Self-avoiding random walks on a family of diamond-type hierarchical lattices

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We have used the exact renormalization-group method proposed by Dhar to study self-avoiding random walks (SAW's) on a family of hierarchical lattices. The generator of the lattices is made of l branches and each branch has m bonds. We expect that since the lattices are infinitely ramified, the critical exponents of SAW's should be different from that on finitely ramified lattices and belong to a new universal class. We calculated the critical exponents α, ν, and γ under the condition $l<m$, and, with D_f the fractal dimension, obtained the scaling law $D_f\nu=2-\alpha$, which agrees with other authors's conclusions. When $l\geq m$, we cannot work out the problem, and some discussion is given.

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The application of fractals in physics was first proposed by Mandelbrot [1]. Since fractal geometry is specially characterized by self-similarity, while Euclidean space is characterized by translational invariance, it has attracted great attention. Almost all statistical physics problems have been studied again on fractals [2–4], and one of them is self-avoiding random walks (SAW's).

SAW's were originally proposed as a model of polymer chains [5], and its connection with other models was found later [6,7]. SAW's on fractals have been studied by several authors [4,8–12]. Their research shows that SAW's on fractals are much different from those on translationally invariant lattices. It is certain that critical exponents of SAW's depend on the geometry properties of their support. On translationally invariant lattices, the Flory formula [13] describes the relation of the critical exponent ν and geometry parameters, while on fractals there may not exist a general formula. People have discussed the influences of fractal dimension, spectral dimension, and other geometry parameters on the SAW critical exponents [4,8–12]. However, we have not yet seen work on the influence of ramification.

In this paper we discussed SAW's on a family of diamond-type hierarchical lattices [14], as shown in Fig. 1, which are infinitely ramified fractals. Many physical problems about the lattices [14–17] have been discussed. The fractal generator is determined by two parameter m and l, where l denotes the number of branches and m the number of bonds in each branch. The fractal dimension is determined by the formula

$$D_f=\ln(ml)/\ln(m)\ . \tag{1}$$

In order to study SAW's, three functions are defined:

$$C(x)=\lim_{N\to\infty}(1/N)\sum_{n=1}^{\infty}C_n(N)x^n\ , \tag{2}$$

$$P(x)=\lim_{N\to\infty}(1/N)\sum_{n=2}^{\infty}P_n(N)x^n\ , \tag{3}$$

$$L(x)=\lim_{N\to\infty}(1/N)\sum_{n=1}^{\infty}\langle R_n^2\rangle C_n(N)x^n/C(x)\ , \tag{4}$$

where x is a weight factor associated with each step of the walks, $C_n(N)$ is the number of distinct n-step SAW's on a lattice with N sites, $P_n(N)$ is the number of distinct SAW loops of n steps, and $\langle R_n^2\rangle$ is the mean-square end-to-end distance for n-step SAW's.

For very large n, we get the asymptotic behavior of the above functions as x tends to $1/\mu$ from below [7]:

$$C(x)\approx K_1(1-x\mu)^{-\gamma}+(\text{less singular terms})\ , \tag{5}$$

$$P(x)\approx K_2(1-x\mu)^{2-\alpha}+(\text{less singular terms})\ , \tag{6}$$

$$L(x)\approx K_3(1-x\mu)^{-\nu}+(\text{less singular terms})\ , \tag{7}$$

where α, ν, and γ are critical exponents; μ is a connective constant; and K_1, K_2, and K_3 are constants.

In the case of a hierarchical lattice, Dhar [18] has proposed a method to get the exact solution of SAW problems. The method depends on the character of the hierarchical model. Its structure at stage r is constructed by stage $r-1$ via an iteration procedure, and we can calculate generating functions stage by stage in terms of the

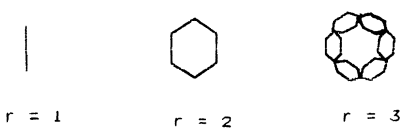

FIG. 1. Growth of the diamond-type hierarchical lattice ($m=3$, $l=2$). The first three stages are shown.

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recursion relation.

In the following, we will use Dhar's method to calculate the critical exponents of SAW's on the above diamond-type lattices. First, we define four restricted partition functions, as shown in Fig. 2, where A_r is the weight of all distinct SAW's starting from one vertex point and terminating at one internal point in the rth stage diamond-type lattice, B_r is the weight of all distinct SAW's starting from one vertex point and terminating at the other vertex point, and D_r and E_r are similarly defined.

