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# Renormalization group theory and variational calculations for propagating fronts

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We study the propagation of uniformly translating fronts into a linearly unstable state, both analytically and numerically. We introduce a perturbative renormalization group approach to compute the change in the propagation speed when the fronts are perturbed by structural modification of their governing equations. This approach is successful when the fronts are structurally stable, and allows us to select uniquely the (numerical) experimentally observable propagation speed. For convenience and completeness, the structural stability argument is also briefly described. We point out that the solvability condition widely used in studying dynamics of nonequilibrium systems is equivalent to the assumption of physical renormalizability. We also implement a variational principle, due to Hadeler and Rothe [J. Math. Biol. **2**, 251 (1975)], which provides a very good upper bound and, in some cases, even exact results on the propagation speeds, and which identifies the transition from "linear marginal stability" to "nonlinear marginal stability" as parameters in the governing equation are varied.

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## I. INTRODUCTION

The steady-state equation for a traveling wave propagating into an unstable state does not uniquely determine the wave speed. As an alternative to directly solving the initial-value problem, velocity selection principles have been sought, which identify *a priori* the wave speed that would be dynamically selected from those permitted by the steady-state equation. Such velocity selection and/or wave-number selection problems [1–14] arise in a wide variety of nonequilibrium systems that exhibit propagation of well-developed patterns and fronts into initially unstable and homogeneous states. Examples occur in such diverse fields as population dynamics [15] and pulse propagation [16] in nerves in biophysical systems, Taylor-Couette flows [11] and Rayleigh-Benard convection [12] in hydrodynamic systems, and crystal growth [14,17], models of solidification and aggregation [18], as well as traveling waves in reaction-diffusion systems [19]. In the present paper, we consider only those systems for which the pattern emerging behind the propagating fronts is homogeneous, or can be described by an envelope curve, and concentrate on predicting the propagation velocities of uniformly translating fronts.

There already exist several proposed criteria [1,2,20] for the dynamical velocity selection mechanism: a minimum speed rule, marginal stability, and structural stability. Among them, the minimum speed principle has in certain cases a rigorous basis in the Aronson-Weinberger theorem [3]. According to the marginal stability hypothesis [1,2,6–8], for most sufficiently localized initial conditions, the propagation velocity of well-developed fronts generically approaches the marginal-stability point which apparently coincides with the minimal velocity out of a family of stable propagating fronts. The marginal-stability point can be determined explicitly from the linearized leading edge approximation, in which only the linearized equation of motion is studied near the front. In the literature, this is sometimes referred to as the *linear-marginal-stability* case or the *pulled* case. However, as emphasized by Ben-Jacob *et al.* [2] and later by van Saarloos [7,8], on the basis of rigorous work by Aronson and Weinberger [3] for a class of simple equations, there exist cases in which the linear-marginal-stability selection fails. This is often referred to as the *pushed* case or *nonlinear-marginal-stability* case. Thus the marginal-stability argument apparently provides a practical method to calculate the selected velocity analytically, but cannot tell when the method is reliable, since there is no general method known to distinguish *a priori* between pushed and pulled cases. This is an important limitation because a given equation may make a transition from pulled to pushed cases as a parameter in this equation is varied.

The most recent proposal is the structural stability hypothesis [21–23], in which the observable fronts are supposed to be stable against small changes in the governing partial differential equation (PDE) itself. When the Aronson-Weinberger theorem is applicable, this hypothesis has the status of a theorem. The hypothesis applies (so far) without counterexamples to both the pushed and pulled cases, but until now it has not been able to yield an analytical means to obtain the selected velocity. Instead, a numerical method, based on the structural stability hypothesis, has been used.

The principal purpose of this paper is to use the perturbative renormalization group (RG) theory to analyze the stability of uniformly translating fronts, and to show that this method can be used to calculate the change in front propagation velocity when the governing equation of a structurally stable front is perturbed. In addition, the RG allows us to predict the uniquely selected velocity by combining the structural stability principle with it. As a biproduct of our study of fronts, we have investigated a

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variational principle due to Hadeler and Rothe [24] for the phase trajectories of steady-state solutions of propagating fronts. We will see that the variational principle can give useful upperbounds on propagation speeds in both pulled and pushed cases and is able to estimate the transition point between those regimes. This work, presented in Sec. VI, is due solely to L.-Y. Chen.

The applicability of RG to traveling fronts is predicated upon its recent successful application to the study of long-time global behavior of physical systems described by nonlinear PDE's: porous medium equations [25–28], convection-diffusion transport with irreversible sorption [29], the propagation of a turbulent burst [30,31], and linear continuum mechanics [32]. In these problems, the global solutions approach the similarity form $u(x,t)=t^{-\alpha}f(xt^{-\beta})$, where $x$ is the spatial coordinate, $t$ the time, and $\alpha$ and $\beta$ are constants. In many cases—the so-called intermediate asymptotics of the second kind [33]—the exponents $\alpha$ and $\beta$ cannot be determined by simple dimensional analysis. In fact, these exponents are analogs of the anomalous dimensions in field theory, and can be evaluated by using renormalized perturbation theory [25–27]. A mathematically rigorous formulation has been given by Bricmont, Kupiainen, and Lin [34]; these methods have also been used to study front propagation in the Ginzburg-Landau equation [35,36]. A detailed and pedagogical discussion of the application of RG to the asymptotics of partial differential equations and its physical interpretation is given in the book by Goldenfeld [37]. We emphasize only that RG can be used to solve the above categories of problems; standard techniques, such as multiple-scale analysis, may also be used (perhaps with more difficulty), and in fact there is a general relationship between multiple-scale analysis and the RG, whose elements are discussed elsewhere [23].

A steady propagating wave solution has the form $\psi(x-ct)$, where $c$ is the propagation speed. If we introduce $X$ and $T$ as $x=\ln X$, $t=\ln T$, then $\psi(x,t)=\Psi(XT^{-c})$ for some $\Psi$. That is, the solutions can be expressed in the form of similarity solutions. Because $X$ and $T$ must already be dimensionless, the velocity $c$ can be viewed as an anomalous dimension, analogous to the exponents $\alpha$, $\beta$ above in the intermediate asymptotics of the second kind. Thus it is a natural guess that there is an RG method to compute the velocity. However, there is an important difference between the similarity solution problems and the propagation problem. In the former, there is a unique asymptotic exponent, whereas in the latter the propagation speed is not unique. Thus, for the selection problem RG is not sufficient. We will see that it must be combined with some new physics—the structural stability hypothesis.

