Fast, accurate algorithm for numerical simulation of Lévy stable stochastic processes

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We propose a fast and accurate algorithm generating Lévy stable stochastic processes of arbitrary index α ranging between 0.3 and 1.99. The scale parameter is also controllable. The algorithm is very fast when α lies between 0.75 and 1.95.

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I. INTRODUCTION

It is well known that there are stochastic processes which are stable [1], i.e., stochastic processes that satisfy the following property: A stochastic variable z, which is a linear combination of several independent stochastic variables x identically distributed, has a probability density of the same form of the x variables. Therefore, stable processes are stable attractors in a functional space of stochastic variables. For example, the sum of n independent stochastic variables of finite variance converges to a well-known stochastic process: the normal (or Gaussian) process. The above statement is a different version of the celebrated central limit theorem.

On the other hand, it has been shown by Lévy [1] that the sum of n independent stochastic variable showing a probability distribution characterized by power-law wings $P(z>u)\propto|z|^{-\alpha}$ converges to a stable process characterized by a probability density, which is now called a Lévy distribution [1,2]. The index α of the Lévy distribution is ranging between zero (excluded) and two (included).

Since the new paradigm of fractal dimension [3] has emerged, an increasing amount of attention has been devoted to stochastic processes with power-law distributions. Theoretical, numerical, and experimental investigations of Lévy stochastic processes have been carried out in different fields as fully developed turbulence [4,5], biological [6–8], polymeric [9], and economic [10] systems.

Lévy stable processes are difficult to manage either theoretically or numerically. In fact, they are characterized by probability density with diverging moments and the analytical form of the symmetrical Lévy stable distribution is not known except for a few special values of the index α. Moreover, an accurate algorithm generating Lévy stable processes of selectable index α and scale parameter γ all over the definition range is known only for $\alpha=2$ (normal process) and $\alpha=1$ (Cauchy process).

In this paper, we propose an algorithm for numerical simulation of a Lévy stable symmetrical stochastic process of any index α, with α ranging continuously from 0.3 to 1.99. The algorithm is very fast for $0.75\le\alpha\le1.95$, where the required Lévy stable stochastic process is generated in a single step.

The paper is organized as follow: in Sec. II we recall the main properties of symmetrical Lévy stable processes, in Sec. III we illustrate the proposed algorithm, and in Sec. IV we draw our conclusions.

II. SYMMETRICAL LÉVY STABLE PROCESSES

The probability density of a symmetrical Lévy stable process is given by [1]

$$L_{\alpha,\gamma}(z)=\frac{1}{\pi}\int_0^\infty \exp(-\gamma q^\alpha)\cos(qz)\,dq\ , \tag{1}$$

where α and γ are two parameters characterizing the distribution. In particular, α defines the index of the distribution and controls the scale properties of the stochastic process $\{z\}$, whereas γ selects the scale unit of the process. Only in a few cases is the analytical form of Eq. (1) known ($\alpha=2$ Gaussian distribution, $\alpha=1$ Cauchy distribution, $\alpha=\frac{2}{3}$, and $\alpha=\frac{1}{2}$). Lévy stable processes can have diverging moments. In fact, it can be shown that $\langle|z|^\eta\rangle$ is diverging for $\eta\ge\alpha$ when $\alpha<2$. It is worth noting that even if some moments of the distribution are in some case diverging, the stochastic process $\{z\}$ is fully defined from a mathematical point of view if $0<\alpha\le2$ and $\gamma>0$ [1].

In the following, for the sake of simplicity, we set $\gamma=1$ unless differently stated; this does not affect the picture of the process because it is always possible to rescale the used units. For our study it is important to consider the series expansion [11] for large arguments ($z\gg0$)

$$L_{\alpha,1}(z)=-\frac{1}{\pi}\sum_{k=1}^{n}\frac{(-1)^k}{k!}\frac{\Gamma(\alpha k+1)}{z^{\alpha k+1}}\sin\left[\frac{k\pi\alpha}{2}\right]+R(z)\ , \tag{2}$$

where $\Gamma(z)$ is the Euler Γ function and

$$R(z)=O(z^{-\alpha(n+1)-1})\ . \tag{3}$$

By using the series expansion [Eq. (2)], we can conclude that the asymptotic approximation of a Lévy stable distribution of index α is for large values of z given by

$$L_{\alpha,1}(z)\approx\frac{\Gamma(1+\alpha)\sin(\pi\alpha/2)}{\pi z^{(1+\alpha)}}=\frac{C_{\rm LS}(\alpha)}{z^{(1+\alpha)}}\ . \tag{4}$$

From the above equation it is evident that Lévy stable distributions are characterized by a power-law behavior on the far wings of the distributions. However, the index

α of the distribution does not control only the wings of distribution, it also affects the value of the distribution at the origin. In fact, starting from Eq. (1), it can be shown that

$$L_{\alpha,1}(0)=\frac{\Gamma(1/\alpha)}{\pi\alpha} . \tag{5}$$

A number of other properties are reported in the literature [12,13] for stable distributions (asymptotic approximations, numerical values, etc.).