Since the fractal is a hierarchical lattice, we can find a set of recursion relations of restricted partition functions between the $(r+1)$th and rth stage diamond-type lattice as follows:

$$A_{r+1}=f_1(B_r)A_r+f_2(B_r)E_r \ , \tag{8}$$

$$B_{r+1}=lB_r^m \ , \tag{9}$$

$$D_{r+1}=f_3(B_r)A_r^2+f_4(B_r)D_r \ , \tag{10}$$

$$E_{r+1}=f_5(B_r)E_r+f_6(B_r)A_r \ , \tag{11}$$

where $f_i(B_r)$ is a function of B_r:

$$f_1(B_r)=l(1+B_r+\cdots+B_r^{m-1}) \ , \tag{12}$$

$$f_2(B_r)=l(1+B_r+\cdots+B_r^{m-2}) \ , \tag{13}$$

$$f_3(B_r)=l(1+2B_r+\cdots+2B_r^{m-2}) +l(l-1)(1+B_r+\cdots+B_r^{m-1})^2 \ , \tag{14}$$

$$f_4(B_r)=mlB_r^{m-1} \ , \tag{15}$$

$$f_5(B_r)=l(l-1)(B_r^m+B_r^{m+1}+\cdots+B_r^{2m-2}) \ , \tag{16}$$

$$f_6(B_r)=l(l-1)(B_r^m+B_r^{m+1}+\cdots+B_r^{2m-1}) \ . \tag{17}$$

In Appendix A we have given an example of the construction of recursion relation (8). The initial values (pertinent to the first stage diamond-type lattice) of these functions are

$$A_1=l(x+x^2+\cdots+x^{m-1}) \ , \tag{18}$$

$$B_1=lx^m \ , \tag{19}$$

$$D_1=l(1+x+\cdots+x^{m-3})x^2 +l(l-1)(x+x^2+\cdots+x^{m-1})^2 \ , \tag{20}$$

$$E_1=l(l-1)(x^{m+1}+x^{m+2}+\cdots+x^{2m-1}) \ . \tag{21}$$

Here we should point out that x is understood to be the weight factor associated with each step of the walks other than each site. Then we can calculate the generating functions stage by stage using the above restricted partition functions.

First we calculate $P(x)$ and solve the critical exponent α. We denote N_r the number of sties in the rth stage diamond-type lattice, and it is easy to obtain:

$$N_r=a(ml)^r+b \ , \tag{22a}$$

where a and b are independent of r and

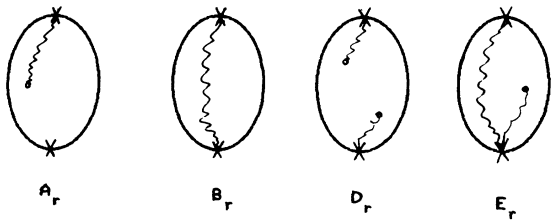

FIG. 2. Four restricted partition functions for an rth stage diamond-type lattice are shown. "×" denotes the vertex point; "○" denotes the ending point.

$$a=\frac{(m-1)l+2}{ml}-\frac{ml+l-2}{ml(ml-1)} \ , \tag{22b}$$

$$b=\frac{ml+l-1}{ml-1} \ . \tag{22c}$$

Then we have the following relation:

$$N=S_rN_r-S_r\frac{ml+l-1}{ml-1} \ , \tag{23}$$

where N is the total number of sites.

We call a closed or open walk L of order r if r is the minimum order such that L can be completely described inside an rth stage diamond-type lattice. Then we find immediately that the sum of the weight of all rth-order SAW loops inside the rth stage diamond-type lattice is $l(l-1)(B_r)^{2m}/2$. We obtain

$$P(x)\approx\sum_{r=2}^{\infty}\frac{l(l-1)(B_{r-1})^{2m}}{2a(ml)^r} \ . \tag{24}$$

Here the contribution of first-order SAW loops and some small constant terms are neglected because the asymptotic form of $P(x)$ is determined by large r terms.

So, from Eq. (24) we can get the connective constant $\mu=1/B^*$ by using the same argument as that of Dhar [18], where B^* is a nontrivial fixed point of Eq. (9) and $B^*=x^*=l^{-1/(m-1)}$.

By using recursion relation (19), Eq. (24) is rewritten as

$$P(x)\approx\frac{l(l-1)(B_1)^{2m}}{2a(ml)^2}+\frac{p(B_1)}{ml} \ . \tag{25}$$

We now turn to the critical exponent α. In order to find α, we should consider that x tends to x^*, i.e., $x=x^*-\delta^{(0)}$, where $\delta^{(0)}$ is a small positive number. Then we get $B_1=B^*-\lambda_1\delta^{(0)}$ under linear approximation, and Eq. (25) becomes

$$P(x^*-\delta^{(0)})=\frac{l(l-1)(x^*-\lambda_1\delta^{(0)})^{2m}}{2a(ml)^2}+\frac{P(x^*-\lambda_1\delta^{(0)})}{ml} \ , \tag{26}$$

where

$$\lambda_1=\left[\frac{\partial B_1}{\partial x}\right]_{x=x^*}=m \ .$$

Considering Eq. (6) and comparing the singular parts in the two sides of Eq. (26), we obtain

$$\alpha=2-\ln(ml)/\ln(m) . \tag{27}$$

Second, we calculate $L(x)$ and the critical exponent ν. The critical exponent ν can be obtained by the scaling argument [19]. $L(x)$ is transformed by the scaling transformation as

$$(1-\mu x')^{-2\nu}=b^{-2}(1-\mu x)^{-\nu} , \tag{28}$$

where $x'=B_1$ is the weight for one-step SAW's after a one-step scaling transformation and $b=m$ is the scaling factor.