The outline of this paper is as follows. In Sec. II, a formal renormalized perturbation theory for the front propagation is formulated. In Sec. III, technical aspects of the renormalization method are discussed. The central issue is the nature of the eigenvalue spectrum of the linearized PDE; this is resolved using considerations of structural stability. In Sec. IV, we discuss the relationship between solvability conditions and renormalizability. Section V contains a number of applications of the formalism, in which we compute the propagation velocity using perturbative RG. Section VI presents results using the variational principle and a simple trial function. In Sec. VII, we use the variational and RG methods to predict the transition between the pushed and pulled cases. We summarize and conclude in Sec. VIII.

## II. FORMAL THEORY OF PERTURBATIVE RENORMALIZATION

In this paper we mostly concentrate on the so-called Fisher–Kolmogorov-Petrovsky-Piskunov (Fisher-KPP) equation [15]:

$$\frac{\partial\psi}{\partial t}=\frac{\partial^2\psi}{\partial x^2}+F(\psi) , \tag{2.1}$$

where $F$ is a continuous function with $F(0)=F(1)=0$. If $F$ satisfies the condition $f(\psi)>0$ for all $\psi\in(0,1)$, then there exists a stable traveling-wave solution interpolating between $\psi=1$ and $\psi=0$ with propagation speed $c$ for each value of $c$ greater than or equal to some minimum value $c^*$, where $c^*\geq\hat{c}\equiv2\sqrt{F'(0)}$ (if $F$ is differentiable at the origin). As suggested by the analogy between similarity solutions and traveling waves, there should be an RG method to compute the propagation speed. Here we show that when a front is structurally stable, we can device a renormalized perturbation method to compute the change in the propagation speed due to a small perturbation in, e.g., the "reaction term" $F$.

### A. Perturbation theory

We consider formally a system described by the following abstract nonlinear (usually parabolic) equation:

$$\frac{\partial\psi}{\partial t}=N\{\psi\} . \tag{2.2}$$

Let $\psi_0(x-c_0t+x_0)$ be a stable traveling front solution of (2.2) with speed $c_0$, where $x_0$ is a free parameter indicating the translational symmetry of the problem explicitly; there is a one-parameter family of traveling-wave solutions with a given speed. Let us add a small structural perturbation $\delta N$ to (2.2); for example, in (2.1) we replace $F$ with $F+\delta F$. We assume that the operator norm of $\delta N$ on an appropriate domain is less than some small positive number $\epsilon$; for example for (2.1), the $C^0$ norm of $\delta F$ is less than $\epsilon$, that is, $\|\delta F\|<\epsilon$, where the norm $\|\delta F\|=\sup_u|\delta F(u)|$. Assume that in response the front solution is modified to $\psi_0+\delta\psi$, where $\|\delta\psi\|$ is of order $\epsilon$. Linearizing (2.2) to order $\epsilon$ in the moving frame with velocity $c_0$, we find the following equation governing the first order correction:

$$\frac{\partial\delta\psi}{\partial t}=\left[c_0\frac{\partial}{\partial\xi}+DN\right]\delta\psi+\delta N\{\psi_0\} , \tag{2.3}$$

where $DN$ is the Fréchet derivative of $N$ at $\psi=\psi_0$, $\xi\equiv x-c_0t+x_0$, and the initial condition satisfies $\delta\psi(\xi,t_0)=0$.

To make our mathematical formalism as simple as possible, first, without loss of generality, we will consider the case for which the highest differential operator is the second order; however, the conclusions to be drawn below apply to other high-order nonlinear operators too.

The formal solution to Eq. (2.3) reads

$$\delta\psi(\xi,t)=\int_{t_0}^{t}dt'\int_{-\infty}^{+\infty}d\xi' G(\xi,t;\xi',t')[\delta N\{u_0\}](\xi') , \quad (2.4)$$

where $G$ is the Green function, which satisfies

$$\frac{\partial G}{\partial t}-\left[c_0\frac{\partial}{\partial\xi}+DN\right]G=\delta(t-t')\delta(\xi-\xi') \quad (2.5)$$

with $G\to 0$ in $|\xi-\xi'|\to\infty$. The Green function may formally be written as

$$G(\xi,t;\xi',t')=\sqrt{\rho(\xi')\rho(\xi)}\sum_{n=0}^{\infty}\frac{1}{C_n^2}e^{-\omega_n(t-t')}u_n(\xi) \times u_n^*(\xi')\Theta(t-t') , \quad (2.6)$$

where we use the summation symbol to represent both summation and integration; $\rho(\xi)$ is an appropriate weight function, $\{C_n^2\}$ are appropriate normalization constants, and $\Theta$ is the Heaviside function. The summation is over the spectrum $\{\omega_n\}$, and $\{u_n\}$ are corresponding appropriately normalized (generalized) eigenfunctions obeying

$$\left[c_0\frac{d}{d\xi}+DN\right]u_n=-\omega_n u_n . \quad (2.7)$$

We do not need to solve completely this seemingly formidable equation. Differentiating Eq. (2.2) with respect to $\xi$, we have

$$\left[c_0\frac{d}{d\xi}+DN\right]\left[\frac{d\psi_0}{d\xi}\right]=0 , \quad (2.8)$$

showing that, due to the translational invariance of the original equation, $u_0\propto d\psi_0/d\xi$ satisfies (2.7). For the fronts in which we are interested, this function is well localized (square integrable), so that 0 is an eigenvalue. We assume that the operator on the left hand side of (2.7) is dissipative, so that 0 is the upper bound of its spectrum.

Thus, only the zeroth eigenfunction $\psi_0$ contributes to the secular term which is proportional to $t-t_0$, and the perturbed solution can be written as

$$\psi(\xi,t)=\psi_0(\xi)-\delta c(t-t_0)\frac{d\psi_0}{d\xi}+(\delta\psi)_r+O(\epsilon^2) . \quad (2.9)$$

Here $\delta c$ is given by

$$\delta c=-\frac{\int_{-\infty}^{+\infty}d\xi\,\rho(\xi)u_0(\xi)[\delta N\{u_0\}](\xi')}{\int_{-\infty}^{+\infty}d\xi\,\rho(\xi)u_0^2(\xi)} \quad (2.10)$$

and $(\delta\psi)_r$ represents the nonsecular terms, which are unimportant for the purpose of determining the modification of the velocity, leading only to finite $O(\epsilon)$ corrections to the profile of the front.

One may immediately guess that this expression for $\delta c$ is the change in the front speed, but there are two problems with this identification. First, the naive perturbation theory is not controlled due to the secularity. A renormalization procedure, given below, will be used to remove the secularity. Second, both the numerator and the denominator of the expression for $\delta c$ may be divergent. This potential difficulty may be treated using the considerations of structural stability given in Sec. III.

