Cancellation exponents and fractal scaling

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We discuss a relationship between cancellation exponents [E. Ott *et al.*, Phys. Rev. Lett. **69**, 2654 (1992); Y. Du and E. Ott, Physica D **67**, 387 (1993)] and the classical Hölder exponents [J. Feder, *Fractals* (Plenum, New York, 1988)] for fractal scaling. We discuss cancellation exponents in deterministic and stochastic settings and present two examples, that of Brownian motion and that of velocity data from fully developed turbulence [K. R. Sreenivasan (experimental data)].

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INTRODUCTION

Recently Ott, Du, and others [1–4] introduced a cancellation exponent to quantitatively characterize properties of fields that vary in sign on small scales. The exponent has proved to be relevant in problems in magnetohydrodynamics (MHD) and turbulence.

In this paper we show that in one dimension the Hölder exponent of a given signal is related to the cancellation exponent of its *derivative* by a simple formula. We discuss the mathematical validity of the cancellation exponents in several examples. We introduce a stochastic setting not considered in previous discussions of cancellation exponents and present two examples, Brownian motion and turbulence.

I. HÖLDER EXPONENTS

Hölder exponents (or Lipschitz-Hölder exponents) arise in many contexts ranging from classical analysis [5] to scaling properties of observable phenomena in nature [6,7].

The classical analysis definition is one of a bound, used to define a modulus of continuity. A function $f(t)$ is Hölder continuous with exponent α if there exists a constant C such that for all t and r,

$$|f(t) - f(t+r)| \leq C|r|^{\alpha}. \tag{1}$$

Any function f satisfying this rule will automatically be continuous, with modulus of continuity determined by the right hand side of (1).

Such a definition determines the most singular behavior of a given signal but does not determine more general scaling properties. For example, the function $f(x) = |x|^{\beta}$ with $0 < \beta < 1$ is smooth everywhere except at the origin. Because of the singularity there it is forced to have a Hölder exponent no larger than β.

A modification of (1), in the context of self-similar scaling, defines a quantity that has relevance to many observable phenomena in nature [6,7]. Also called by the name Hölder exponent (or singularity strength), the exponent α associated with the process $X(t)$ satisfies the scaling law

$$|X(t+r) - X(t)| \sim |r|^{\alpha}. \tag{2}$$

To see if a given signal scales with a particular exponent α, all one has to do is to take many samples of $|X(t+r) - X(t)|$ over many orders of magnitude of $|r|$ and see if one can fit a straight line to the data

$$\ln|X(t+r) - X(t)| \quad \text{vs} \quad \ln|r| \ .$$

The slope of the line determines α. This Hölder exponent, as opposed to definition (1), is particularly relevant to cancellation exponents.

It may be that a particular signal scales with many different exponents α on corresponding different sets of fractal dimension $f(\alpha)$. In this case the signal is called *multifractal* [7].

A similar scaling definition [8] is relevant for stochastic processes. The random variable X has probabilistic Hölder exponent h if the variance of increments $V(t-r)$ [6] scales like $|r|^{2h}$. That is, h satisfies

$$E(|X(t) - X(t+r)|^2) \sim (r)^{2h}. \tag{3}$$

Here E denotes expected value. In particular, the most basic stochastic process, Brownian motion, has a probabilistic Hölder exponent h of 1/2. Also, fractional Brownian motion, $B_H(t)$ has exponent H, by definition [9]. We define a more general Hölder exponent h_q [10,11] by

$$E(|X(t+r) - X(t)|^q) \sim |r|^{qh_q}. \tag{4}$$

The fractal Brownian motions all have trivial h_q, in that

$h_q = H$ for all q. If a given signal shows h_q to vary as a function of q then it possesses nontrivial multifractality.

II. CANCELLATION EXPONENTS

While Hölder exponents are associated with continuous processes, the cancellation exponents introduced by Ott, Du, and others are associated with violently discontinuous processes [12]. A typical example is the magnetic field induced by a turbulent velocity field in an electrically conducting medium.