Then, considering again the linear approximation around the fixed point, i.e., $x'-x^*=\lambda_1(x-x^*)$, one finds

$$\nu=\ln(m)/\ln(\lambda_1)=1 . \tag{29}$$

Combining the expressions of α and ν, we acquire the scaling law $D_f\nu=2-\alpha$. This relation has been proved for many kinds of fractals [9–11,18] and may hold for all fractals.

Finally we calculate $C(x)$ and the critical exponent γ. In Appendix B we have argued that $C(x)$ has the following form:

$$\begin{aligned} C(x)\approx \sum_{r=2}^{\infty} \frac{1}{a(ml)^r}[&q_1(B_{r-1})A_{r-1}^2+q_2(B_{r-2})A_{r-1}E_{r-1} \\ &+q_3(B_{r-1})E_{r-1}^2+q_4(B_{r-1})D_{r-1} \\ &+q_5(B_{r-1})A_{r-1}+q_6(B_{r-1})E_{r-1}] , \end{aligned} \tag{30}$$

where $q_i(B_{r-1})$ is a polynomial in B_{r-1}.

Again, we consider $x=x^*-\delta^{(0)}$ and choose a small positive number ε such that

$$1\gg\varepsilon\gg\delta^{(0)} \tag{31}$$

and

$$r_0\equiv\ln(\varepsilon/\delta^{(0)})/\ln(\lambda_1)\gg 1 . \tag{32}$$

Then we have two regions: For $r<r_0$, $\delta^{(r)}$ is less than ε and B_r can be approximated as B^*. For $r>r_0$, B_r rapidly approaches zero. Here,

$$\delta^{(r)}\equiv x^*-B_r\approx\lambda_1^r\delta^{(0)} . \tag{33}$$

For $r<r_0$, A_r and B_r can be approximately written as

$$A_r\approx G_1\lambda_2^r , \tag{34}$$

$$E_r\approx G_2\lambda_2^r , \tag{35}$$

where G_1 and G_2 are some constants of proportionality and λ_2 is the larger eigenvalue of the matrix

$$\begin{bmatrix} f_1 & f_2 \\ f_6 & f_5 \end{bmatrix} .$$

Inserting Eq. (35) into Eq. (10) and considering λ_2^2 to be numerically larger than $f_4(x^*)$, we get [9]

$$D_r\approx G_3\lambda_2^{2r} , \tag{36}$$

where G_3 is constant.

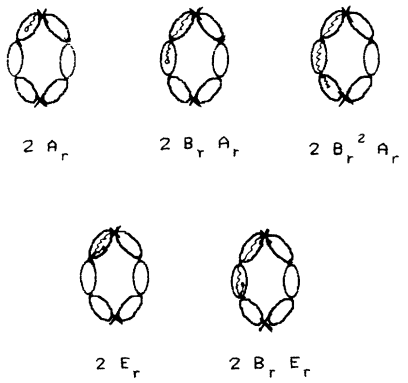

FIG. 3. All possible ways of constructing the restricted partition function A_{r+1} are shown ($m=3$, $l=2$). "○" denotes the rth stage diamond-type lattice.

For $r>r_0$, B_r is approximated as zero. In this case, E_r is always zero, A_r is

$$A_r=G_1(\lambda_2)^{r_0}(l)^{r-r_0} , \tag{37}$$

and

$$D_r\approx[A_r]^2 . \tag{38}$$

From the expressions (30), (37), and (38), we can find easily that $C(x)$ is divergent for any nonzero x when $l\geq m$ and finite for $x<x^*$ when $l<m$. Since $C(x)$ is divergent when $l\geq m$, we cannot obtain an asymptotic form such as expression (5), so we cannot get γ by using this method. When $l<m$, $C(x)$ can be approximately expressed by its largest term, which gives

$$C(x)\sim K_4\left[\frac{\lambda_2^2}{ml}\right]^{r_0} . \tag{39}$$

Here we have only considered the terms proportional to $(\lambda_2)^{2r_0}$ and neglected the terms proportional to $(\lambda_2)^{r_0}$, since $r_0\gg 1$.

