Lévy statistics in a Hamiltonian system

J. Klafter[1] and G. Zumofen[2]
[1]*School of Chemistry, Tel-Aviv University, Tel-Aviv 69978, Israel*
[2]*Laboratorium für Physikalische Chemie, Eidgenössische Technische Hochschule Zentrum, CH-8092 Zürich, Switzerland*
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Enhanced diffusion in a Hamiltonian system is studied in terms of the continuous-time random walk formulation for Lévy walks. The previous Lévy-walk scheme is extended (i) to include interruptions by periods of temporal localization and (ii) to describe motion in two dimensions. We analyze a case of conservative motion in a two-dimensional periodic potential. Numerical calculations of the mean-squared displacements and the propagators for intermediate energies are consistent with the Lévy-walk description.

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Transport in Hamiltonian systems has been of much interest as it is one of the fundamental processes in many nonlinear conservative systems, especially since it has been noticed that trajectories in such systems do not necessarily follow a simple Gaussian behavior, but may rather obey different statistical rules [1–5]. One of the problems which has been actively studied is motion due to a periodic two-dimensional potential, mainly because it can model, among other relaxations, atom diffusion on a surface in the classical limit [6,7]. In the latter case both molecular dynamics calculations and experiments have pointed towards the existence of long flights, or correlated walks. Although various models have been proposed to explain these correlations, the nature of this diffusional process is still not completely understood [8].

Trajectories with long flights on all scales are by now common in many nonlinear systems and have been shown to lead to anomalous diffusion [2–5, 9–11]. Examples for anomalous diffusion in dynamical systems cover both dissipative and Hamiltonian systems. One finds diffusion anomalies in numerical studies in one-dimensional maps [5,9], in the Chirikov-Taylor standard map [1,11–14], in stochastic webs [4], and in experiments on tracer diffusion in flow systems [15–17], where enhanced diffusion has been recently directly observed [17]. Such anomalies have been analyzed in terms of a new statistical description based on Lévy stable distributions, which generalizes the central limit theory [18,19].

In this paper we study the classical motion of a particle subject to the potential surface

$$V(x,y)=A+B(\cos x+\cos y)+C\cos x\cos y\ , \qquad (1)$$

where the third term is responsible for the nonintegrability of the corresponding Hamiltonian. This problem has been studied by Geisel, Zacherl, and Radons (GZR) [3] and by Chernikov *et al.* [4] and has been shown to exhibit both regular and enhanced diffusion, namely, mean-squared displacements which grow as $\langle r^2(t)\rangle \sim t^\alpha$, $1\le\alpha<2$. Here we show that this problem also belongs to the class of Lévy walks.

We begin with the equations of motion of the particle:

$$\begin{aligned}\partial_t x=p_x\ ,\quad \partial_t p_x=(B+C\cos y)\sin x\ ,\\ \partial_t y=p_y\ ,\quad \partial_t p_y=(B+C\cos x)\sin y\ ,\end{aligned} \qquad (2)$$

which we studied numerically. In the numerical calculations, instead of the ordinary differential equations given in Eq. (2), we used second order differential equations and applied a multistep, Störmer type, predictor-corrector formula of order 11 [20]. With this method and with an integration step of $\Delta t=0.01$, the energy was conserved as $|\Delta E/E|\lesssim 10^{-10}$ after 10^{10} integration steps.

Our numerical experiments indicate that the accuracy provided by this method is appropriate for the energy range $3\lesssim E\lesssim 4$; however, for energies $E\lesssim 3$ and $E\gtrsim 4$ the lengths of laminar motion are restricted by the accuracy so that a power law for the mean free paths is observed only on scales of $t\lesssim 10^3$ and a crossover for the mean-square displacement to regular diffusion takes place at longer times. Similar restrictions on the lengths of laminar phases are expected to arise when noise is added to the equations of motion, which is required when more realistic systems have to be modeled. Depending on the strength of the noise, the lengths of the laminar phases are limited and, consequently, regular Brownian motion is expected at long times. In this respect the finite computation accuracy and the noise have the same effect on the motion.