III. THE ALGORITHM

During the past years, a huge number of numerical simulations of power-law distributed stochastic processes have been carried out [3,13,14]. Obviously it is quite simple to write down an algorithm for numerical simulation of stochastic variables characterized by a power-law distribution, however, the processes obtained with a simple algorithm are not Lévy stable because the probability density is different from the value expected starting from Eq. (1) for $z=0$ when $z\approx 0$. We already stated in the Introduction that a sum of several independent variables having the same power-law distribution will eventually converge to the Lévy stable process characterized by the same asymptotic power law. In the following, we will show that this convergence is usually quite slow, moreover, it does not allow us to control the scale factor γ of the obtained stochastic process. In several simulations concerning random processes, it is very important to control the exact nature of the investigated stochastic process and the exact value of the scale factor. Below we present an algorithm that allows us to generate a stochastic variable whose probability density is arbitrary close to a Lévy stable distribution characterized by arbitrary chosen control parameters ($0.3\leq\alpha\leq 1.99$, $\gamma>0$).

To illustrate the algorithm, we divide it in three successive steps. The first step is to calculate

$$v=\frac{x}{|y|^{1/\alpha}} , \tag{6}$$

where x and y are two normal stochastic variables with standard deviation σ_x and σ_y. By using the probability theory, it can be shown that the probability distribution of the stochastic process $\{v\}$ is given by

$$P(v)=\frac{1}{\pi\sigma_x\sigma_y}\int_0^\infty y^{1/\alpha}\exp\left[-\frac{y^2}{2\sigma_y^2}-\frac{v^2y^{2/\alpha}}{2\sigma_x^2}\right]dy . \tag{7}$$

This probability density has very interesting properties, in fact, for large arguments of the stochastic variable ($|v|\gg 0$), it is well described by the asymptotic approximation

$$P(v)\approx\frac{\alpha 2^{(\alpha-1)/2}\sigma_x^\alpha\Gamma((\alpha+1)/2)}{\pi\sigma_y z^{(1+\alpha)}}=\frac{C_v(\sigma_x,\sigma_y,\alpha)}{z^{(1+\alpha)}} , \tag{8}$$

whereas its value at the origin is

$$P(v=0)=\frac{2^{(1-\alpha)/2\alpha}\sigma_y^{1/\alpha}\Gamma((\alpha+1)/2\alpha)}{\pi\sigma_x} . \tag{9}$$

The analogy between Eqs. (8) and (9) and Eqs. (4) and (5) is quite remarkable. Unfortunately, it is not possible to choose a couple of values for σ_x and σ_y that satisfy the following conditions simultaneously for an arbitrary value of α:

$$\begin{aligned} C_{LS}(\alpha)&=C_v(\sigma_x,\sigma_y,\alpha) , \\ L_{\alpha,1}(0)&=P(v=0) . \end{aligned} \tag{10}$$

These conditions are jointly satisfied for $\alpha=1$ only by a couple of values ($\sigma_x=\sigma_y=1$). In this case, the distribution $P(v)$ coincides with a Cauchy distribution characterized by $\gamma=1$ [$L_{1,1}(v)$]. As the standard deviations σ_x and σ_y cannot be chosen independently for an arbitrary value of α, we set $\sigma_y=1$. After this setting, we determine the value of σ_x by requiring that the asymptotic values of $P(v)$ coincide with the asymptotic values of $L_{\alpha,1}(v)$, i.e., we determine σ_x by solving the equation

$$C_{LS}(\alpha)=C_v(\sigma_x,1,\alpha) . \tag{11}$$

By using Eqs. (4) and (8), we obtain

$$\sigma_x(\alpha)=\left[\frac{\Gamma(1+\alpha)\sin(\pi\alpha/2)}{\Gamma((1+\alpha)/2)\alpha 2^{(\alpha-1)/2}}\right]^{1/\alpha} . \tag{12}$$

With this choice, the distributions of Eqs. (7) and (1) have the same asymptotic behavior for large values of the stochastic variable v. In Fig. 1, we compare $P(v)$ obtained for $\alpha=1.5$, $\sigma_x=0.696\,575$, $\sigma_y=1$ with $L_{1.5,1}(v)$; the two curves, obtained by numerical integration of Eqs. (1) and (7), are different in the region close to the origin but coincide on the wings. The inset of the figure shows the two curves in a semilogarithmic plot; from this inset, it is evident that the two distributions almost coincide for $|v|\geq 10$.

By using Eq. (12) we obtain the asymptotic coincidence of the two distributions for large values of the stochastic variable v; the second step is to ensure that the probability density of the generated stochastic process $\{v\}$ coincides all over the range with the Lévy stable distribution

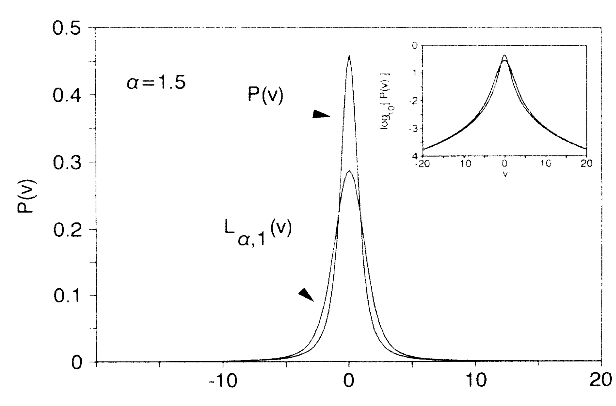

FIG. 1. Comparison of the Lévy stable distribution of index $\alpha=1.5$ and scale factor $\gamma=1$ $L_{1.5,1}(v)$ with the probability density of Eq. (7) obtained by setting $\alpha=1.5$, $\sigma_y=1$, and $\sigma_x=0.696\,575$ [Eq. (12)]. The semilogarithmic inset shows that the two functions are almost coincident for $|v|\geq 10$.

of the same index α and scale factor $\gamma=1$. In analogy with the normal case, one can think that it is sufficient to sum up a limited number n of independent variables each of them distributed in accord with Eq. (7), i.e., it is sufficient to consider the stochastic variable

$$z_n=\frac{1}{n^{1/\alpha}}\sum_{k=1}^{n} v_k \ . \tag{13}$$

In Eq. (13) the scaling factor $n^{1/\alpha}$ ensures that the variable $\{z_n\}$ is characterized by the same scale factor γ characterizing the stochastic variable v [2].

