Variational bounds on energy dissipation in incompressible flows: Shear flow

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A variational principle for upper bounds on the time averaged rate of viscous energy dissipation for Newtonian fluid flows is derived from the incompressible Navier-Stokes equations. When supplied with appropriate test "background" flow fields, the variational formulation produces explicit estimates for the energy dissipation rate. This dissipation rate is related to the drag of the fluid on the boundaries, and so these estimates translate into bounds on the drag. We analyze the problem of boundary-driven shear flow in detail, comparing the rigorous estimates obtained from the variational method with both recent experimental results and predictions of a conventional closure approximation from statistical turbulence theory.

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I. INTRODUCTION

One of the theoretical challenges presented by the problem of turbulence in fluid systems is to derive quantitative results directly from the equations of motion—the Navier-Stokes equations. In the absence of exact analytic solutions corresponding to turbulent flows, most theoretical approaches consist of approximate treatments of one sort or another, ranging from the imposition of statistical assumptions and moment hierarchy truncations, to the introduction of scaling hypotheses [1]. Rigorous results following directly from the primary model are thus important as checks on the validity of approximations, and for quantitative evaluation of the quality of predictions of secondary theories.

In this paper we focus on a specific physical quantity, the rate of viscous energy dissipation, and establish a framework for its practical, rigorous estimation directly from the equations of motion—the incompressible Navier-Stokes equations for a Newtonian fluid—for boundary-driven flows. Our approach is to derive variational principles for bounds on the time averaged energy dissipation rate, utilizing a decomposition that we refer to as the "background flow" method. The variational principles apply to both laminary and turbulent flows and, with appropriate modifications, to externally forced or thermally driven systems.

The energy dissipation rate in boundary-driven flows is of fundamental interest for applications because in a steady state situation it is the rate at which work must be done against viscous drag forces in order to enforce the boundary conditions. Bounds on the energy dissipation rate thus translate into bounds on the magnitude of the drag forces exerted by the fluid on the boundaries. For example, if a viscous fluid is sheared between parallel plates as illustrated in Fig. 1, then power must be expended by an external agent to maintain one plate in a state of motion with respect to the other. If the bottom plate is fixed and the top plate is moving uniformly at speed U, then the average rate of viscous energy dissipation in the fluid is the product FU, where F is the average drag force exerted by the fluid on the plate. Another fundamental example problem is sketched in Fig. 2. A sphere moving at constant speed U through a viscous medium must be subject to a force counteracting the net force of the fluid on its surface. On average in a steady state, the energy dissipation rate is just FU, where F is the average magnitude of the drag force.

One important point that is often ignored in theoretical work on problems like these is that *it is not generally known whether the primary model admits unique solutions, or if the solutions which do exist are smooth* [2]. That is, in three spatial dimensions it has not been ruled out that solutions of the incompressible Navier-Stokes equations can exhibit singularities within a finite time starting from arbitrarily smooth initial conditions. This mathematical problem has its roots in the physical

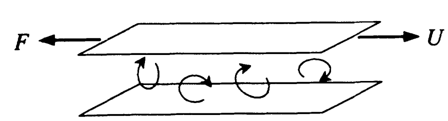

FIG. 1. Fluid is sheared between parallel plates. The velocity difference is maintained by a force opposing the viscous drag, and the average energy dissipation rate is the average power expended by this force, i.e., the product of the average force times the speed FU.

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FIG. 2. A sphere moving at speed U through a viscous fluid must have a force acting on it to maintain its motion. The energy dissipation rate in the fluid is, on average, the product of the average force times the speed FU.

phenomenon of vortex stretching, one of the fundamental mechanisms of turbulent dynamics, and this unresolved issue leaves open the question of the validity of these hydrodynamic equations in the turbulent regime; the macroscopic equations of motion are derived from microscopic considerations under the assumption that their solutions are free from singularities [3]. These singularities—if they do exist, which has also not been established—correspond to a violation of the hypothesis of a separation of scales between the microscopic dynamics and the macroscopic structures. These considerations mean that rigorous results, free from secondary hypotheses or additional uncontrolled approximations, are important for the evaluation of the basic model.

