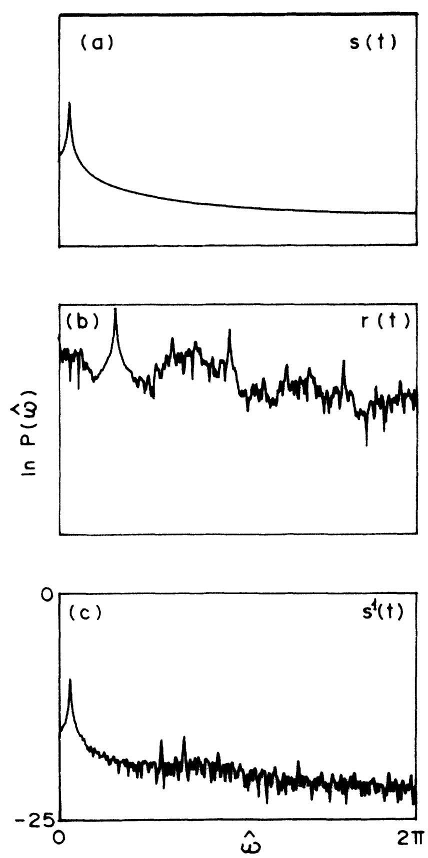

FIG. 5. Power spectra of the signals (a) $s(t) = F\sin(\Omega t)$ ($F = 0.02$, $\Omega = 0.2$, $\varepsilon = 1.0$); (b) $r(t)$; (c) $s^1(t)$.

Here the $x(t)$ signal from Eqs. (4a) and (4b) acts as the masking signal and $r(t)=x(t)+s(t)$. By assuming that the information signal of the form $s(t)=f\sin(\Omega t)$ (single tone, $f=0.02$, $\Omega=0.2$), Fig. 5 shows the power spectra of the information signal $s(t)$, the actual transmitted signal $r(t)$, and the recovered signal $s^1(t)$. Also, for the experimental feasibility of investigating the secure transmission of analog signals, a suitable analog simulation circuit for the Duffing oscillator can be employed. We have tested this method of chaos synchronization in a wide variety of second-order nonautonomous nonlinear systems also including the forced *LCR* circuit equations, van der Pol oscillator, second-order nonautonomous VPD oscillator.

In summary, we have discussed an alternative way of investigating chaos synchronization in identical van der Pol–Duffing and Duffing oscillators with a one-way coupling element. A criterion for synchronization of chaos based on the asymptotic stability analysis for the case of the van der Pol–Duffing oscillator has been discussed. In addition, we demonstrated the possibility of sending signals in a secure way with this kind of identical chaotic system with a one-way coupling element.

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