They introduced the cancellation exponent in order to measure the small scale cancellations of the magnetic field lines. The definition, introduced by Du, Tél, and Ott recently in [3], is the following. Given a "signal" $Y(t)$,

$$\kappa_q = \limsup_{\epsilon \to 0} \frac{\ln \chi(\epsilon)}{\ln(1/\epsilon)} , \tag{5}$$

where

$$\chi(\epsilon) = \sum_{I_j} \left| \int_{I_j} Y(t)dt \right|^q \tag{6}$$

and $\{I_j\}$ represents a partition of the domain into boxes of length ϵ. The first notion of a cancellation exponent introduced in [4] has $q = 1$.

We point out that in order for (5) to have a nontrivial limit the sum in (6) must become infinite as $\epsilon \to 0$. For $q = 1$, this implies that $\int |Y(t)|dt$ is infinite. If one views Y as a "signed singular measure" Y does not have bounded total variation, a necessary condition for a signed measure to be σ additive [5]. In particular, the integral $\int Y(t)dt$ may not be well defined. A simple example comes from the paper of Du, Tél, and Ott [3]. Define $\rho(x) = \sum_{n=1}^{\infty} f_n \delta(x - x_n)$ where $f_n = (-1)^n/n$, $x_n = 1/n$. One then obtains

$$\int_0^1 \rho(x)dx = \sum_{n=1}^{\infty} (-1)^n \Big/ n, \tag{7}$$

which if summed in this order produces a bounded result: $-\ln 2$. On the other hand, it is a well known fact that the alternating sum (7) can be summed in a different order to produce a different limit.

In previous computations of cancellation exponents, the ambiguity discussed above is overcome in two ways. (a) For the case of a physically observable quantity like a magnetic field, $B(x)$, there is always a small diffusive length scale present below which $B(x)$ is completely smooth. Hence one interprets (5) and (6) as a scaling relationship over a range of ϵ much the same as one interprets the Hölder exponent in (2) or (3). (b) Another possibility, employed in [3], is to use the natural ordering of the line in one dimension to prescribe an ordering for an alternating sum of the form (7). However, it is unclear how to handle such a situation in higher dimensions. For the purpose of the examples discussed below, we interpret the integral $\int_{I_j} Y(t)dt$ in the same sense as (a) above, that is, via a cutoff length scale.

We now show a simple direct link between the Hölder exponent of a signal and the cancellation exponent of its derivative. We consider the one-dimensional (1D) case. Let $X(t)$ be a signal with Hölder exponent α and let $X'(t)$ denote it's derivative dX/dt. Then κ_1 satisfies

$$\begin{aligned} \epsilon^{-\kappa_1} &= \chi(\epsilon) \\ &= \sum_{I_j} \left| \int_{I_j} X'(t)dt \right| \\ &= \sum_{I_j} |X(t_j + \epsilon) - X(t_j)| \\ &\sim \sum_{I_j} \epsilon^{\alpha} = \epsilon^{\alpha - 1}. \end{aligned}$$

Hence

$$k_1 = 1 - \alpha.$$

In a stochastic setting, define

$$\chi(\epsilon) = E\left(\sum_{I_j} \left| \int_{I_j} Y(t)dt \right| dt \right) = \sum_i E\left(\left| \int_{I_j} Y(t)dt \right|^q \right) .$$

Such a definition is intended for signals $Y(t)$ that are singular enough to suffer the same ambiguity as discussed above. A similar derivation for the probabilistic Hölder exponent h then gives

$$2h = 1 - \kappa_2. \tag{8}$$

Using the definition of Hölder exponent h_q from (4) we obtain

$$qh_q = 1 - \kappa_q. \tag{9}$$

III. EXAMPLES

A. Brownian motion

For the first example we choose $W(t)$ to be the standard Wiener process $W(t)$ (Brownian motion). The expected value of $|W(t) - W(t+r)|^{2q}$ scales as r^q for all values of q due to the simple scaling properties of the probability density function for $W(t)$ [13]. Hence the probabilistic Hölder exponent h_q of the signal is $h_q = 1/2$ for all q. Equation (8) then predicts that the *velocity* of a particle undergoing a Brownian motion has a 1D cancellation exponent of $\kappa_q = 1 - q/2$.