The rest of this paper is organized as follows. In the next section we formulate variational principles for bounds on time average energy dissipation rates for a class of flows driven by inhomogeneous boundary conditions. The analysis is general enough to encompass both closed, bounded systems and open, unbounded systems. The basis of the upper bound principle is a decomposition of the flow field into a "background" and a "fluctuation," reminiscent of, but distinct from, the Reynolds decomposition into mean and fluctuating components. Section II culminates with the statement of a minimization problem for upper bounds which, when supplied with appropriate trial background flows, can be used to derive explicit estimates. As an example application of the technique, in Sec. III we analyze the boundary-driven shear flow geometry illustrated in Fig. 1. Combined with elementary functional estimates, the variational approach yields explicit bounds on the energy dissipation rate and the drag, including the high Reynolds number turbulent drag for this problem.

The discussion in Sec. IV covers a number of points. We compare the rigorous results obtained for the shear-flow problem with both the predictions of a conventional statistical closure model and recent experimental results. This comparison leads naturally to questions of how the upper bounds might be lowered, and we present some possible directions for improvement, including a discussion of the Euler-Lagrange equations leading to the optimal estimates that this approach can yield. Open flow problems like those suggested in Fig. 2 present other challenges which we also outline in this section. We close the discussion by pointing out a connection between our upper bound variational principle and a heuristic marginal stability hypothesis for turbulent flows. For completeness and because many of the conventional models, approximations, and results of classical turbulence theory are unfamiliar to the physics community, we include an appendix on the elements of statistical turbulence theory along with a sample closure appropriate for wall-bounded turbulence. Finally, a brief appendix explaining one of the functional estimates (Poincare's inequality) is included.

This paper is the follow up to a previous short presentation of some of these results [4], and it is the first in a planned series of three papers developing variational bounds and the background flow method. In the subsequent paper we intend to apply the same general approach to a body-force driven problem, namely channel flow [5]. The third paper in the series is planned to deal with the problem of thermal convection, where bounds on the energy dissipation rate lead to estimates for the rate of convective heat transport [6].

II. VARIATIONAL PRINCIPLES FOR ENERGY DISSIPATION BOUNDS

Suppose an incompressible Newtonian fluid is confined to a region of space Ω with stationary boundary $\partial\Omega$. We denote the kinematic viscosity by ν, and without loss of generality mass units can be chosen so that the density $\rho=1$. The fluid's velocity vector field $\mathbf{u}(\mathbf{x},t)$ satisfies the Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\nu\Delta\mathbf{u} , \tag{2.1}$$

$$\nabla\cdot\mathbf{u}=0 , \tag{2.2}$$

where $p(\mathbf{x},t)$ is the pressure field, along with appropriate boundary conditions and square-integrable initial conditions $\mathbf{u}_0(\mathbf{x})$.

The region Ω may or may not be compact, and part of the boundary $\partial\Omega$ may correspond to rigid no-slip boundaries, boundaries at $|\mathbf{x}|=\infty$, and periodic conditions. We presume that the velocity field is prescribed on the part of $\partial\Omega$ corresponding to rigid boundaries, and in particular we take zero flux across those boundaries. That is, if $\mathbf{n}$ is the normal to the boundary at a rigid boundary, then

$$\mathbf{n}\cdot\mathbf{u}|_{\mathbf{x}\in\partial\Omega}=0 . \tag{2.3}$$

All components of the fluid velocity vector field are specified at $|\mathbf{x}|=\infty$, if applicable. For the purposes of this paper, we consider boundary and initial condition setups such that finite kinetic energy solutions exist (in some Galilean frame) and possess enough regularity for us to manipulate the equations of motion and perform operations like integrations by parts at will. We will comment on this assumption more in Sec. IV.