The above argument is correct but the convergence of the stochastic process $\{z_n\}$ is quite slow. To quantify how slow the convergence is, we simulate an entire family of stochastic processes $\{z_n\}$ characterized by the same control parameters, but with n varying from 1 to 125. The calculations are done by using Eqs. (13) and (6). The closeness between the generated stochastic process and the Lévy stable process of the same index is quantified by calculating the error sum of squares between the two distributions,

$$\epsilon^2(n)=\sum_{z,z_n=-10}^{+10}[P(z_n)-L_{\alpha,1}(z)]^2 \ , \tag{14}$$

by using probability density with 100 points ranging between -10 and $+10$ usually. In Fig. 2, we show the result of a numerical simulation performed by setting $\alpha=1.5$, $\sigma_y=1$, and $\sigma_x=0.696\,575$. As expected, the error sum of squares $\epsilon^2(n)$ (upper sets of points in the figure) is a monotonically decreasing function of the number n of intermediate independent stochastic variables $\{v\}$. However, the convergence of $\{z_n\}$ towards a Lévy stable process of index α is very slow. Even using more than 100 intermediate stochastic variables $\{v\}$, we are still quite far from a complete matching between the two processes. To reach a faster convergence to the Lévy stable process of index α, we propose to perform an appropriate nonlinear transformation. We will show that the nonlinear transformation

$$w=\{[K(\alpha)-1]\exp(-v/C(\alpha)]+1\}v \tag{15}$$

allows an almost immediate convergence towards the Lévy stable process of index α, if the two parameters $K(\alpha)$ and $C(\alpha)$ are properly determined. This statement is fully supported from the numerical results reported in Fig. 2. In this figure, in addition to the set of points already illustrated, we report the $\epsilon_c^2(n)$ obtained by comparing $P(z_{cn})$ with $L_{\alpha,1}(z)$ (lower set of points in the figure), where

$$z_{cn}=\frac{1}{n^{1/\alpha}}\sum_{k=1}^{n} w_k \tag{16}$$

is a weighted average of n independent stochastic variables w generated by using Eq. (15). The points in the figure are obtained with $\alpha=1.5$, $\sigma_x=0.696\,575$, $K(\alpha)=1.599\,22$, and $C(\alpha)=2.737$. From the figure it is evident that by performing the additional nonlinear transformation of Eq. (15), the convergence to the Lévy stable stochastic process is very fast and a high accuracy is reached even when $n=1$. In fact, the value of $\epsilon_c^2(1)$ is well below the value of $\epsilon^2(125)$ and the scattering observed in the curve of $\epsilon_c^2(n)$ is related to the finiteness of the number of realizations (2×10^4 realizations) used to determine the $P(z_{cn})$.

We determine the optimal value of $K(\alpha)$ by requiring

$$P(w=0)=L_{\alpha,1}(0) \ . \tag{17}$$

Close to the origin ($w\approx0$), Eq. (15) is well approximated by

$$w=K(\alpha)v \ , \tag{18}$$

and then Eq. (17) will be satisfied if

$$K(\alpha)=\frac{P(v=0)}{L_{\alpha,1}(0)} \ . \tag{19}$$

By using Eqs. (5) and (7), we obtain

$$K(\alpha)=\frac{\alpha\Gamma((\alpha+1)/2\alpha)}{\Gamma(1/\alpha)}\left[\frac{\alpha\Gamma((\alpha+1)/2)}{\Gamma(1+\alpha)\sin(\pi\alpha/2)}\right]^{1/\alpha} \ . \tag{20}$$

In Table I, we report a set of values of $K(\alpha)$ as a function of α. To illustrate the procedure we used to determine the optimal value of $C(\alpha)$, we first need to analyze the first derivative of Eq. (15). The first derivative of Eq. (15) assumes the value $\dot{w}=1$ for $v=C(\alpha)$, and it is always higher (lower) than 1 for $v<C(\alpha)$ and lower (higher) than 1 for $v>C(\alpha)$ when $\alpha>1$ ($\alpha<1$). So that the point

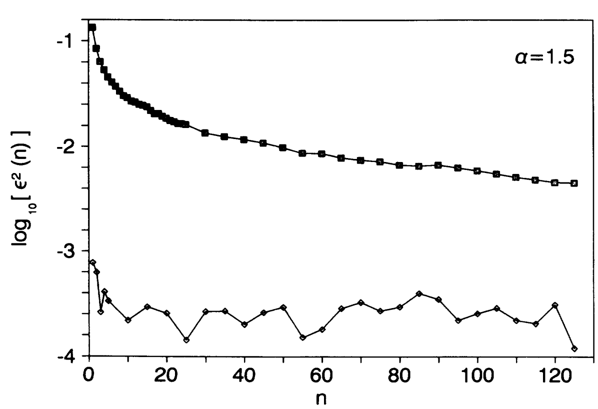

FIG. 2. Error sum of squares between the Lévy stable distribution $L_{1.5,1}(z)$ and the distribution of the stochastic processes simulated as a function of the number of independent stochastic variables n obtained by using Eq. (13) (□) with $\alpha=1.5$, $\sigma_y=1$, and $\sigma_x=0.696\,575$ and Eq. (16) (◇) with $\alpha=1.5$, $\sigma_y=1$, $\sigma_x=0.696\,575$, $K(1.5)=1.599\,22$, and $C(1.5)=2.737$.