The paths of a Brownian motion are nowhere differentiable, hence one cannot compute the derivative of such a function. However, we can approximate Brownian motion as follows. Consider time intervals of length Δt and over each such time interval take a step of size $S(i) = \pm\sqrt{\Delta t}$ with equal probability of $\pm\sqrt{\Delta t}$. Then define inductively $W_\epsilon(t_i + \delta t) = W_\epsilon(t_i) + S(i)\delta t$ where $t_i = (\Delta t)i$, and $\delta t < \Delta t$. The central limit theorem implies that $W_\epsilon(t) \to W(t)$ in the sense of distributions as $\epsilon = \Delta t \to 0$ [13].

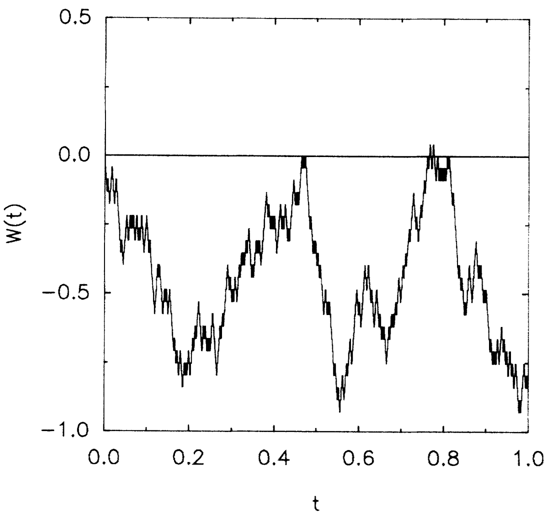

FIG. 1. An approximation to pure Brownian motion. $\Delta t = 1/512$.

$W_\epsilon(t)$ is differentiable almost everywhere. Given a realization of W_ϵ, we can compute the cancellation exponent, κ_1, associated with dW_ϵ/dt. The scaling (5) will be visible up to the cutoff length scale ϵ. We use a simple random number generator to create a candidate W_ϵ. Figure 1 shows a candidate $W_\epsilon(t)$. Figure 2 shows $\ln[\chi(r)]$ vs $-\ln r$ where $\chi(r) = \sum_i |\int_{I_i} (dW(t)/dt)dt|$, and $\{I_i\}$ is a partition into boxes of size r. The straight line has slope 0.47.