### B. Renormalization and renormalization group

For a certain class of sufficiently localized initial conditions, all the transient solutions to the unperturbed PDE of the form $u(x,t)=u(x-x_c(t,c_0,x_0),t)$ asymptotically converge to the same uniformly translating propagating front, $u(x,t)=u(x-c_0t+x_0)$ as $t\to\infty$. After the perturbation has been switched on, $x_0$ is no longer a constant of motion of the perturbed system, and is therefore not observable at long times. From a measurement of $x_0$ at late times, the initial value of $x_0$ cannot be deduced; hence, it must be renormalized from the RG point of view [37]. As we saw in the Introduction, if we make transformations $x=\ln X$, $t=\ln T$, $x_0=\ln A_0$, and $u(x,t)=U(X,T)$, the traveling-wave solution can be expressed in the form of a similarity solution

$$U(X,T)=U(\ln A_0X/T^{c_0})=U_1(A_0X/T^{c_0}) =U_2(T/(A_0X)^{1/c_0}) ,$$

for appropriate functions $U$, $U_1$, and $U_2$. It now becomes clear that the role of the constant of motion $x_0$ is the same as that played by the initial total amount of mass or energy $Q_0$ in the nonlinear diffusion problems studied previously [31–33]. Although it is possible to renormalize traveling-wave problems by regarding them as similarity solutions [38], here we will perform an alternative type of RG analysis directly on the traveling-wave solution.

The divergence of the secular term $(\delta u)_s\propto(t-t_0)$ as $t\to\infty$ can be removed order by order in $\epsilon$ by regarding $t_0$ as a regularization parameter and introducing an additive renormalization constant $Z=Z(t_0,\mu,\epsilon)$ with $\mu$ an arbitrary time. Note that the limit $t-t_0\to\infty$ can be achieved in two equivalent ways: either by keeping $t_0$ fixed and letting $t\to\infty$, or by keeping $t$ fixed and letting $t_0\to-\infty$. We will use the latter method in the following text, which corresponds to the conventional treatment of logarithmic divergences in the zero cutoff limit of the bare perturbation series for, e.g., $\varphi^4$ field theory. $t$ corresponds to $\ln T$ in the analogy discussed in the preceding section, so that $t-t_0$ corresponds to $\ln(T/T_0)$, where $\ln T_0$ corresponds to $t_0$. In the standard Gell-Mann–Low RG [39], we introduce an arbitrary length scale (or rather time scale in this context) $L$ and split $T/T_0$ into $(T/L)(L/T_0)$; the term $\ln(T_0/L)$ is absorbed into the (multiplicative) renormalization constant often denoted by $Z$. The corresponding procedure here is to split $t-t_0$ into $t-\mu-(t_0-\mu)$, where $\mu$ corresponds to $\ln L$ in the standard approach. Since multiplications in the standard multiplicative renormalization procedure correspond to addition in our case, we renormalize $x_0(t_0)$ by $x_0^R + Z(t_0,\mu,\epsilon)$, and Taylor expand $Z=\sum_{n=1}^{\infty} a_n(t_0,\mu)\epsilon^n$. The coefficients $a_n(n \geq 1)$ are determined order by order in $\epsilon$ in such a way that all the secular divergences in $\psi(x,t)$ are cancelled out. In this way, the renormalized solution $\psi(x,t)$ remains finite even in the limit $t_0 \to -\infty$. Hence, to $O(\epsilon)$, the solution $\psi(x,t)$ can be rewritten as

$$\begin{aligned}\psi(x,t) &= \psi_0(\xi_0) - \delta c(t-t_0)\frac{d\psi_0}{d\xi_0} + O(\epsilon)\ , \\ &\qquad \xi_0 = x - c_0 t + x_0\ , \\ &= \psi_0(\xi + \epsilon a_1) - \delta c(t-t_0)\frac{d\psi_0}{d\xi} + O(\epsilon)\ , \\ &\qquad \xi = x - c_0 t + x_0^R\ , \\ &= \psi_0(\xi) + \epsilon a_1 \frac{d\psi_0}{d\xi} - \delta c(t-t_0)\frac{d\psi_0}{d\xi} + O(\epsilon)\ , \end{aligned} \tag{2.11}$$

where $O(\epsilon)$ refers to all finite terms of order $\epsilon$, regular in the limit $t_0 \to -\infty$. By choosing $a_1\epsilon = \delta c(\mu - t_0)$, the secular divergence is removed. We obtain the renormalized solution

$$\psi(x,t) = \psi_0(\xi) - \delta c(t-\mu)\frac{d\psi_0}{d\xi} + O(\epsilon)\ , \qquad \xi = x - c_0 t + x_0^R(\mu)\ . \tag{2.12}$$

It is impossible that the actual solution $\psi(x,t)$ can depend on the arbitrary time scale $\mu$, because (as with $L$ in the standard procedure) $\mu$ is not present in the original problem. This is expressed by the renormalization group equation

$$\left.\frac{\partial \psi}{\partial \mu}\right|_{x,c_0,t,\epsilon} = 0\ . \tag{2.13}$$

Hence, to order $\epsilon$ the RG equation yields, after equating $\mu$ with $t$,

$$\frac{\partial \psi}{\partial t} + \delta c \frac{\partial \psi}{\partial \xi} = 0\ . \tag{2.14}$$

This has the form of an amplitude equation, an observation discussed further in Sec. IV and in Ref. [23]. Thus the speed of the renormalized wave is indeed $c = c_0 + \delta c$, and the leading long-time asymptotic behavior, to $O(\epsilon)$, is

$$\psi(x,t) \sim \psi_0(x - ct + x_0') + O(\epsilon)\ , \tag{2.15}$$

where $x_0'$ is the new constant of motion for the perturbed system.

### C. Application to Fisher's equation

Now we consider the application of the above formalism to Fisher's equation. An important technical aspect of this discussion is deferred to Sec. III. We consider only a perturbation $\delta F$ to the nonlinear reaction term $F$ of (2.1), so that (2.4) reads

$$\delta\psi(\xi_0,t) = e^{-c_0\xi_0/2}\int_{t_0}^{t} dt' \int_{-\infty}^{+\infty} d\xi' G(\xi_0,t;\xi',t') \times e^{c_0\xi'/2}[\delta F\{\psi_0\}](\xi')\ . \tag{2.16}$$

Here, $G$ is the Green function, satisfying

$$\frac{\partial G}{\partial t} - \mathcal{L}G = \delta(t-t')\delta(\xi - \xi')$$

with $G \to 0$ in $|\xi - \xi'| \to \infty$, and

$$\mathcal{L} \equiv \frac{\partial^2}{\partial \xi^2} + F'(\psi_0(\xi)) - \frac{c_0^2}{4}\ .$$