TABLE I. Values of the control parameters $\sigma_x(\alpha)$, $K(\alpha)$, and $C_2(\alpha)$ used with the algorithm of Eqs. (6), (15), and (16) to generate Lévy stable stochastic processes of index α and scale factor ($\gamma=1$). The parameters $\sigma_x(\alpha)$ and $K(\alpha)$ are obtained by calculating Eqs. (12) and (20), respectively, whereas $C_2(\alpha)$ is obtained by solving numerically the integral equation given in Eq. (22).

α	$\sigma_x(\alpha)$	$K(\alpha)$	$C_2(\alpha)$
0.1	9.922 44	0.000 032	
0.2	3.138 2	0.021 243	
0.3	2.104 11	0.124 698	
0.4	1.700 47	0.273 51	
0.5	1.479 34	0.423 607	
0.6	1.333 91	0.560 589	
0.7	1.226 37	0.683 435	
0.8	1.139 99	0.795 112	2.483
0.9	1.066 18	0.899 389	2.767 5
1	1	1	
1.1	0.938 291	1.100 63	2.945
1.2	0.878 829	1.205 19	2.941
1.3	0.819 837	1.318 36	2.900 5
1.4	0.759 679	1.446 47	2.831 5
1.5	0.696 575	1.599 22	2.737
1.6	0.628 231	1.793 61	2.612 5
1.7	0.551 126	2.064 48	2.446 5
1.8	0.458 638	2.501 47	2.206
1.9	0.333 819	3.461 5	1.791 5
1.95	0.241 176	4.806 63	1.392 5
1.99	0.110 693	10.498	0.608 9

$v=C(\alpha)$, i.e., the point $w(C)=\{[K(\alpha)-1]/e+1\}C(\alpha)$ does not need correction if

$$P[w=w(C)]=P(v=c)=L_{\alpha,1}[w(C)] \ . \tag{21}$$

We can write this last equation as an integral equation by using Eqs. (1) and (7):

$$\frac{1}{\pi\sigma_x}\int_0^\infty q^{1/\alpha}\exp\left[-\frac{q^2}{2}-\frac{q^{2/\alpha}C(\alpha)^2}{2\sigma_x^2(\alpha)}\right]dq$$

$$=\frac{1}{\pi}\int_0^\infty\cos\left[\left[\frac{K(\alpha)-1}{e}+1\right]C(\alpha)\right]\exp(-q^\alpha)dq \ . \tag{22}$$

We solve this integral equation numerically. We find two different solutions $C_1(\alpha)$ and $C_2(\alpha)$ in the interval $0.75\le\alpha\le1.99$. In Fig. 3, we plot the values of $C_1(\alpha)$ and $C_2(\alpha)$ together with the numerical estimation of the interval of $C(\alpha)$, which speeds up the convergence towards the Lévy stable process of the selected index α. This interval is determined by simulating for several values of $C(\alpha)$ the stochastic process for a selected value of α (in this analysis we use to obtain each $\epsilon^2[C(\alpha)]$ 2×10^4 realizations of the stochastic process). For each value of α, the best interval is determined by using the measure defined in Eq. (14). In the figure, for several values of α, we show $\epsilon^2_{\min}$ as a small box and a bar which indicates the interval of the $C(\alpha)$ values providing ϵ^2 values, which differs by less than 10% from the minimum value $\epsilon^2_{\min}$. A direct analysis of Fig. 3 allows us to conclude that $C_2(\alpha)$ is the value that speeds up the convergence of the generated stochastic process towards the Lévy stable stochastic process of index α and scale factor $\gamma=1$. The values of $C_2(\alpha)$ for several values of α are summarized in Table I.

The effectiveness of our algorithm is shown in Fig. 4 where we present the probability density of the stochastic process obtained by using the algorithm of Eqs. (6), (15), and (16). The control parameters are $\alpha=1.5$, $n=1$, $\sigma_x=0.696\,575$, $K(\alpha)=1.599\,22$, and $C(\alpha)=2.737$. In the figure, boxes are the result of the simulation (10^6 real-

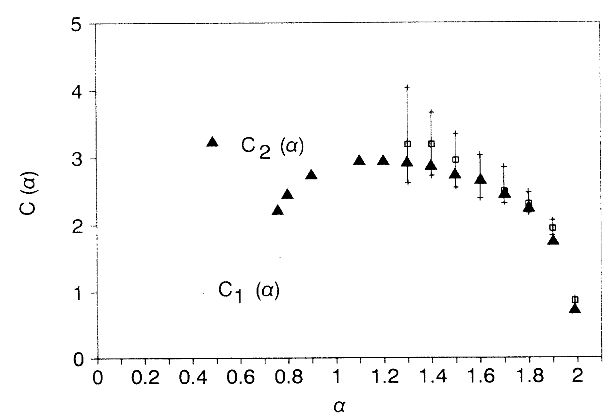

FIG. 3. $C_1(\alpha)$ (□) and $C_2(\alpha)$ (▲) are the numerical solutions of the integral Eq. (22). The bars are the intervals of $C(\alpha)$ that speed up the convergence towards a Lévy stable process. These intervals are obtained by performing numerical simulations of the process with several different values of $C(\alpha)$ and by studying the error sum of squares [Eq. (14)] for each of them. The $\epsilon^2_{\min}$ is indicated with a black box.