The instantaneous energy dissipation rate (per unit mass) in the fluid is defined as

$$\nu\|\nabla\mathbf{u}\|_2^2=\nu\sum_{i,j=1}^{d}\left\|\frac{\partial u_i}{\partial x_j}\right\|_2^2 , \tag{2.4}$$

where $\|f\|_2$ denotes the L_2 norm of a function $f(\mathbf{x})$ on Ω:

$$\|f\|_2=\left[\int_\Omega |f(\mathbf{x})|^2 d\mathbf{x}\right]^{1/2} . \tag{2.5}$$

We will be concerned with time averaged energy dissipation rates,

$$\langle \nu\|\nabla \mathbf{u}\|_2^2\rangle_T=\frac{1}{T}\int_0^T \nu\|\nabla \mathbf{u}(\cdot,t)\|_2^2 dt \ . \tag{2.6}$$

Note that a long-time limit of the finite-time average need not exist, even though finite-time averages may be bounded. Moreover, long-time averages need not be unique, for even if the limit $T\to\infty$ did exist, it would generally depend on the initial conditions.

A variational principle for lower bounds on $\langle \nu\|\nabla \mathbf{u}\|_2^2\rangle_T$ is easy to formulate:

Theorem 1 (lower bound principle). For every solution $\mathbf{u}(\mathbf{x},t)$ starting from every initial condition $\mathbf{u}_0(\mathbf{x})$,

$$\lim_{T\to\infty}\inf\langle \nu\|\nabla \mathbf{u}\|_2^2\rangle_T \ge \inf\{\nu\|\nabla \mathbf{U}\|_2^2 | \nabla\cdot\mathbf{U}=0 \ , \ \mathbf{U}(\mathbf{x}) \text{ satisfies } \mathbf{u}\text{'s boundary conditions}\} \ . \tag{2.7}$$

Proof. At each instant $t>0$ the solution $\mathbf{u}(\mathbf{x},t)$ is divergence free and satisfies the boundary conditions. Hence at each instant,

$$\nu\|\nabla \mathbf{u}(\cdot,t)\|_2^2 \ge \inf\{\nu\|\nabla \mathbf{U}\|_2^2 | \nabla\cdot\mathbf{U}=0 \ , \ \mathbf{U}(\mathbf{x}) \text{ satisfies } \mathbf{u}\text{'s boundary conditions}\} \ . \tag{2.8}$$

The result follows by taking the time average over $[0,T]$ and letting $T\to\infty$. ■

This theorem simply states that the smallest possible long-time average energy dissipation rate for a solution of the Navier-Stokes equations is at least as large as the smallest possible value consistent with the constraints imposed by the vanishing divergence and boundary conditions, disregarding the dynamical equations. Its utility comes from the fact that we may derive partial differential equations for the minimizing field. Indeed, the velocity vector field $\mathbf{U}(\mathbf{x})$ minimizing $\|\nabla \mathbf{U}\|_2$ subject to the constraints in Eq. (2.7) satisfies Stokes equations,

$$0=-\nu\Delta\mathbf{U}+\nabla P \ , \tag{2.9}$$

$$0=\nabla\cdot\mathbf{U} \ , \tag{2.10}$$

along with the same boundary conditions as $\mathbf{u}$ on $\partial\Omega$. This is because the Stokes system in Eqs. (2.9) and (2.10) are the Euler-Lagrange equations corresponding to the variational problem of minimizing the functional,

$$F\{\mathbf{U}\}=\int_\Omega \{\nu|\nabla\mathbf{U}|^2-2P\nabla\cdot\mathbf{U}\}d\mathbf{x} \ , \tag{2.11}$$

where $-2P(\mathbf{x})$ is the Lagrange multiplier for the $\nabla\cdot\mathbf{U}=0$ constraint.