Substituting expression (32) into (39), we get

$$C(x)\sim K_4(\varepsilon/\delta^{(0)})^{\gamma} , \tag{40}$$

with

$$\gamma=\ln(\lambda_2^2/ml)/\ln(m) . \tag{41}$$

By now we have solved the three critical exponents α, ν, and γ in the condition $l<m$, with $\alpha=2-\ln(ml)/$

TABLE I. Some specific results for the critical exponents (α, ν, and γ) are shown, together with the fractal dimension D_f, the eigenvalue λ_2, and the connective constant μ.

m	l	D_f	α	ν	γ	μ	λ_2
3	2	1.63	0.37	1	2.77	1.41	5.6
4	2	1.50	0.5	1	2.46	1.26	7.8
5	2	1.43	0.57	1	2.28	1.19	9.9
5	3	1.68	0.32	1	2.92	1.32	13.5
5	4	1.86	0.14	1	3.36	1.41	16.7

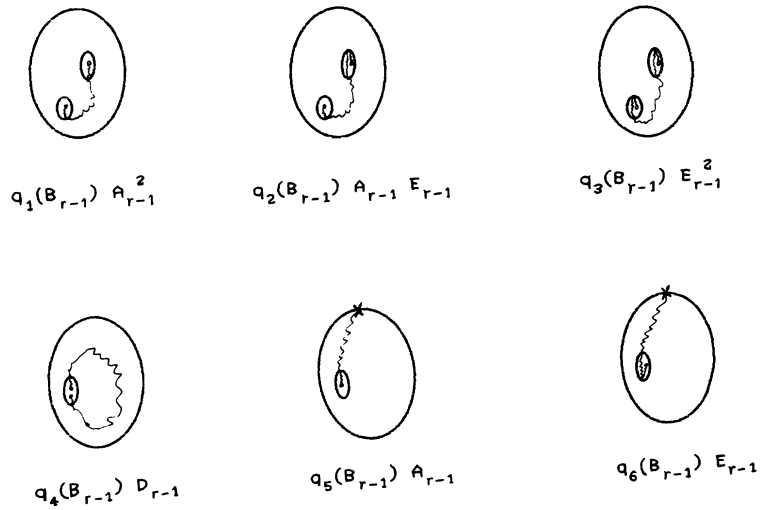

FIG. 4. The six possible rth-order SAW paths within an rth stage diamond-type lattice are shown. "×" denotes the vertex point; "○" denotes the $(r-1)$th stage diamond-type lattice; and the larger circle one denotes the rth stage.

$\ln(m)$, $\nu=1$, and $\gamma=\ln(\lambda_2^2/ml)/\ln(m)$, and got the relation $D_f\nu=2-\alpha$, which agrees with the conclusions of other authors. In Table I we have shown specific results for α, ν, and γ, together with the fractal dimension, the eigenvalue λ_2, and the connective constant μ.

We notice that the critical exponent ν is always 1 and is equal to the results of SAW's on nonloop structures. It can be understood as follows: On the infinite lattice the number of going-far-away SAW's is much larger than that of those going back along the loops, so we obtained the same critical exponent ν as that on a nonloop structure.

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APPENDIX A: THE DERIVATION OF EQ. (8) IN SEC. II

In the following we show the construction of the recursion relation (8). As an example, we consider the case $m=3$, $l=2$. All possible ways of constructing A_{r+1} are shown in Fig. 3. By summing all contributions, we get

$$A_{r+1}=2(1+B_r+B_r^2)A_r+2(1+B_r)E_r\ . \tag{A1}$$

Then for general m and l, we can similarly get

$$A_{r+1}=f_1(B_r)A_r+f_2(B_r)E_r\ , \tag{A2}$$

where

$$f_1=l(1+B_r+\cdots+B_r^{m-1})\ ,$$

$$f_2=l(1+B_r+\cdots+B_r^{m-2})\ .$$

This is just Eq. (8) in Sec. II.

APPENDIX B: THE DERIVATION OF $C(x)$

In the following we express $C(x)$ by the four restricted partition functions. In Fig. 4, we have shown all possible rth-order SAW paths within an rth stage diamond-type lattice, where $q_i(B_{r-1})$ is a polynomial in B_{r-1}. Then, summing all the contributions shown in Fig. 4 and using the same approximation as in Eq. (24), we get

$$C(x)\approx\sum_{r=2}^{\infty}\frac{1}{a(ml)^r}[q_1(B_{r-1})A_{r-1}^2+q_2(B_{r-1})A_{r-1}E_{r-1}+q_3(B_{r-1})E_{r-1}^2+q_4(B_{r-1})D_{r-1}+q_5(B_{r-1})A_{r-1}+q_6(B_{r-1})E_{r-1}]\ . \tag{B1}$$

This is just Eq. (30).

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