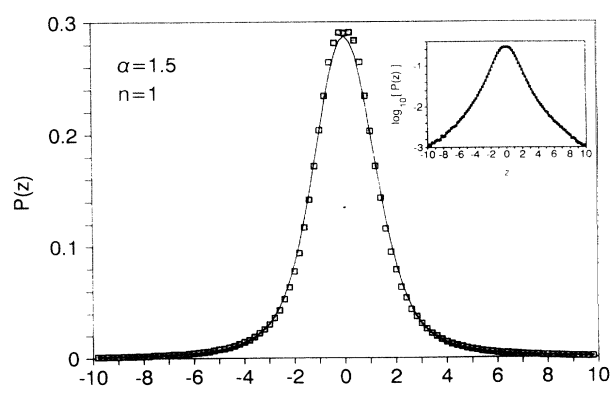

FIG. 4. Probability density (□) of the stochastic process obtained with the algorithm of Eqs. (6) and (15) ($n=1$) together with the Lévy stable distribution $L_{1.5,1}(z)$ (continuous line). The control parameters of the stochastic process are $\alpha=1.5$, $\sigma_y=1$, $\sigma_x=0.696\,575$, $K(1.5)=1.599\,22$, and $c(1.5)=2.737$ and the number of realizations are 10^6. In the inset, the two distributions are plotted by using a logarithmic scale to evidence the agreement on the wings of the distributions.

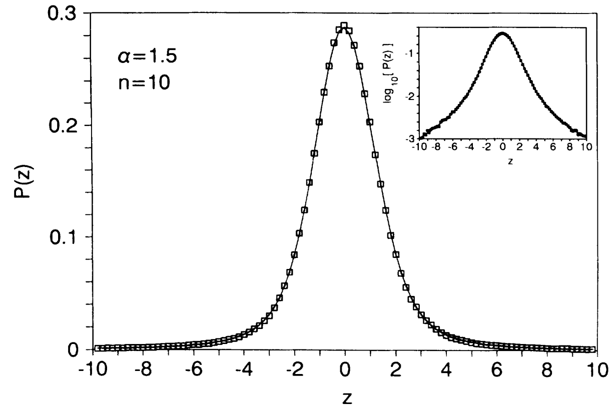

FIG. 5. Probability density (□) of the stochastic process obtained with the algorithm of Eqs. (6), (15), and (16) with $n=10$ together with the Lévy stable distribution $L_{1.5,1}(z)$ (continuous line). The other parameters are the same as those reported in the caption of Fig. 4.

izations), whereas the continuous line is the Lévy stable distribution calculated by performing a numerical integration of Eq. (1). In the inset, the same data are plotted by using a logarithmic scale for the distributions. The figure shows that the probability distribution of the generated stochastic process is in very good agreement with the Lévy stable distribution of index $\alpha=1.5$ and scale factor $\gamma=1$ all over the investigated range except for a few points very close to the origin. This small discrepancy can be eliminated if we generate the stochastic processes by using Eq. (16) with $n>1$; a small number of intermediate independent variables is sufficient to reach a very good agreement all over the entire range. In Fig. 5, we show the result of a numerical simulation performed with the same control parameters used to obtain Fig. 4, but with $n=10$. The agreement between the distribution of the simulated process and the distribution of the Lévy stable process of the same index is excellent all over the entire range.

In Fig. 6, we show a family of probability density obtained by simulating stochastic processes by using Eqs. (6), (15), and (16). In all the simulations $n=1$, while α is varying between 0.8 and 1.9, the other control parameters are selected for each value of α from Table I. In Fig. 7, we show the contour lines of the above figure (noisy lines) together with the contour lines obtained by analyzing the related family of Lévy stable distributions calculated by performing numerical integrations of Eq. (1). From the figure it is evident that our algorithm is accurate all over the investigated ranges.

In the above presentation, the scale factor has always been set to the value $\gamma=1$. However, in numerical simulations of Lévy stable processes it can be useful to control the scale factor γ also. A control of the scale factor is easily obtainable if we multiply the stochastic process z, obtained by using Eq. (13) or (16), by an appropriate multiplicative factor. The linear transformation

$$z_s=\gamma^{1/\alpha}z \tag{23}$$

allows us to obtain a Lévy stable process z_s of the same index of the process z and scale factor equals to γ. In

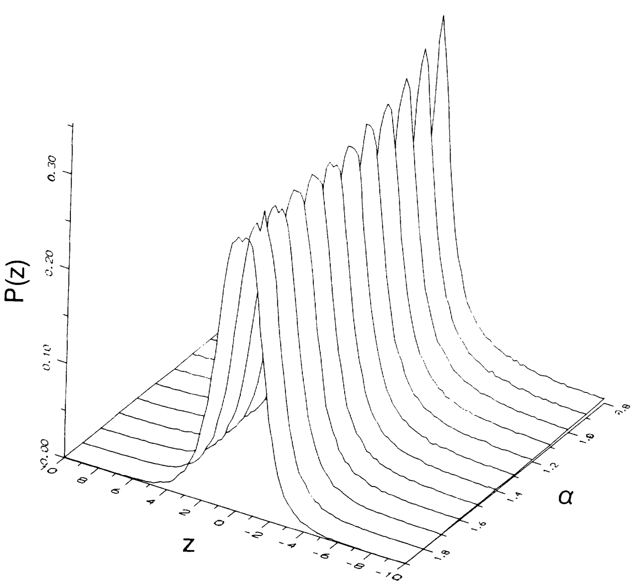

FIG. 6. Ensemble of probability density obtained by performing numerical simulations with the algorithm of Eqs. (6) and (15) ($n=1$). The control parameters for each value of α are reported in Table I. To obtain each distribution we use 10^6 different realizations of the stochastic process.