Formally, $G$ reads

$$G(\xi,t;\xi',t') = u_0(\xi)u_0^*(\xi') + \sum e^{-\lambda_n(t-t')}u_n(\xi)u_n^*(\xi')\ ,$$

where $\mathcal{L}u_0 = 0$, and $\mathcal{L}u_n = \lambda_n u_n$. The summation symbol, which may imply appropriate integration, is over the spectrum other than the point spectrum $\{0\}$. Since the system is translationally symmetric, $u_0 \propto e^{c_0\xi/2}\psi_0'(\xi)$. Due to the known stability of the propagating wave front, the operator $\mathcal{L}$ is dissipative, so that zero is the least upper bound of its spectrum. Hence, only $u_0$ contributes to the secular term in $\delta\psi$. A similar argument as in the general case gives the explicit formula for (2.10):

$$\delta c = -\frac{\int_{-\infty}^{+\infty} d\xi\, e^{c_0\xi}\psi_0'(\xi)\delta F\{\psi_0\}(\xi)}{\int_{-\infty}^{+\infty} d\xi\, e^{c_0\xi}\psi_0'^2(\xi)}\ . \tag{2.17}$$

## III. JUSTIFICATION OF PERTURBATIVE RG AND STRUCTURAL STABILITY

Before proceeding further, we must now address the issue that the formula for $\delta c$, for example, may not be finite. The formal perturbation approach may fail if zero is not an isolated eigenvalue of the linearized operator. In this section, we will discuss when the restricted formula (2.17) is meaningful. To this end, we need results from our structural stability analysis, and so we begin with a brief summary of this topic [21–23].

Equation (2.1) can be classified into two cases: ambiguous and unambiguous. We say that the equation is *unambiguous* if it allows a unique propagating speed for its traveling-wave solutions (here, we consider the waves traveling in the positive direction only), and is ambiguous otherwise. A necessary and sufficient condition for (2.1) to be unambiguous is that $F$ does not have any isolated minimum at the origin. Notice that by a certain indefinitely small $C^0$ perturbation $\delta F$ of $F$, we can convert an ambiguous equation into an unambiguous one. Hence, at most one propagating speed can be stable against this modification of the system. Because we are perturbing $F$, that is, the equation itself, we call such a perturbation a structural perturbation. Thus we may say that there is at most one structurally stable propagating speed for (2.1). It has been proved that the slowest propagating speed $c^*$ which allows stable (in the usual sense) propagating waves is a continuous functional of $F$ so long as

$Q(F) \equiv \sup_{\psi\in(0,1]} F(\psi)/\psi$ is a continuous functional of $F$. This is satisfied if $\delta F$ is $C^0$-small and $\sup_{\psi\in(0,1]}\delta F(\psi)/\psi$ is smaller than some positive number which converges to zero as $\|\delta F\|$ goes to zero. We call such a perturbation a physically small ($p$-small) perturbation. Thus we may say that the wave with speed $c^*$ is structurally stable against $p$-small structural perturbations.

We have conjectured that only structurally stable solutions of a model equation correspond to the physically observable phenomena. This structural stability hypothesis is closely related to, but not identical with the idea originally proposed by Andronov and Pontrjagin [40] for dynamical systems. The conjecture implies for (2.1) that $c^*$ is the selected propagation speed. It seems to be widely believed that $c^*$ is indeed the unique observable speed for (2.1), but a proof does not exist for general $F$ [41]. However, our numerical studies have so far failed to uncover counterexamples. The structural stability hypothesis is usually redundant, but has predictive power in cases where one's model of a physical phenomenon inadvertently includes unphysical features. This is the case for the overidealized models [such as (2.1)] of propagation phenomena considered here.

Prompted by our structural stability analysis, we have used the following numerical method to estimate the selected velocity. We apply the shooting method to the ordinary differential equation (ODE) governing the propagating wave front shape $\psi$,

$$\frac{d^2\psi}{d\xi^2} = -c\frac{d\psi}{d\xi} + F(\psi) , \tag{3.1}$$

where $c$ is the propagating speed, and $\xi$ corresponds to $x-ct$ after modifying $F$ by adding a $p$-small perturbation to convert the equation into the unambiguous class. We know that the unique speed is almost identical to the true selected velocity, since $c^*$ is continuously dependent on $F$. Therefore, there is only one $c$ which allows the shooting method to give a solution. This $c$ is the selected speed. However, in practice, the method may not be very accurate, because the true unique solution and other solutions which do not reach $\psi=0$ for large $\xi$ may not be easy to distinguish when the perturbations are small.

There are several important consequences of our structural stability study. First, notice that even if $\delta F$ is $p$ small, $-\delta F$ need not be. Furthermore, it is easy to construct an indefinitely $C^0$-small $\delta F$ which increases $F'(0)$ indefinitely, but $-\delta F$ is $p$-small. An example of such $\delta F$ is a very tiny but very sharp spike well localized near the origin. This implies that for this $\delta F$, $\delta c$ must be indefinitely large, while $\delta c$ for $-\delta F$ is infinitesimal. Now (2.17) is a linear functional of $\delta F$, so that the sign change of $\delta F$ cannot cause such a drastic change. This clearly demonstrates that the formula (2.17) is not meaningful, if either $\delta F$ or $-\delta F$ is not $p$-small. More explicitly, $\sup_u|\delta F(u)/u|$ must vanish as $\|\delta F\|$ goes to zero. If $\delta F$ is differentiable at the origin, the condition is satisfied.

The perturbation approach outlined above is legitimate only when zero is an isolated point spectrum of the operator $\mathcal{L}$. Unfortunately, this is not the case for many reaction-diffusion equations. For (2.1) the essential spectrum has the range $(-\infty,\eta)$, where $\eta=\max\{F'(1),F'(0)\}-c_0^2/4$, as is easily seen from Rota's theorem (a generalization of Weyl's theorem) [42]. Hence, if $c_0=2\sqrt{F'(0)}$, which is the important case for the pulled equations, then the eigenvalue zero is not isolated. Nevertheless, we can justify the formula or rather its augmented version as follows.

Notice that we can always find a sequence $\{\delta f_k\}$ of piecewise differentiable perturbations such that $\delta f_k$ converges to zero in $C^0$ norm, but $\delta f_k'(0)$ is always $-1$. This is a sequence of $p$-small perturbations, so that $c^*(F+\delta f_k)$ converges to $c^*(F)$. However, the essential spectrum of the operators $\mathcal{L}$ is bounded from above by $-1$, so for these perturbed systems, 0 is always isolated. Now consider the perturbed system with $\delta F$. Instead of the system with $F+\delta F$, we again consider a sequence of systems $F+\delta F+\delta f_k$. For this sequence, $c^*$ converges to $c^*(F+\delta F)$. Thus we may study the perturbation of the system with $\delta F+\delta f_k$ instead of the original system with $\delta F$ to compute the charge in velocity. Thus we may conclude that the ordinary perturbation theory can be used, as done formally in Sec. II.