Note that the Stokes equations are the linear part of the stationary Navier-Stokes equations for the problem at hand. Normally one expects a unique solution to such a linear elliptic system, so that it provides a direct approach to developing explicit lower bounds on the smallest possible long-time average rate of energy dissipation.

Formulating a variational principle for upper bounds on $\langle \nu\|\nabla \mathbf{u}\|_2^2\rangle_T$ is more involved. We have the following theorem:

Theorem 2 (upper bound principle). For every solution $\mathbf{u}(\mathbf{x},t)$ starting from every square integrable initial condition $\mathbf{u}_0(\mathbf{x})$,

$$\lim_{T\to\infty}\sup\langle \nu\|\nabla \mathbf{u}\|_2^2\rangle_T \le \inf\left\{\nu\|\nabla\mathbf{U}\|_2^2+\int_\Omega \mathbf{U}\cdot\nabla\mathbf{V}\cdot\mathbf{U}\,d\mathbf{x}\,\middle|\,\nabla\cdot\mathbf{U}=0 \ , \mathbf{U} \text{ satisfies the boundary } \textit{and} \text{ spectral conditions}\right\} , \tag{2.12}$$

where the vector field $\mathbf{V}(\mathbf{x})$ is the solution of the linear inhomogeneous system

$$0=-\nu\Delta\mathbf{V}+2(\nabla\mathbf{U})_{\text{sym}}\cdot\mathbf{V}+\nabla P+\mathbf{U}\cdot\nabla\mathbf{U} \ , \tag{2.13}$$

$$0=\nabla\cdot\mathbf{V} \ , \tag{2.14}$$

with vanishing boundary conditions on the rigid portions of $\partial\Omega$ and for $|\mathbf{x}|\to\infty$, and periodic boundary conditions on the periodic boundaries. Here, $(\nabla\mathbf{U})_{\text{sym}}$ is the symmetric part of the tensor $\nabla\mathbf{U}$. If $\mathbf{U}\cdot\nabla\mathbf{U}$ is not a gradient, the *spectral constraint* on $\mathbf{U}(\mathbf{x})$ is the condition that the self-adjoint operator acting on $\mathbf{V}$ in Eqs. (2.13) and (2.14) is positive and invertible. If $\mathbf{U}\cdot\nabla\mathbf{U}$ is a gradient the operator need only be non-negative, in which case the spectral constraint can be cast as an eigenvalue problem,

$$\lambda\mathbf{W}=-\nu\Delta\mathbf{W}+2(\nabla\mathbf{U})_{\text{sym}}\cdot\mathbf{W}+\nabla P \ , \tag{2.15}$$

$$0=\nabla\cdot\mathbf{W} \ , \tag{2.16}$$

with $\mathbf{W}$ satisfying the same vanishing or periodic boundary conditions as $\mathbf{V}$, and demanding that the eigenvalues satisfy $\lambda\ge 0$.

Proof. Suppose that $\mathbf{u}(\mathbf{x},t)$ satisfies the Navier-Stokes equations on Ω and the boundary conditions on $\partial\Omega$. Decompose $\mathbf{u}$ into the sum of square-integrable components according to

$$\mathbf{u}(\mathbf{x},t)=\mathbf{U}(\mathbf{x})+\mathbf{v}(\mathbf{x},t) \ , \tag{2.17}$$

$$\nabla\cdot\mathbf{U}=0=\nabla\cdot\mathbf{v} \ , \tag{2.18}$$

where the stationary divergence-free "background flow" $\mathbf{U}(\mathbf{x})$ satisfies $\mathbf{u}$'s boundary conditions and the divergence-free fluctuation $\mathbf{v}(\mathbf{x},t)$ satisfies the homogeneous version of $\mathbf{u}$'s boundary conditions, i.e., it vanishes on boundaries where $\mathbf{u}$ is specified and is periodic on $\mathbf{u}$'s periodic boundaries. The initial condition on the fluctuation field is $\mathbf{v}_0(\mathbf{x})=\mathbf{u}_0(\mathbf{x})-\mathbf{U}(\mathbf{x})$, and the background field is assumed to satisfy the spectral condition.