We investigated for different energies E the time dependence of the moments of displacement, the propagator, the probability distribution of the mean free paths, and related quantities. The numerical results demonstrate consistently that there is a range of parameters in the problem for which enhanced diffusion is observed. Here we concentrate on the set of parameters studied previously by GZR [3]: $A=2.5$, $B=1.5$, and $C=0.5$. GZR reported on enhanced diffusion for energies $2\lesssim E\lesssim 4.5$ and unraveled the characteristic island-around-islands structure in the phase space. Furthermore, GZR studied the distribution of the mean free paths and the power spectrum of the time-dependent velocities and proposed an enhanced diffusion behavior characterized by an energy-dependent exponent. Part of these results were also reported by Chernikov *et al.* [4]. Our study recovers these findings on enhanced diffusion and establishes that the Lévy-walk approach is appropriate.

In Fig. 1 we present a typical trajectory obtained from the numerical solution of the equations of motion for the energy $E=4$. We notice the self-similar nature of the trajectory with laminar phases on all scales. The inset supports this impression and also shows that on the scale of a unit cell the trajectory follows a wiggly curve.

Lévy distributions have been introduced in order to describe the enhanced diffusion on the basis of broad distributions of single motion events [5]. These distributions are consistent with the self-similar properties of the trajectories. Lévy distributions $L_\gamma(x)$, however, exhibit diverging moments for $\gamma<2$, a fact which makes the description of anomalous diffusion in terms of stable laws problematic. Therefore, space-time correlations were proposed by, for instance, assuming motion at constant velocity. The latter assumption can be formulated in terms of a δ function for the space-time correlation in the probability distribution of the motion events [21,22].

In our statistical analysis we first discuss the continuous-time random walk (CTRW) framework of the Lévy walk process in one dimension. We then extend the original derivation to include (i) the case when temporary spatial localization takes place intermittently and (ii) the case of motion in two dimensions.

We briefly outline the main ingredients in the Lévy-walk process. We choose the velocity picture in which the particle moves continuously at a constant velocity and changes directions at random [5]. Let $v(t)$ denote a stochastic time-dependent function such that the motion takes place at a constant velocity for some time after which the direction and the length of the next motion event is chosen randomly but at the same velocity. The motion events are uncorrelated and the duration times of the motion events follow a probability distribution $\psi(t)$.

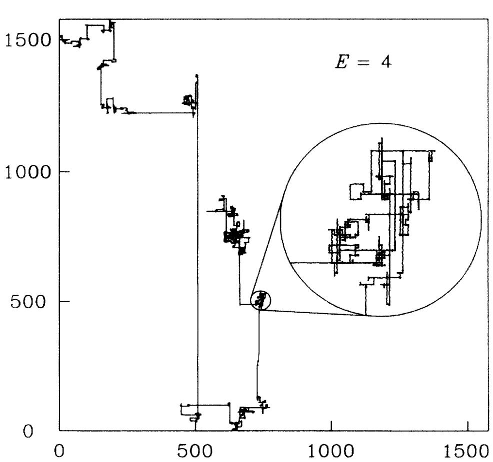

FIG. 1. A typical two-dimensional trajectory $\mathbf{r}(t)$ resulting from the equations of motion for $E=4$ after time $t=10^5$. The inset shows a part of the trajectory on an enlarged scale to strengthen the impression of the self-similarity.

This idealized picture of the motion defines the probabilistic description and has been introduced within the CTRW approach [5,21,22]. To account for the space-time correlation mentioned above, the approach is based on a space-time coupled memory function $\psi(r,t)$, the probability to move a distance r in time t in a single motion event and to stop at r for initiating a new motion event at random:

$$\psi(r,t)=\tfrac{1}{2}\delta(|r|-t)\psi(t) , \tag{3}$$

where length and time are given in dimensionless units. In our analysis we use the power law [3,5,9]

$$\psi(t)\sim t^{-\gamma-1} , \quad \gamma>0 . \tag{4}$$

An equivalent approach to $\psi(r,t)$ would be to rewrite expression (3) in terms of the δ function and the distribution of mean free path $\psi(r)$ [5]. It is the broadness in the distributions $\psi(t)$ and $\psi(r)$ which is responsible for the anomalous behavior.