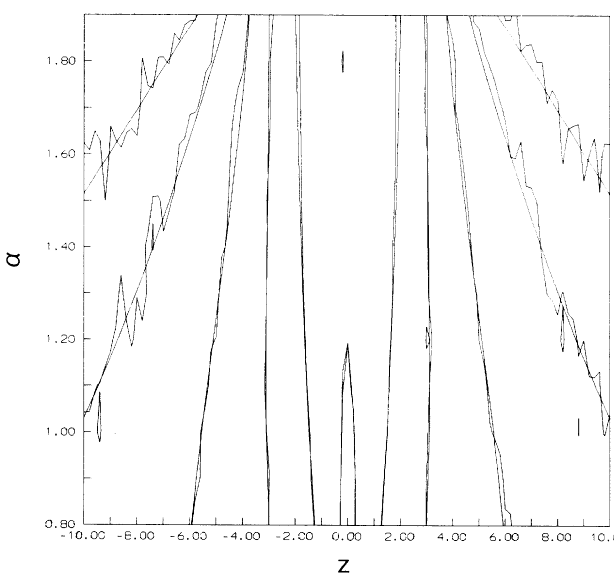

FIG. 7. Contour lines of the probability density shown in Fig. 6 (noisy lines) together with the contour lines of the ensemble of Lévy stable distributions (smooth lines) calculated by performing numerical integration of Eq. (1) ($\gamma=1$). The contour lines are obtained by setting the following z values: 0.001, 0.003, 0.01, 0.03, 0.1, and 0.3 (from left to the middle of the picture, respectively).

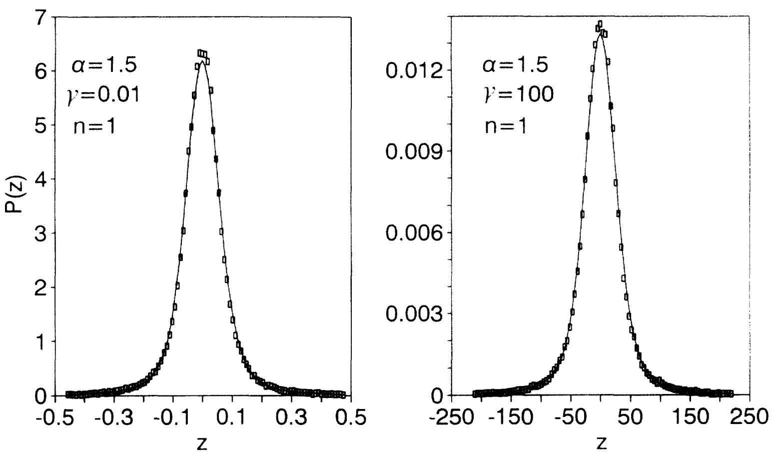

FIG. 8. Probability densities (black boxes) of Lévy stable processes of index $\alpha=1.5$ and scale factor $\gamma=0.01$ (a) and $\gamma=100$ (b). The stochastic processes are simulated by using Eqs. (6), (15), and (23) and we use 10^6 realizations to obtain each distribution. The control parameters are reported in Table I for $\alpha=1.5$, $n=1$. The continuous lines are the Lévy stable distributions of index α and corresponding scale factor γ.

Fig. 8, we show two probability distributions obtained for $\alpha=1.5$ and $n=1$, the usual control parameters (Table I, but for the following values of γ: $\gamma=0.01$ [Fig. (8a)] and $\gamma=100$ [Fig. (8b)]. Both simulations (black boxes in the figure) are in full agreement with the Lévy stable distributions obtained by numerical integration of Eq. (1) for $\alpha=1.5$ and $\gamma=0.01$, 100 (continuous lines in the figure).

The algorithm of Eqs. (6) and (16) is fast and efficient within the interval $0.75\leq\alpha\leq1.95$. The upper limit is determined by the fact that for $\alpha>1.95$ the function of Eq. (15) is not invertible, and due to this, the probability density has local minima. On the other hand, the integral equation [Eq. (22)] has no real solutions for $\alpha<0.75$ so that this value fixes the lower limit of maximal efficiency of our algorithm. It is worthwhile to point out that the algorithm of Eqs. (6) and (16) is effective even outside this limit. The only problem outside the maximal efficiency interval is that to reach a given degree of accuracy, it could be necessary to use a relatively high number n of intermediate independent stochastic variables. The best value of $C(\alpha)$ in this case must be determined by using a heuristic approach. The effectiveness of our algorithm outside the interval $0.75\leq\alpha\leq1.95$ is illustrated in Figs. 9 and 10. In Fig. 9, we show the probability distribution of the stochastic process generated by setting $\alpha=0.3$ and $n=100$, whereas in Fig. 10, we show the probability distribution obtained by setting $\alpha=1.99$ and $n=10$; the other control parameters are reported in Table I. The agreement between the probability distributions of the simulated processes and the calculated Lévy stable distributions is excellent over the entire range. We calculate the Lévy stable distributions either by numerical integration of Eq. (1) ($\alpha=1.99$) or by using the polynomial expansion provided in [12] ($\alpha=0.3$).