For the pulled case, $c^*=2\sqrt{F'(0)}$, so that both the numerator and the denominator of (2.17) diverge. This difficulty is removed if we consider the sequence mentioned above instead of the original system. In this case, all the members of the sequence are unambiguous equations (pushed cases), so the decay rate of the eigenfunction belonging to zero is much faster than $e^{-c^*\xi}$ near the tip. That is, we have a natural regularizing factor for the member of the sequence. Taking the limit is thus equivalent to computing the limit with the aid of l'Hôpital's rule,

$$\delta c = -\lim_{l\to\infty} \frac{\int_{-l}^{+l} d\xi\, e^{c_0\xi}\psi_0'(x)[\delta F\{\psi_0\}](\xi)}{\int_{-l}^{+l} d\xi\, e^{c_0\xi}\psi_0'^2(\xi)} . \tag{3.2}$$

Thus with the aid of the structural stability consideration, we can give the correct form (3.2) for the change of velocity and a sufficient condition for its validity: $\pm\delta F$ must be $p$-small and the right and left derivatives at the origin must exist.

## IV. AMPLITUDE EQUATIONS, SOLVABILITY, AND RG

The general formula (2.10) for the modification of velocity due to the external perturbations can also be constructed from a solvability condition, analogous to that occurring in the dynamics of defects and dislocations in nonequilibrium patterns [43–46]. We start with the perturbed equation

$$\frac{\partial\psi}{\partial t} = N\{\psi\} + \delta N\{\psi\} , \tag{4.1}$$

which is assumed to have a steady-state solution of the propagating front $\psi(x,t)=\psi(\xi)$, with $\xi=x-c_0t+x_0$. We further assume that the operator norm of $\delta N$ is of order $\epsilon$. Suppose that the perturbed velocity and solution can be written as $c=c_0+\delta c$ and $\psi=\psi_0+\delta\psi$, where $c_0$ and $\psi_0$ correspond to the unperturbed velocity and solution. Then to order $\epsilon$ we get

$$-\left[DN+c_0\frac{d}{d\xi}\right]\delta\psi=\delta c\frac{d\psi_0}{d\xi}+\delta N\{\psi_0\} . \qquad (4.2)$$

Because the linear operator on the left-hand side of Eq. (4.2) has a zero eigenvalue corresponding to an eigenfunction $u_0 \propto d\psi_0/d\xi$, the condition for the existence of a nontrivial solution to (4.2) is that the right-hand side should be orthogonal to the null space of the operator: $\tilde{\psi}_0 \equiv \rho(\xi)\psi_0=\rho(\xi)du_0/d\xi$, where we have included the appropriate weight function $\rho$,

$$\delta c(\rho\psi_0,\psi_0)+(\rho\psi_0,\delta N\{\psi_0\})=0 . \qquad (4.3)$$

We immediately recover the formula (2.10) for $\delta c$.

In particular, let us now consider a case in which the change in the speed $\delta c$ results from the change of some parameter $\gamma$ in the nonlinear operator $N(\psi,\gamma)$. If we assume that the traveling-wave solution exists for some $\gamma$, and that we can write $\delta N=(dN/d\gamma)\delta\gamma$, and further assume that the order of the differentiation and the integration can be exchanged, then from (2.10) we obtain the formally exact result

$$\frac{dc}{d\gamma}=-\frac{\int_{-\infty}^{+\infty}d\xi\, e^{c\xi}\psi_0'(\xi)dN/d\gamma}{\int_{-\infty}^{+\infty}d\xi\, e^{c\xi}\psi_0'^2(\xi)} . \qquad (4.4)$$

Unfortunately, we have not found a way to use this formula in cases where exact results are not already available.

The reader may well wonder why we make a fuss about the RG in Sec. III, given that the formula for $\delta c$ is obtained trivially from the solvability condition of (4.2). Our point is to demonstrate that the solvability conditions (order by order) and the perturbative renormalization procedure are equivalent. The use of solvability conditions is reminiscent of pattern selection in dendritic crystal growth phenomena [14], the derivation of amplitude equations, such as the time-dependent Ginzburg-Landau equation, or more generally the equations of motion on the slow or center manifold (in the loose sense of the word) [47]. Indeed, (2.14) is the equation of motion governing the slow motion relative to the original unperturbed front. It is actually a general feature that amplitude equations and slow motion equations are RG equations. These general features of the renormalization group approach are discussed more thoroughly elsewhere [23,38].

## V. EXAMPLES

We now apply the perturbation approach to a variety of front propagation problems.

### A. Generalized Fisher's equation

Consider the model

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+u(u-\mu)(1-u) , \quad 0\le\mu\le\tfrac{1}{2} . \qquad (5.1)$$

Here, we identify $F=u^2(1-u)$ and $\delta F=-\mu u(1-u)$ and regard $\mu$ as a small perturbation parameter. The equation is unambiguous for all $\mu\ge-\frac{1}{2}$. In particular, the unperturbed case is unambiguous and its unique propagation front shape is described by

$$u_0=\frac{1}{1+e^{1/\sqrt{2}\xi}} , \quad \xi=x-c_0t+x_0 , \qquad (5.2)$$

where $c^*\equiv c(\mu=0)=1/\sqrt{2}$ and $x_0$ is an arbitrary constant. From (2.17) we obtain $\delta c=-\sqrt{2}\mu$. Thus, the selected velocity of the perturbed equation, to $O(\mu)$, is $c^*=1/\sqrt{2}-\sqrt{2}\mu$, which is the same as the exact result from the variational method we will describe below. Presumably higher-order terms in $\mu$ will spoil this result.

### B. Van Saarloos' fourth order equation

Van Saarloos considered the following equation [8],

$$\frac{\partial\phi}{\partial t}=\frac{\partial^2\phi}{\partial x^2}-\gamma\frac{\partial^4\phi}{\partial x^4}+\frac{\phi}{b}(b+\phi)(1-\phi) , \qquad (5.3)$$

where $\gamma<\frac{1}{12}$ and $0\le b\le 1$. He numerically solved both this full time-dependent equation for $\psi(x,t)$ and the corresponding ordinary differential equation for the steady propagating wave state $\phi(\xi)$ in the moving frame with velocity $c$ for various values of $b$ and $\gamma$ [8]. For $\gamma=0.08$ and $b=0.1$, he observed that the velocity $c\simeq 2.715$.