We further introduce $\Psi(r,t)$, the probability density to move a distance r in time t in a single motion event and not necessarily stop at r:

$$\Psi(r,t)=\tfrac{1}{2}\delta(|r|-t)\int_t^\infty d\tau\,\psi(\tau) . \tag{5}$$

$\psi(r,t)$ and $\Psi(r,t)$ are the relevant quantities for the description of the motion. The motion consists of a sequence of these events and the propagator $P(r,t)$ can be cast in the following way:

$$P(r,t)=\Psi(r,t)+\int_{-\infty}^{\infty}dr'\int_0^t dt'\,\psi(r',t')\Psi(r-r',t-t')+\cdots . \tag{6}$$

The first term denotes the probability to reach location r in time t in a single motion event. The second term is the probability to reach r at time t with one stop and so on to include all combinations of motion events. In the Fourier-Laplace space $(r\to k, t\to u)$, the convolution integrals simplify and the series in Eq. (6) can be given in a closed form as [5]

$$P(k,u)=\Psi(k,u)/[1-\psi(k,u)] . \tag{7}$$

Here and in what follows we make use of the convention that the variables denote in which space (Fourier and/or Laplace), or for which spatial dimension, the function is thought to hold.

Based on the propagator, the calculation of the mean-squared displacement is straightforward:

$$\langle r^2(u)\rangle=-\partial_k^2 P(k,u)|_{k=0} , \tag{8}$$

and depending on γ, ballistic-type, intermediate-enhanced, and regular diffusion are obtained [5]:

$$\langle r^2(t)\rangle\sim\begin{cases} t^2 , & 0<\gamma<1 \\ t^{3-\gamma} , & 1<\gamma<2 \\ t , & 2<\gamma .\end{cases} \tag{9}$$

For the intermediate-enhanced diffusion regime $1<\gamma<2$

relevant to the behavior discussed in this paper, the propagator is well represented by the Lévy stable distribution [5]

$$P(r,t)\sim\begin{cases} t^{-1/\gamma}L_\gamma(\xi)\ , & r<t \\ 0\ , & r>t\ ,\end{cases} \tag{10}$$

where ξ is the scaling variable $\xi=cr/t^{1/\gamma}$.

We now extend this derivation to the case where the motion is interrupted by phases of spatial localization, which means that the particle does not move at a constant velocity at all times but that the phases of laminar motion are intermittently interrupted by periods of no motion on the scale of typically one unit cell. These localization periods result from multiple scattering of the particle within a unit cell. A similar phenomenon has been proposed by Solomon, Weeks, and Swinney for tracer diffusion in a flow system [17]. We denote the probability distribution of the localization times by $\tilde{\psi}(t)$ and consider the power-law behavior

$$\tilde{\psi}(t)\sim t^{-\tilde{\gamma}-1}\ . \tag{11}$$

We also introduce

$$\tilde{\Psi}(t)=\int_t^\infty d\tau\,\tilde{\psi}(\tau)\ , \tag{12}$$

the probability distribution for not having moved until time t. For the description of the propagator we assume that the observation starts with an event of motion at constant velocity and we can thus write

$$P(r,t)=\Psi(r,t)+\int_0^t\psi(r,t')\tilde{\Psi}(t-t')dt' + \int_{-\infty}^{\infty}dr'\int_0^t dt'\int_0^{t'}dt''\psi(r',t'')\tilde{\psi}(t'-t'')\Psi(r-r',t-t')+\cdots\ , \tag{13}$$

where the first term denotes the probability to reach location r in time t in a single motion event. The second term is the probability to reach r at an earlier time and to stay localized until time t. The third term is the probability to reach r in time t in two motion events interrupted by one period of localization. The sum has to be extended over all possible combinations of motion events interrupted by periods of localization. Taking the Fourier-Laplace transform and summing over even and odd terms independently, we obtain

$$P(k,u)=[\Psi(k,u)+\tilde{\Psi}(u)\psi(k,u)]/[1-\psi(k,u)\tilde{\psi}(u)]\ . \tag{14}$$