IV. CONCLUSIONS

In this paper we present an algorithm generating Lévy stable stochastic processes of arbitrary choosen index α

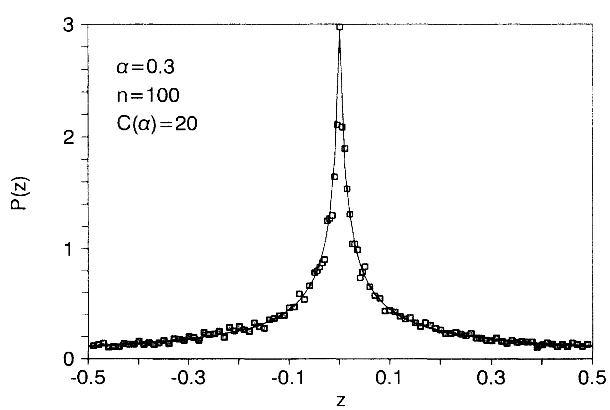

FIG. 9. Probability density (□) of the stochastic process obtained with the algorithm of Eqs. (6), (15), and (16) with $\alpha=0.3$ and $n=100$ together with the related Lévy stable distribution $L_{0.3,1}(z)$ (continuous line). The other parameters are as reported in Table I. The heuristic choice of the parameter $C(0.3)=20$ is not critical.

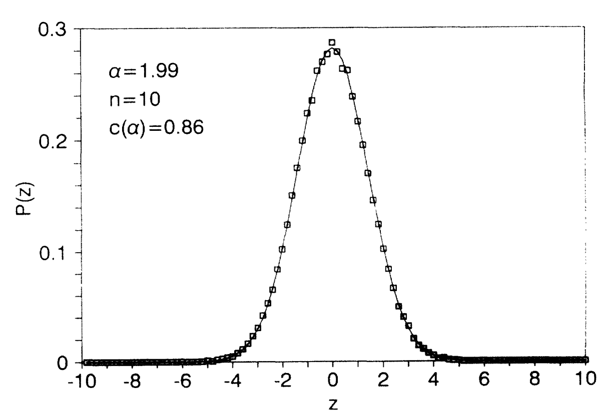

FIG. 10. Probability density (□) of the stochastic process obtained with the algorithm of Eqs. (6), (15), and (16) with $\alpha=1.99$ and $n=10$ together with the related Lévy stable distribution $L_{1.99,1}(z)$ (continuous line). The other parameters are as reported in Table I. The heuristic choice of the parameter $C(1.99)=0.86$ is critical.

and scale factor γ. The algorithm is effective for any value of γ and for α lying in the interval $0.3 \leq \alpha < 2$. The algorithm is very fast when α is selected between 0.75 and 1.95. In our opinion the availability of a fast and accurate algorithm will be useful to perform simulations of several systems where Lévy stable processes are involved, especially when one is interested in the details of the dynamics of a system where a well-characterized Lévy stable process is present.

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[1] P. Lévy, *Théorie de l'Addition des Variables Aléatoires* (Gauthier-Villars, Paris, 1937).
[2] B. V. Gnedenko and A. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables* (Addison-Wesley, Cambridge, MA, 1954).
[3] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).
[4] H. Takayasu, Prog. Theor. Phys. **72**, 471 (1984).
[5] M. F. Shlesinger, B. J. West, and J. Klafter, Phys. Rev. Lett. **58**, 1100 (1987).
[6] A. Ott, J. P. Bouchaud, D. Langevin, and W. Urbach, Phys. Rev. Lett. **65**, 2201 (1990).
[7] C. K. Peng, J. Mietus, J. M. Hausdorff, S. Havlin, H. E. Stanley, and A. L. Goldberger, Phys. Rev. Lett. **70**, 1343 (1993).
[8] S. V. Buldyrev, A. L. Goldberger, S. Havlin, C. K. Peng, M. Simons, and H. E. Stanley, Phys. Rev. E **47**, 4514 (1993).
[9] J. Moon and H. Nakanishi, Phys. Rev. A **42**, 3221 (1990).
[10] R. N. Mantegna, Physica A **179**, 232 (1991).
[11] H. Bergstrom, Ark. Mat. Astron. Fys. **2**, 375 (1952).
[12] E. W. Montroll and J. T. Bendler, J. Stat. Phys. **34**, 129 (1984).
[13] J. P. Bouchaud and A. Georges, Phys. Rep. **195**, 128 (1990).
[14] S. Havlin and D. Ben-Avraham, Adv. Phys. **36**, 695 (1987).

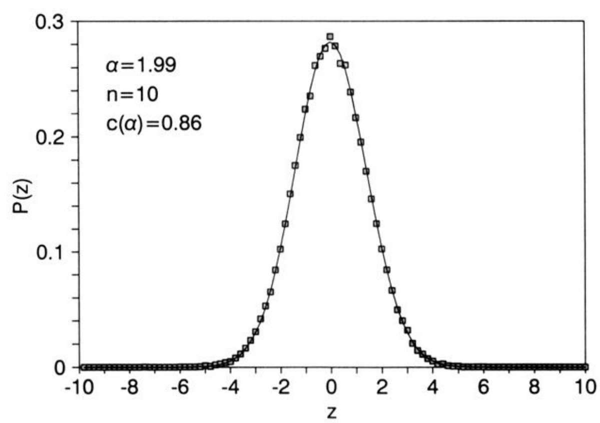

FIG. 10. Probability density (□) of the stochastic process obtained with the algorithm of Eqs. (6), (15), and (16) with $\alpha=1.99$ and $n=10$ together with the related Lévy stable distribution $L_{1.99,1}(z)$ (continuous line). The other parameters are as reported in Table I. The heuristic choice of the parameter $C(1.99)=0.86$ is critical.