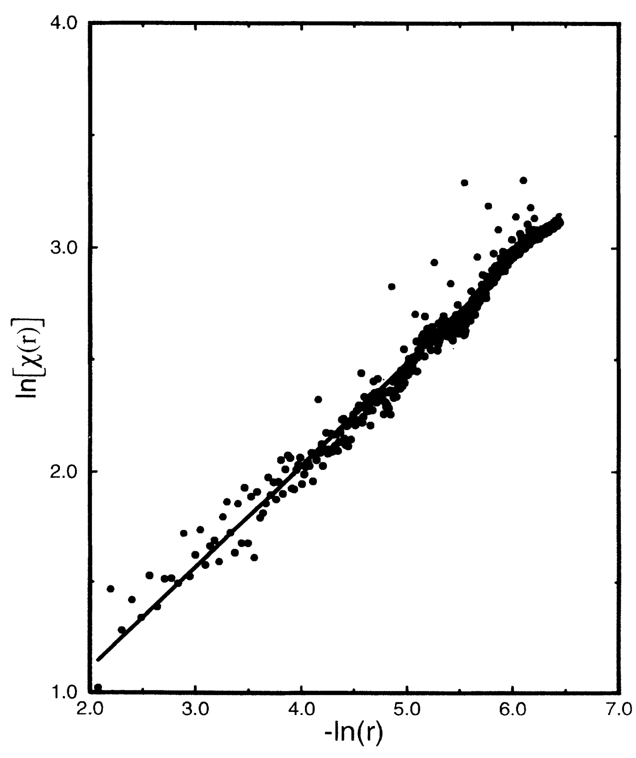

FIG. 2. The cancellation exponent κ_1 for the derivative of Brownian motion.

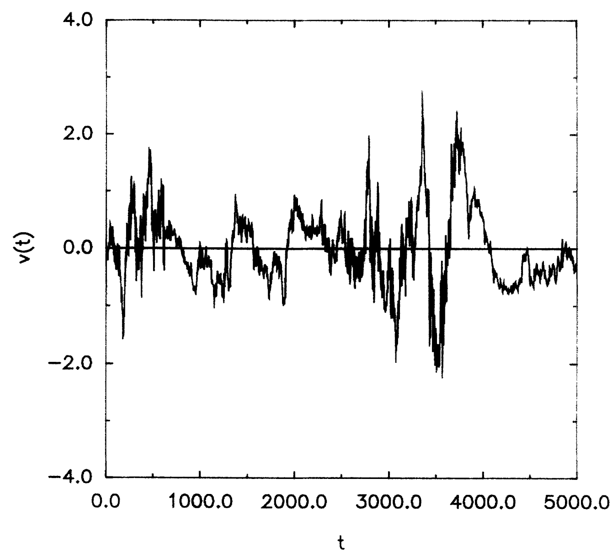

FIG. 3. A sample of velocity data from fully developed turbulence.

B. Turbulence

The second example is a velocity signal (fixed point in space with variation in time) $v(t)$ from experimental data from fully developed turbulence [14]. The data comes from a "single point" hot wire experiment with a sampling rate sufficient to resolve Kolmogorov scales in the

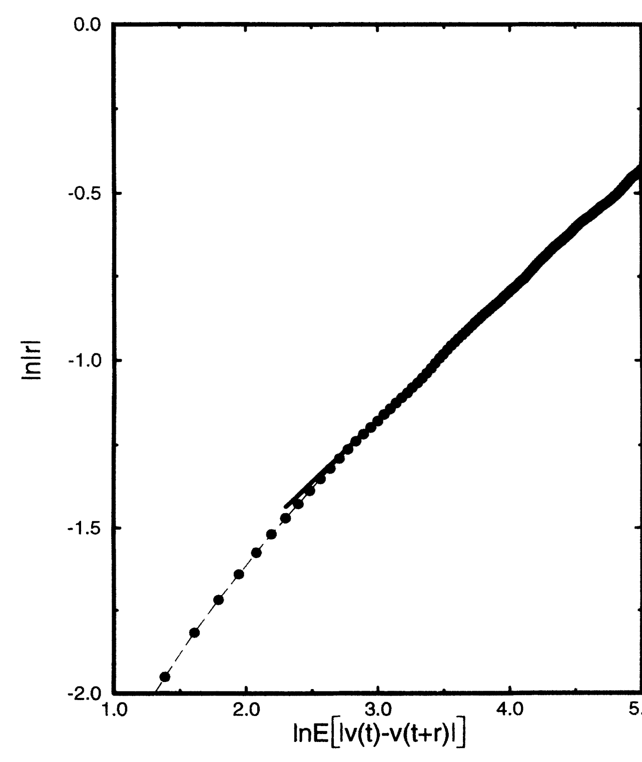

FIG. 4. The Hölder exponent h_1 for fully developed turbulence.