A similar expression is obtained when the walks are initiated by a localization event followed by motion at constant velocity. Furthermore, for the analysis of iterated maps in terms of CTRW's we have demonstrated that stationary conditions are an important issue. The consideration of stationary conditions is beyond the scope of this paper; details will be given elsewhere [23]. For the derivation of the mean-squared displacement we make use of Eq. (8) which leads to

$$\langle r^2(u)\rangle=-[1-\psi(k,u)\tilde{\psi}(u)]^{-1}[\partial_k^2\Psi(k,u)+\tilde{\Psi}(u)\partial_k^2\psi(k,u)] - [\Psi(k,u)+\tilde{\Psi}(u)\psi(k,u)][1-\psi(k,u)\tilde{\psi}(u)]^{-2}\psi(u)\partial_k^2\psi(k,u)|_{k=0}\ . \tag{15}$$

From this expression we obtain for the dominant term at long times:

$$\langle r^2(t)\rangle\sim\begin{cases} t^{2+\tilde{\gamma}-\gamma}\ , & 1<\gamma<2\ ,\ \tilde{\gamma}<1 \\ t^{3-\gamma}\ , & 1<\gamma<2\ ,\ \tilde{\gamma}>1\ .\end{cases} \tag{16}$$

Equation (16) indicates that depending on the two exponents γ and $\tilde{\gamma}$, the motion shows enhanced, regular, or dispersive behavior. It should be noted that for $\tilde{\gamma}>1$, for which $\tilde{\psi}(t)$ in Eq. (11) has a finite first moment, the enhancement reduces to the result obtained by the original Lévy-walk scheme. A different time dependence for the combined laminar-localized case has been proposed by Zaslavsky *et al.* [24].

We now present a schematic derivation of the propagator for motion in two dimensions. According to the structure of the potential, Eq. (1), we assume that the motion takes place exclusively parallel to the axes. Again we consider motion at a constant velocity and we assume that the previously discussed periods of localization are absent which would be justified for example for values $\tilde{\gamma}>1$. For this motion we formulate the probability distribution for a single motion even analogously to Eq. (3) as

$$\psi(\mathbf{r},t)\sim\tfrac{1}{4}[\delta(y)\delta(|x|-t)+\delta(x)\delta(|y|-t)]\psi(t)\ , \tag{17}$$

where $\mathbf{r}=(x,y)$. In Fourier-Laplace space this expression is cast as

$$\psi(\mathbf{k},u)=\tfrac{1}{2}\psi(k_x,u)+\tfrac{1}{2}\psi(k_y,u)\ , \tag{18}$$

and the probability distribution in analogy to Eq. (4) is

$$\Psi(\mathbf{k},u)=\tfrac{1}{2}\Psi(k_x,u)+\tfrac{1}{2}\Psi(k_y,u)\ . \tag{19}$$

For the small $(\mathbf{k},u)$ expansion we obtain for the intermediate-enhanced diffusion case the asymptotic behavior

$$P(\mathbf{r},t)\simeq P(x,t/2)P(y,t/2)\ ,\quad 1<\gamma<2 \tag{20}$$

where $P(x,t)$ and $P(y,t)$ correspond to the one-dimensional problem given in Eq. (10). Thus we obtain

$$P(\mathbf{r},t)\sim\begin{cases} t^{-2/\gamma}L_\gamma(\xi)L_\gamma(\eta)\ , & |x|+|y|<t \\ 0\ , & |x|+|y|>t\ ,\end{cases} \tag{21}$$

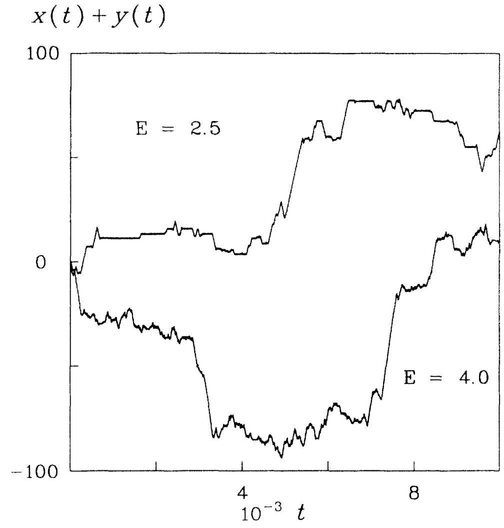

FIG. 2. Typical realizations of the one-dimensional trajectory $x(t)+y(t)$ for energies as indicated.