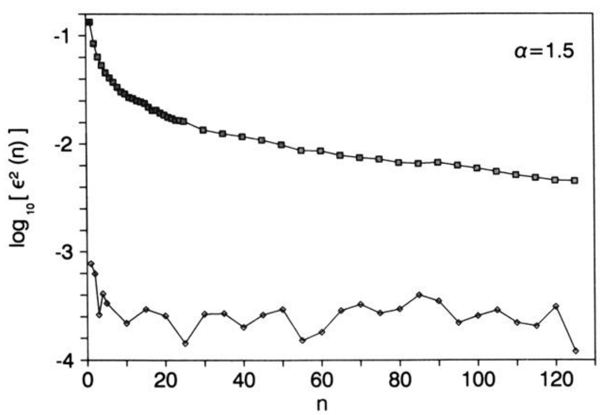

FIG. 2. Error sum of squares between the Lévy stable distribution $L_{1.5,1}(z)$ and the distribution of the stochastic processes simulated as a function of the number of independent stochastic variables n obtained by using Eq. (13) (□) with $\alpha=1.5$, $\sigma_y=1$, and $\sigma_x=0.696\,575$ and Eq. (16) (◇) with $\alpha=1.5$, $\sigma_y=1$, $\sigma_x=0.696\,575$, $K(1.5)=1.599\,22$, and $C(1.5)=2.737$.

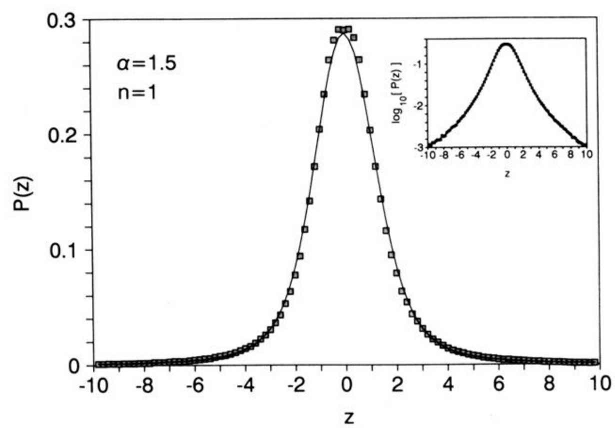

FIG. 4. Probability density (□) of the stochastic process obtained with the algorithm of Eqs. (6) and (15) ($n=1$) together with the Lévy stable distribution $L_{1.5,1}(z)$ (continuous line). The control parameters of the stochastic process are $\alpha=1.5$, $\sigma_y=1$, $\sigma_x=0.696\,575$, $K(1.5)=1.599\,22$, and $c(1.5)=2.737$ and the number of realizations are 10^6. In the inset, the two distributions are plotted by using a logarithmic scale to evidence the agreement on the wings of the distributions.

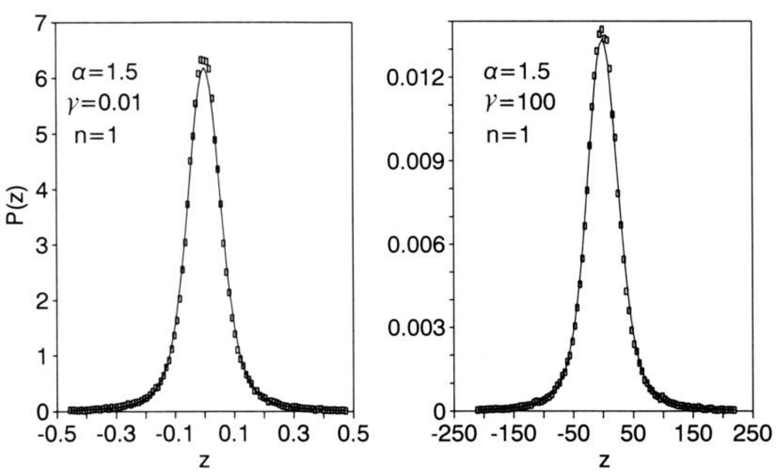

FIG. 8. Probability densities (black boxes) of Lévy stable processes of index $\alpha = 1.5$ and scale factor $\gamma = 0.01$ (a) and $\gamma = 100$ (b). The stochastic processes are simulated by using Eqs. (6), (15), and (23) and we use 10^6 realizations to obtain each distribution. The control parameters are reported in Table I for $\alpha = 1.5$, $n = 1$. The continuous lines are the Lévy stable distributions of index α and corresponding scale factor γ.

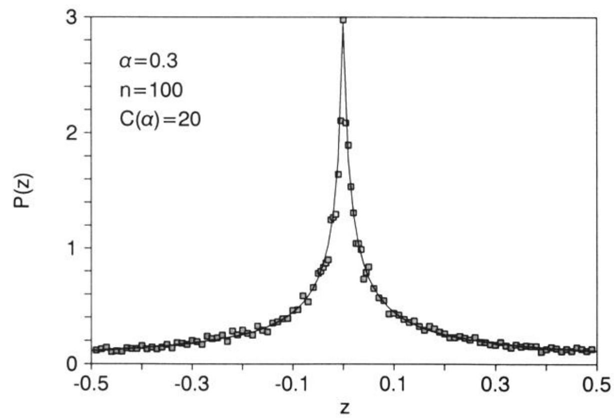

FIG. 9. Probability density (□) of the stochastic process obtained with the algorithm of Eqs. (6), (15), and (16) with $\alpha=0.3$ and $n=100$ together with the related Lévy stable distribution $L_{0.3,1}(z)$ (continuous line). The other parameters are as reported in Table I. The heuristic choice of the parameter $C(0.3)=20$ is not critical.