Here we present analytical results obtained by treating the fourth-order term $-\gamma\partial^4\phi/\partial x^4$ as a small perturbation. The unperturbed equation has the propagating-front solution [8] $\phi_0(\xi)=(1+e^{\kappa\xi})^{-1}$, $\xi=x-c_0t+x_0$, where $\kappa=[c_0-\sqrt{c_0^2-4}]/2$, and $c_0=2$ for $\frac{1}{2}<b<1$; $\kappa=[c_0+\sqrt{c_0^2-4}]/2$, and $c_0=\sqrt{2b}+1/\sqrt{2b}$, for $0<b\le\frac{1}{2}$. The transition point from the pulled to the pushed case is $b=b_c=\frac{1}{2}$. It is worth mentioning that it is straightforward to obtain these results by applying the variational method presented in Sec. VI. From the formula (2.10) we obtain, for $0<b\le\frac{1}{2}$,

$$\delta c=-\gamma\kappa^3\left\{1-\left[2-\frac{c_0}{\kappa}\right]\left[\frac{7}{2}-\frac{9}{5}\left[3-\frac{c_0}{\kappa}\right]+\frac{1}{5}\left[3-\frac{c_0}{\kappa}\right]\left[4-\frac{c_0}{\kappa}\right]\right]\right\} , \qquad (5.4)$$

and $\delta c=-\gamma$ for $\frac{1}{2}<b<1$.

Note that the linear-marginal-stability velocity $c_l$ switches from $c_l=2$ to $c_l\approx 2-\gamma\approx 2\sqrt{1-\gamma}$ as $b$ crosses the value $\frac{1}{2}$. We find that the $O(\gamma)$ RG prediction agrees very well with the numerical calculation. For instance, for $\gamma=0.08$, we obtain $c=2.696+O(\gamma^2)$ for $b=0.1$, which is close to the numerical result $c\approx 2.715$ by van Saarloos [8].

### C. Newman's population equation

The second example which we consider is a modified porous medium equation which has been extensively discussed by Newman [48] in the contexts of population genetics and combustion. The equation can be represented as

$$\frac{\partial u}{\partial t}=\frac{1}{2}\frac{\partial}{\partial x}\left[u^n\frac{\partial u}{\partial x}\right]+u(1-u)\ ,\quad n\geq 0\ . \tag{5.5}$$

For $n=0$, it reduces to the Fisher-KPP equation and the propagation velocity is $c_0=\sqrt{2}$. For $n=1$, it has a unique traveling-wave solution,

$$\begin{aligned}&u_0(x,t)=(1-e^{\xi})\Theta(-\xi)\ ,\\&\xi=x-c_0t+x_0\ ,\\&c_0=\tfrac{1}{2}\ ,\end{aligned}\tag{5.6}$$

where $\Theta$ is the Heavyside function.

Numerical results by Newman [48] suggest that for any $n>0$, solutions evolve asymptotically into traveling-wave solutions with a unique velocity $c\approx(n+1)^{-1}$, depending only on the value of $n$.

To study this problem analytically, we write $n=1+\delta$ and perform a perturbative RG calculation, regarding $\delta$ as a small parameter. Expanding Eq. (5.5) in $\delta$, we obtain, to $O(\delta)$, the perturbation $\delta N\{u_0\}=(\delta/2)\partial_x(u_0\partial_x u_0\ln u_0)$. The weight function is $\rho(\xi)=e^{\xi}$ and the ground state is $\psi_0(\xi)=du_0/d\xi=-e^{\xi}$, $-\infty<\xi\leq 0$. Using formula (2.10) we have $\delta c=-(\frac{13}{48})\delta$, and the perturbed velocity to $O(\delta)$ is $c=\frac{1}{2}-(\frac{13}{48})\delta$. A consistent formula $c=[2+(\frac{13}{12})\delta+O(\delta^2)]^{-1}$ works remarkably well even for $\delta$ as large as 1. Setting $\delta=1$, i.e., $n=2$, we obtain $c\approx 0.3243$, which is in excellent agreement with the numerical result $c\approx 0.32$ obtained by Newman [48].

### D. Pulled case revisited

The renormalized perturbation theory result (2.10) can also be used to calculate heuristically the selected velocity of the unperturbed system by imposing the structural stability principle. Within the perturbation theory, a necessary and sufficient condition that $c^*$ be the selected speed is that $\delta c(c^*)$ must vanish as $\delta F$ vanishes. For concreteness, we consider the Fisher-KPP equation (2.1) with $F=\psi(1-\psi)$. It is found that the change in the velocity $\delta c(c)$ is zero as $\|\delta F\|\to 0$ for all perturbations $\delta F$, which are both $p$-small and differentiable at the origin, only for $c=c^*=2$; for $c>c^*$ there exist such perturbations for which $\delta c$ does not vanish as $\|\delta F\|\to 0$. A simple example of the latter is the perturbation $\delta F=\psi(1-\psi)-(\psi-\Delta)(1-\psi)\Theta(\psi-\Delta)$, where $\Theta$ is the step function, and we let $\Delta\to 0+$. The RG calculation shows that $\delta c\sim\sqrt{c^2-4}$ as $\Delta\to 0+$. Obviously, only for $c=c^*=2$, $\delta c$ goes to zero, but for $c>c^*=2$, $\delta c$ does not vanish. Therefore, only the wave with $c=c^*=2$ is structurally stable, and $c=2$ is identified as the selected velocity of the unperturbed system.

## VI. VARIATIONAL PRINCIPLES

In this section, we digress briefly to report results of L.-Y. Chen using the variational principle of Hadeler and Rothe [24]. With relatively little effort, useful estimates for front speeds and transition points are obtained.

We consider (2.1) in the variable $u(x,t)$, where the reaction term $F$ is continuously differentiable and satisfies the condition $F(0)=F(u_s)=0$ (where $u=0$ and $u_s$ are two steady-state solutions of the PDE); other generic conditions will be imposed later. Hadeler and Rothe's principle is that [24]

$$c_0=\inf_{\rho}\sup_{0\leq u\leq u_s}\left\{\rho'(u)+\frac{F(u)}{\rho(u)}\right\}\ , \tag{6.1}$$

where supremum means least upper bound, etc., ′ denotes differentiation with respect to the argument, and the function $\rho(u)\equiv-\partial_x u$ satisfies the conditions

$$\begin{aligned}&\rho(u)>0\ ,\quad 0<u<u_s\ ;\\&\rho(0)=\rho(u_s)=0\ ,\quad \rho'(0)>0\ ,\quad \rho'(u_s)<0\ .\end{aligned}\tag{6.2}$$

It was found empirically that if $F$ is of the form $F(u)=u(u_s^n-u^n)\Phi(u)$, where $\Phi(u)$ is some rational function which does not have zeros at $u=0$ or $u_s$, then the following choice very often gives the correct $c^*$:

$$\rho(u)=u(u_s^n-u^n)\psi(u,\{\kappa\})\ , \tag{6.3}$$

where $\psi$ is a function positive for $0\leq u\leq u_s$ and $\{\kappa\}$ is a set of variational parameters. Often one parameter $\kappa$, which is the scaling factor of $\rho$, is sufficient. Even when the choice fails to give the exact $c^*$, it gives a very good upper bound of $c^*$.