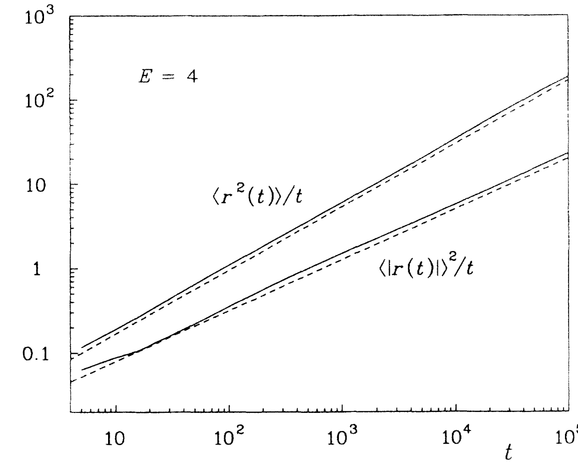

FIG. 4. The time evolution of the moments of the displacement calculated from a trajectory of length $t=10^8$ for the energy $E=4$. The ratios of the mean-squared displacement and the square of the mean absolute value over time t are given by solid lines. For $\gamma=1.25$ the upper dashed line indicates the slope of $2-\gamma=0.75$, and the lower dashed line, the slope of $(2/\gamma)-1=0.6$, respectively.

where ξ,η are the two scaling variables $\xi=cx/t^{1/\gamma}$, $\eta=cy/t^{1/\gamma}$. From the symmetry in Eq. (21) it is quite natural to consider for the purpose of presentation the projection of the walk onto the axes x or y so that the one-dimensional case is recovered. We apply

$$P(r,t)=\tfrac{1}{2}\delta(r-y)\int P(\mathbf{r},t)dx+\tfrac{1}{2}\delta(r-x)\int P(\mathbf{r},t)dy \sim t^{-1/\gamma}L_\gamma(\xi)\ ,\quad r<t\ ,\ 1<\gamma<2 \tag{22}$$

where the scaling variable is $\xi=cr/t^{1/\gamma}$, as in Eq. (10).

As a consequence of the fact that the motion takes place exclusively along the x or y axes, we may represent the two-dimensional trajectories $\mathbf{r}(t)$ as a one-dimensional trajectory $x(t)+y(t)$ as a function of time. In Fig. 2 the trajectory $x(t)+y(t)$ is displayed for energies $E=2.5$ and 4. For $E=2.5$ we notice that phases of laminar motion are interrupted by periods of localization. For $E=4$ these periods ar hardly visible and a pattern is observed which is very similar to the one found for one-dimensional iterated maps [5].

In Fig. 3 the distribution $\psi(r)$ of mean free paths and the probability distribution $\psi(t)$ of laminar phases are presented. At longer times the two probabilities follow approximately the same power law. This indicates that the δ-function correlation introduced in Eq. (3) is ap-

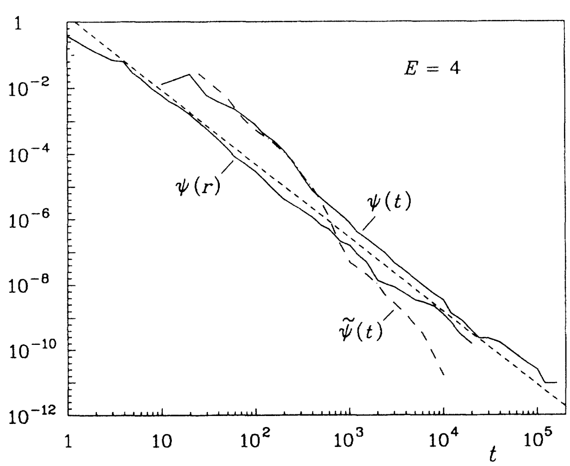

FIG. 3. Relevant distributions of individual motion events calculated from a trajectory of length $t=10^8$ for the energy $E=4$. Plotted are the distribution $\psi(t)$ of times of laminar motion and the distribution $\psi(r)$ of mean free path as solid lines. The distribution $\tilde{\psi}(t)$ of times of localization is given by the dot-dashed line. The dashed line indicates the slope of -2.25 corresponding to the exponent $\gamma=1.25$.