Inserting this decomposition into the Navier-Stokes equations and using the boundary conditions in the relevant integrations by parts, we find that the kinetic energy in the fluctuations evolves according to

$$\frac{d}{dt}\tfrac{1}{2}\|\mathbf{v}\|_2^2+\int_\Omega \mathbf{v}\cdot(\nabla\mathbf{U})_{\mathrm{sym}}\cdot\mathbf{v}\,d\mathbf{x}+\int_\Omega \mathbf{U}\cdot\nabla\mathbf{U}\cdot\mathbf{v}\,d\mathbf{x} = -\nu\|\nabla\mathbf{v}\|_2^2-\nu\int_\Omega \nabla\mathbf{v}:\nabla\mathbf{U}\,d\mathbf{x}\ , \qquad (2.19)$$

where $\nabla\mathbf{v}:\nabla\mathbf{U}$ means $\sum_{i,j} v_{i,j}U_{i,j}$. The instantaneous energy dissipation rate is expressed in terms of the background and fluctuation fields as

$$\nu\|\nabla\mathbf{u}\|_2^2=\nu\|\nabla\mathbf{v}\|_2^2+2\nu\int_\Omega \nabla\mathbf{v}:\nabla\mathbf{U}\,d\mathbf{x}+\nu\|\nabla\mathbf{U}\|_2^2\ . \qquad (2.20)$$

Equation (2.20) can be used to replace half of the first term and all of the second term on the right-hand side of Eq. (2.19), and the result rearranged into

$$\frac{d}{dt}\tfrac{1}{2}\|\mathbf{v}\|_2^2+\tfrac{1}{2}\nu\|\nabla\mathbf{u}\|_2^2=-\left\{\tfrac{1}{2}\nu\|\nabla\mathbf{v}\|_2^2+\int_\Omega \mathbf{v}\cdot(\nabla\mathbf{U})_{\mathrm{sym}}\cdot\mathbf{v}\,d\mathbf{x}+\int_\Omega \mathbf{U}\cdot\nabla\mathbf{U}\cdot\mathbf{v}\,d\mathbf{x}\right\}+\tfrac{1}{2}\nu\|\nabla\mathbf{U}\|_2^2\ . \qquad (2.21)$$

Now average Eq. (2.21) over time from $t=0$ to T:

$$\frac{1}{T}\|\mathbf{v}(\cdot,T)\|_2^2+\langle \nu\|\nabla\mathbf{u}\|_2^2\rangle_T=\nu\|\nabla\mathbf{U}\|_2^2-2\left\langle\int_\Omega\{\tfrac{1}{2}\nu|\nabla\mathbf{v}|_2^2+\mathbf{v}\cdot(\nabla\mathbf{U})_{\mathrm{sym}}\cdot\mathbf{v}+\mathbf{U}\cdot\nabla\mathbf{U}\cdot\mathbf{v}\}d\mathbf{x}\right\rangle_T+\frac{1}{T}\|\mathbf{v}_0\|_2^2\ . \qquad (2.22)$$

Dropping the first term on the left-hand side at the expense of the quality sign and taking $T\to\infty$,

$$\limsup_{T\to\infty}\langle \nu\|\nabla\mathbf{u}\|_2^2\rangle_T\le\nu\|\nabla\mathbf{U}\|_2^2-2\limsup_{T\to\infty}\left\langle\int_\Omega\{\tfrac{1}{2}\nu|\nabla\mathbf{v}|_2^2+\mathbf{v}\cdot(\nabla\mathbf{U})_{\mathrm{sym}}\cdot\mathbf{v}+\mathbf{U}\cdot\nabla\mathbf{U}\cdot\mathbf{v}\}d\mathbf{x}\right\rangle_T\ . \qquad (2.23)$$