### A. Fisher's population model

We consider the equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+u(1-u)(1+\nu u^n)\ ,\qquad n\geq 1\ ,\quad -1\leq\nu<+\infty\ , \tag{6.4}$$

where $u=0,1$ are two steady-state points. When $n=1$, (2.7) reduces to the original Fisher's population model that Hadeler and Rothe discussed in Ref. [24]. They indirectly obtained correct results for $n=1$ by taking advantage of the known exact solution of the front profile, although they did not choose a proper trial function. Here we show that for $n>1$, the velocities and values of the parameter $\nu$ at the transition from pushed to pulled case are the same as in the $n=1$ case, although we do not know how to obtain the exact solution of the front profile in any $n>1$ case. In this sense, all $n\geq 1$ models belong to the same universality class, as far as the velocities and transition parameters are concerned. The numerical calculations which we have performed in several $n\geq 1$ cases by the shooting method are in excellent agreement with our analytical results and support the assertions above.

We chose

$$\rho(u)=\kappa u(1-u)\ , \tag{6.5}$$

where $\kappa>0$ is some adjustable parameter related to the decay rate of the front profile. If we define $g(u)\equiv\rho'(u)+F(u)/\rho(u)$, then

$$g(u)=\kappa+\frac{1}{\kappa}+\frac{\nu}{\kappa}u^n-2\kappa u \ . \tag{6.6}$$

First we consider the case $\nu>0$ and $n>1$, and study the behavior of the function $\phi(u)=\nu u^n/\kappa-2\kappa u$ with $\kappa>0$ fixed. It has an extremum at the point $u_0=(2\kappa^2/n\nu)^{1/n-1}$, which is determined by the condition $\phi'(u_0)=n\nu u_0^{n-1}/\kappa-2\kappa=0$, and further, since $\phi''(u_0)=n(n-1)\nu u_0^{n-2}/\kappa>0$, $\phi$ has a minimum at $u=u_0$. No matter whether $u_0$ lies in or out of $0\le u\le 1$, the supreme bound or maximum is always reached at either of the two boundary sides, i.e., $u=0$ or 1. Thus, we have

$$G(\kappa)=\max_{0\le u\le 1}\{g(u)\}=\kappa+\frac{1}{\kappa}+\max\left\{0,\frac{\nu}{\kappa}-2\kappa\right\} . \tag{6.7}$$

In the case of $n=1$ and $\nu>0$, we have $g(u)=\kappa+1/\kappa+(\nu/\kappa-2\kappa)u$, so we have the same result (6.7) as in the cases of $n>1$. There are two separate cases to consider below. If $\nu/\kappa-2\kappa\ge 0$, i.e., $\kappa\le\sqrt{\nu/2}$, then we have

$$G(\kappa)=\kappa+\frac{1}{\kappa}+\left[\frac{\nu}{\kappa}-2\kappa\right]$$
$$=\frac{1+\nu}{\kappa}-\kappa \ . \tag{6.8}$$

Since $G'(\kappa)=-(1+\nu)/\kappa^2-1<0$ for any $\kappa$, $G(\kappa)$ is a monotonically decreasing function of $\kappa$, and the minimum is attained at $\kappa=(\nu/2)^{1/2}$. Thus, in the interval $\nu\ge 2$, our estimate for the minimal velocity is

$$c_0=\min_{\kappa>0}\{G(\kappa)\}=\frac{2+\nu}{\sqrt{2\nu}} \ . \tag{6.9}$$

If $\nu/\kappa-2\kappa\le 0$, i.e., $\kappa\ge\sqrt{\nu/2}$, then we have

$$G(k)=\kappa+\frac{1}{\kappa}+0\ge 2 \ , \tag{6.10}$$

with equality only when $\kappa=1$. Also, we must satisfy the condition $1\ge\sqrt{\nu/2}$, i.e., $\nu\le 2$. Thus, the minimum velocity is

$$c_0=\min_{\kappa>0}\{G(\kappa)\}=2 \ . \tag{6.11}$$

Secondly, we consider the case of $-1\le\nu\le 0$ and $n\ge 1$. Because $\nu/\kappa-2\kappa$ is always negative for any $\kappa>0$, we have the same results as Eqs. (6.10) and (6.11). In summary, in all cases of $n\ge 1$, we have the following minimal velocities:

$$c_0=\begin{cases}2 & \text{for } -1\le\nu\le 2 \ ,\\ (2+\nu)/\sqrt{2\nu} & \text{for } \nu\ge 2 \ .\end{cases} \tag{6.12}$$

### B. Other examples

A second more complicated example is the partial differential equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+u+du^{n/2+1}-u^{n+1} \ , \quad n>0 \ . \tag{6.13}$$

For $n=4$, it reduces to an equation considered by van Saarloos [8]. By using the method of reduction of order of ODE, he obtained exact results for front velocities $c_0$ and the transition parameter value $d_c$:

$$c_0=\begin{cases}2 & \text{for } d\le d_c=2/\sqrt{3} \ ,\\ (-d+2\sqrt{d^2+4})/\sqrt{3} & \text{for } d\ge d_c \ .\end{cases} \tag{6.14}$$

The nonlinear source term in Eq. (6.13) can be written as

$$F(u)=u(u_s^{n/2}-u^{n/2})(u_r^{n/2}+u^{n/2}) \ , \tag{6.15}$$

where $u_s^{n/2}=[d+(d^2+4)^{1/2}]/2$ and $u_r^{n/2}=[-d+(d^2+4)^{1/2}]/2$. It always has an unstable steady state $u=0$ and an absolutely stable state $u=u_s>0$. For simplicity, we only consider the propagation front connecting the above two steady states.

In this case, we choose the trial function

$$\rho(u)=\kappa u(u_s^{n/2}-u^{n/2}) \ , \quad \kappa>0 \ , \quad 0\le u\le u_s \ , \tag{6.16}$$

but *not* $\rho(u)=\kappa u(u_s-u)$ as in the former example. As a result, we obtain the complete result

$$c_0=\begin{cases}2 & \text{for } d\le d_c\equiv n/\sqrt{2(n+2)} \ ,\\ [-nd+(n+2)\sqrt{d^2+4}]/2\sqrt{2(n+2)} \ , & \text{for } d\ge d_c \ .\end{cases} \tag{6.17}$$

As a check, when $n=4$, the result (6.14) by van Saarloos [8] is recovered.

We remark that in these examples, only when $g(u)$ reaches its maximum at $u=u_s>0$ with $\kappa$ fixed is it possible for a transition to the pushed case to occur, while when $g(u)$ reaches its maximum at $u=0$ it is the pulled case that occurs. We conjecture that this is typical. Based on this naive picture, we have also studied several other interesting cases.