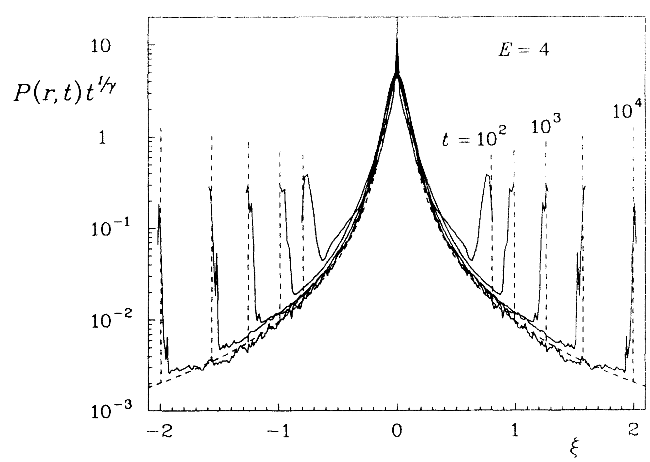

FIG. 5. The propagator $P(r,t)$ calculated from a trajectory of length $t=10^8$ for the energy $E=4$ according to Eq. (22). Plotted as solid lines are $P(r,t)$ in the scaling representation for times t as indicated and for the scaling variable $\xi=r/t^{1/\gamma}$ with $\gamma=1.25$. The bell shaped dashed line is the stable law $cL_\gamma(cx)$ with c adjusted to the experimental results. The vertical dashed lines at $r=\pm vt$ denote the probabilities of motion at constant velocity v in a single laminar phase.

propriate. The exponent $\gamma = 1.25$ was obtained, a value found previously by GZR [3]. Shown in this figure is also $\tilde{\psi}(t)$, the distribution of localization times. For $E = 4$ the distribution $\tilde{\psi}(t)$ drops quickly for times $t > 10^2$. With decreasing energy E the distribution $\tilde{\psi}(t)$ gets broader which is consistent with the energy-dependent periods of no motion in Fig. 2. However, for longer times the exponent $\tilde{\gamma}$ was always larger than 1, $\tilde{\gamma} > 1$.

In Fig. 4 the time evolution of the displacements is shown. Plotted are the absolute value as $\langle |r(t)| \rangle^2/t$ and the mean-squared displacement as $\langle r^2(t) \rangle /t$ for $E = 4$. The denominator t was chosen to strengthen the impression of the enhancement in the diffusion relative to the simple Brownian motion. We notice a small but significant difference between the behaviors of the two quantities. The predicted power-law behaviors are $\langle r^2(t) \rangle \sim t^{3-\gamma}$, according to Eq. (16),and $\langle |r(t)| \rangle^2 \sim t^{2/\gamma}$, according to the scaling property of the underlying Lévy process [24,25]. With the exponent $\gamma = 1.25$, both quantities follow reasonably the assumed power-law behavior.

Finally, in Fig. 5 we give the results for the propagator corresponding to Eq. (19) in the scaling representation. For $\gamma = 1.25$ the scaling is obeyed on progressively larger scales with increasing time. Furthermore, the calculated $P(r,t)$ follows reasonably the stable law. The peaks at the center and at the outermost wings result primarily from stationary condition effects, which were analyzed in detail for one-dimensional maps [5,26]. The theoretical locations and probabilities of the peaks in the wings were obtained from the analysis in Ref. [26] and from the step units used in the histograms.

To summarize, our numerical analyses indicate that the Lévy-walk model based on motion at a constant velocity is very appropriate and that the power law assumed for the probability distributions characterizing single motion events works reasonably well. The corresponding exponent γ was observed to vary only in a small range $1 < \gamma \lesssim 1.3$ for energies $2.5 \le E \le 4.5$. We conclude that the Lévy-walk approach provides a promising method for the description of anomalous transport behavior in dynamical systems.

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