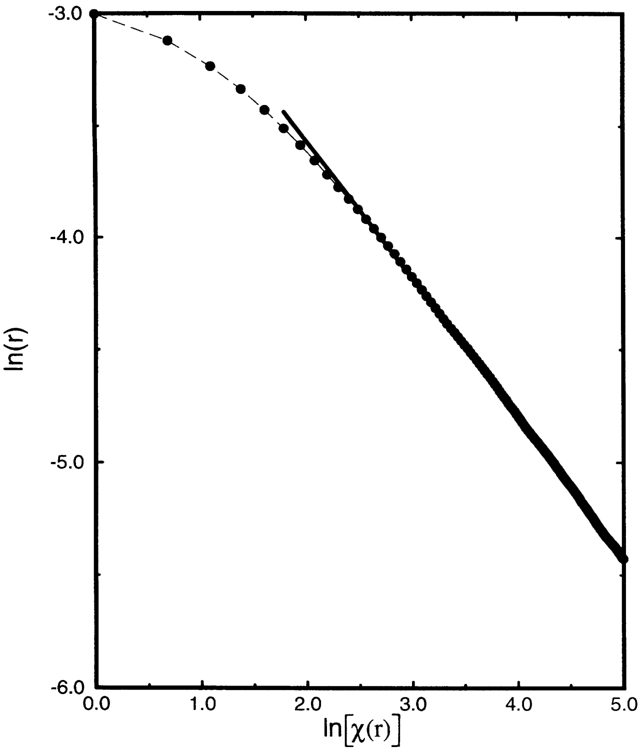

FIG. 5. The cancellation exponent κ_1 for the same data used to compute h_1.

flow. The details of the experiment are discussed in [4]. A sample of this data is depicted in Fig. 3. This example was considered in [4]. In particular, they noted that the cancellation exponent κ_1 (computed from a 1D integral) for the derivative of the velocity signal was approximately 0.6. This is consistent with a Hölder exponent (first order structure function) larger than 1/3 for the velocity field in fully developed turbulence [15].

To check this relationship directly, we compute both h_1 and κ_1 for a sample of the velocity data from [14]. Figure 4 shows the plot of $\ln E(|v(t) - v(t+r)|)$ vs $\ln |r|$ for the velocity field. Figure 5 shows a plot of $\ln[\chi(r)]$ vs $\ln r$ for the velocity field. The slopes yield, respectively, $\kappa_1 = 0.65$ and $h_1 = 0.375$. The discrepancy is due to variation in the choice of data points used to compute the slope. This data is well known to possess multifractal behavior [8] with different values for h_q as q varies. The relationship (9) between h_q and κ_q provides a method of deriving κ_q from h_q and vice versa.

CONCLUSION

In conclusion, we show that in one dimension, the cancellation exponents can be computed from quantities (specifically Hölder exponents) associated with an integral of the chosen field. This fact should be useful for those wishing to calculate scaling properties associated with cancellation or multifractality.

Our last remark concerns a fully three-dimensional quantity $Y(x)$ such as the vorticity field in turbulence or the magnetic field in MHD. In this case, a 1D cancellation exponent, computed by taking data samples along a given line is, by the above arguments, related to a Hölder exponent of a quantity whose directional derivative along that line is the same field $Y(x)$. Such a quantity has no physical interpretation that we know of. It would be interesting to understand how cancellation exponents associated with quantities like vorticity and magnetic field are related to scaling properties of other physical quantities such as the velocity field.

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[12] That is, a Hölder exponent can only be computed for a continuous function [for example, see Eq. (1)] while a nontrivial cancellation exponent will only result from a signal that is even more singular than a measure. See in particular the discussion of total bounded variation.
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[15] The relationship between the first order structure function and the cancellation exponent is noted in footnote [16] in [4]. See also the following related work done independently of this manuscript. Samuel I. Vainshtein, Yunson Du, and K R. Sreenivasan (unpublished).