(i) The Fisher-KPP equation [15], where $F(u)=u-u^n$, $n\ge 2$. If we choose $\rho(u)=\kappa u(1-u^{n-1})$, $0\le u\le 1$, we obtain $c_0=2$.

(ii) Schlogl's second model for chemical reactions [49], where $F(u)=\gamma-\beta u+3u^2-u^3=(1-u)(u-u_1)(u-u_2)$, $\beta=\gamma+2$, $0<\gamma<1$, $u_1=1+(1-\gamma)^{1/2}$, $u_2=1-(1-\gamma)^{1/2}$. We choose $\rho(u)=\kappa(u-1)(u_1-u)$, $1\le u\le u_1$, and find that the exact result $c_0=3\sqrt{(1-\gamma)/2}$ is obtained.

(iii) A generalized version of Fisher's model [24], where $F(u)=u(1-u)(u-\mu)$, $0\le u\le\frac{1}{2}$. If $\rho=\kappa u(1-u)$ is chosen, $c_0=1/\sqrt{2}-\sqrt{2}\mu$ is obtained. This result is exactly the same as the perturbative RG result we obtained in Sec. IV.

## VII. TRANSITION FROM PUSHED TO PULLED CASES

In this section, we consider how the transition point between pulled and pushed cases is changed when a *p*-small perturbation is present. Consider the following indicative example,

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+u(u_s-u)(u_r+u+\epsilon u^n)\ , \quad n\geq 0\ , \tag{7.1}$$

where $u_s=[d+\sqrt{d^2+4}]/2$, $u_r=[-d+\sqrt{d^2+4}]/2$, $d>0$, and $\epsilon$ is a perturbation parameter. For $\epsilon=0$ the transition from the pulled to the pushed case occurs [8] as $d$ is increased from zero at $d=d_c\equiv 1/\sqrt{2}$. We can apply the perturbation approach only for $d>d_c$, because we need the explicit formula for the unperturbed wave front: $u_0(\xi)=u_s(1+Ae^{\kappa\xi})^{-1}$, where $\kappa=[c_0+\sqrt{c_0^2-4}]/2$ and $c_0=[3\sqrt{d^2+4}-d]/2\sqrt{2}$, and $A$ is a constant of integration. For $d<d_c$, there is no known formula for the unperturbed wave front.

The zeroth eigenfunction is $u_0'(\xi)\propto -\kappa u_0(1-u_0/u_s)$ and the weight function $\rho(\xi)=(u_s/u_0-1)^{c_0/\kappa}$. To determine the new transition point $d_c$, we substitute into the formula (3.2) and obtain, to $O(\epsilon)$,

$$\delta c=\epsilon\frac{u_s^{n+1}}{\kappa}\frac{\Gamma(n+4)\Gamma(2-c_0/\kappa)}{3!\Gamma(n+2-c_0/\kappa)}\ . \tag{7.2}$$

We also have $\hat{c}=2$ for $n>0$ and $\hat{c}=2\sqrt{1+\epsilon u_s}$ for $n=0$. Setting $c^*(d_c)=\hat{c}(d_c)$, we find that

$$2\sqrt{1+\epsilon u_s(d_c)}=c_0(d_c)+\delta c(d_c)\ . \tag{7.3}$$

For $n=0$, we have $\delta c=\sqrt{2}\epsilon$ and

$$d_c=\frac{\sqrt{2+\epsilon^2}+3\epsilon}{2}\ , \tag{7.4}$$

while for $n=1$, we have $\delta c=\epsilon/2\sqrt{c_0^2-4}$ and

$$d_c=\frac{(1+3/2\epsilon)}{\sqrt{2}}\ . \tag{7.5}$$

For $n=2$, we have

$$\delta c=\epsilon/20u_s\left[c_0+3\sqrt{c_0^2-4}\right]\left[c_0-\sqrt{c_0^2-4}\right]\sqrt{c_0^2-4}\ . \tag{7.6}$$

We can also apply the variational methods to determine the velocity $c^*$ and transition point for any $n\geq 0$ and $d$. We choose the trial function

$$\rho(u)=\kappa u(u_s-u)\ , \quad 0\leq u\leq u_s\ , \quad \kappa>0\ . \tag{7.7}$$

For $n=0$, we find $d_c=[\sqrt{2+\epsilon^2}+3\epsilon]/2$ and

$$c=\begin{cases}2\sqrt{1+u_s\epsilon} & \text{for } d\leq d_c\ ,\\ c=c_0(d)+\sqrt{2}\epsilon & \text{for } d\geq d_c\ .\end{cases} \tag{7.8}$$

For $n=1$, we have the result $d_c=(1-\epsilon)/\sqrt{2(1+\epsilon)}$ and

$$c=\begin{cases}2 & \text{for } d\leq d_c\ ,\\ \sqrt{2/(1+\epsilon)}\sqrt{d^2+4}+(\epsilon-1)/\sqrt{2(\epsilon+1)}(d+\sqrt{d^2+4})/2 & \text{for } d\geq d_c\ .\end{cases} \tag{7.9}$$

If we expand the above results to $O(\epsilon)$, we find that they coincide with the results from perturbative RG. We suspect that the results from variational methods are actually the exact ones for $n=0$ and 1.

For $n=2$, we obtain

$$c=\begin{cases}2 & \text{for } d\leq d_c\ ,\\ \sqrt{2/(1+\epsilon)}(\sqrt{d^2+4}+\epsilon u_s^2)-\sqrt{(1+\epsilon u_s)/2}u_s & \text{for } d\geq d_c\ ,\end{cases} \tag{7.10}$$

where $d_c$ is the positive root of the equation $\epsilon u_s^3(d_c)+u_s^2(d_c)-2=0$. If we expand the velocities in (7.10) to $O(\epsilon)$, we observe that these results are slightly greater than those from perturbative RG and yet give good upper bounds. In fact, for $n\geq 2$ and small $\epsilon$, perturbative RG gives more accurate results on velocities and transition points than the trial function we used.

## VIII. CONCLUDING REMARKS

In this paper, we have shown that for structurally stable fronts, a renormalization group method can be used to compute the change in the front speed when the governing equation is perturbed by a marginal operator; further, by combining the structural stability principle with RG, we are able to predict the uniquely selected front itself. Our results apply to both the pulled and pushed cases. We demonstrated that the solvability condition widely used in studying pattern selection in nonequilibrium systems is identical to the physical renormalizability (observability) condition. We have also implemented a variational principle which gives very good upper bounds and sometimes exact results on front speeds, and which identifies the transition between the pulled and pushed cases.

In future work, we will investigate the application of these methods for systems where a spatial pattern forms behind the front.

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