The last term above is bounded uniformly in time:

$$\left\langle\int_\Omega\{\tfrac{1}{2}\nu|\nabla\mathbf{v}|_2^2+\mathbf{v}\cdot(\nabla\mathbf{U})_{\mathrm{sym}}\cdot\mathbf{v}+\mathbf{U}\cdot\nabla\mathbf{U}\cdot\mathbf{v}\}d\mathbf{x}\right\rangle_T\ge\inf_{\nabla\cdot\mathbf{w}=0}\int_\Omega\{\tfrac{1}{2}\nu|\nabla\mathbf{w}|_2^2+\mathbf{w}\cdot(\nabla\mathbf{U})_{\mathrm{sym}}\cdot\mathbf{w}+\mathbf{U}\cdot\nabla\mathbf{U}\cdot\mathbf{w}\}d\mathbf{x}\ , \qquad (2.24)$$

where the infimum is taken over divergence-free vector fields $\mathbf{w}(\mathbf{x})$ satisfying the fluctuation's boundary conditions. The minimizing vector field $\mathbf{V}(\mathbf{x})$ for the right-hand side of Eq. (2.24) satisfies the Euler-Lagrange equations,

$$0=-\nu\Delta\mathbf{V}+2(\nabla\mathbf{U})_{\mathrm{sym}}\cdot\mathbf{V}+\nabla P+\mathbf{U}\cdot\nabla\mathbf{U}\ , \qquad (2.25)$$

$$0=\nabla\cdot\mathbf{V}\ . \qquad (2.26)$$

Note that if $\mathbf{U}\cdot\nabla\mathbf{U}$ is a gradient, then it can be absorbed into the pressure term resulting in a homogeneous equation for $\mathbf{V}$, the solution of which is $\mathbf{V}\equiv 0$. If $\mathbf{U}\cdot\nabla\mathbf{U}$ is not a gradient, the spectral constraint is the invertibility condition guaranteeing that this extremizing vector field exists, and it is the convexity condition guaranteeing that the solution is indeed a minimum. Using the Euler-Lagrange equations for the minimizing field, we have

$$\left\langle\int_\Omega\{\tfrac{1}{2}\nu|\nabla\mathbf{v}|_2^2+\mathbf{v}\cdot(\nabla\mathbf{U})_{\mathrm{sym}}\cdot\mathbf{v}+\mathbf{U}\cdot\nabla\mathbf{U}\cdot\mathbf{v}\}d\mathbf{x}\right\rangle_T\ge\int_\Omega\{\tfrac{1}{2}\nu|\nabla\mathbf{V}|_2^2+\mathbf{V}\cdot(\nabla\mathbf{U})_{\mathrm{sym}}\cdot\mathbf{V}+\mathbf{U}\cdot\nabla\mathbf{U}\cdot\mathbf{V}\}d\mathbf{x} =\tfrac{1}{2}\int_\Omega \mathbf{U}\cdot\nabla\mathbf{U}\cdot\mathbf{V}\,d\mathbf{x}=-\tfrac{1}{2}\int_\Omega \mathbf{U}\cdot\nabla\mathbf{V}\cdot\mathbf{U}\,d\mathbf{x}\ , \qquad (2.27)$$

where we have exploited $\mathbf{V}$'s boundary conditions in the course of performing the integrations by parts in the last two steps. Putting together Eq. (2.23) and Eq. (2.27), we have shown that

$$\limsup_{T\to\infty}\langle \nu\|\nabla\mathbf{u}\|_2^2\rangle_T\le\nu\|\nabla\mathbf{U}\|_2^2+\int_\Omega \mathbf{U}\cdot\nabla\mathbf{V}\cdot\mathbf{U}\,d\mathbf{x}\ , \qquad (2.28)$$

which establishes the result. ■

We will introduce a Lagrange multiplier for the spectral constraint and derive the associated Euler-Lagrange equations in Sec. IV below, but we do not solve them here. However, even without solving the Euler-Lagrange equations this variational formulation is useful for establishing rigorous upper estimates. To derive such bounds we must produce a trial background field $\mathbf{U}(\mathbf{x})$ satisfying the boundary conditions and the spectral constraint, and—if $\mathbf{U}\cdot\nabla\mathbf{U}$ is not a gradient—solve the linear (typically nonconstant coefficient) inhomogeneous system in Eqs. (2.13) and (2.14) for $\mathbf{V}$. Ideally one solves the linear eigenvalue problem in Eqs. (2.15) and (2.16) to ensure the spectral constraint for a candidate trial background flow, but this is not always completely necessary. This process of manufacturing upper bounds is to be contrasted with that required to establish lower bounds. There we must find the exact minimizing velocity field by solving the Stokes system in Eqs. (2.9)–(2.10). In the next section we use these theorems, along with elementary functional methods, to derive explicit bounds for a particular example.

III. BOUNDARY-DRIVEN SHEAR FLOW

Consider the problem shown in Fig. 3. A fluid of viscosity ν is confined between parallel planes located at $z=0$ and $z=h$, and we take periodic boundary conditions on the intervals $[0,L_x]$ and $[0,L_y]$ in, respectively, the x and y directions. The velocity at the lower plate is zero, and along the upper plate it is $\mathbf{i}U$ ($\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors in the x, y, and z directions). Define the Reynolds number

$$R=\frac{Uh}{\nu}\ . \qquad (3.1)$$

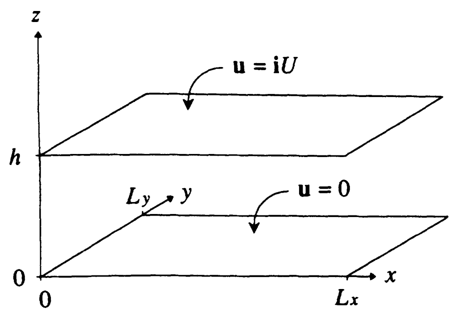

FIG. 3. Plates of dimension $L_x \times L_y$ are separated by gap h in the z direction. The plate at $z=0$ is stationary and the plate at $z=h$ is moving at speed U in the x direction. Boundary conditions are periodic in the x and y directions.

The solution of the Stokes equations for this setup is "planar Couette flow,"

$$\mathbf{U}(\mathbf{x})=\mathbf{i}\frac{U}{h}z \ , \tag{3.2}$$

$$P(\mathbf{x})=\text{const} \ . \tag{3.3}$$

The energy dissipation rate in this flow is

$$\nu\|\nabla\mathbf{U}\|_2^2=\nu\frac{U^2}{h^2}L_xL_yh \ , \tag{3.4}$$

and so for any solution of the Navier-Stokes equations with these boundary conditions,

$$\liminf_{T\to\infty}\langle \nu\|\nabla\mathbf{u}\|_2^2\rangle_T \ge \nu\frac{U^2}{h^2}L_xL_yh \ . \tag{3.5}$$

The force required to slide the top plate over the bottom lubricated by the fluid is the total energy dissipation rate per unit velocity, which is also the wall shear stress τ times the area of the plates. When the fluid is in the laminar state given by Eq. (3.2), the minimum force is

$$F_{\min}=\tau_{\min}L_xL_y=\frac{\nu UL_xL_y}{h} \ . \tag{3.6}$$

This is the familiar expression for the drag force from elementary physics texts, proportional to the product of the viscosity, the shear rate, and the contact area. It is useful to express this force in nondimensional terms. Measuring length, mass, and time in units of the geometric parameter h and the material parameters $\rho\equiv 1$ and ν, we have

$$\frac{h^2\tau_{\min}}{\nu^2}=\frac{Uh}{\nu}=R \ . \tag{3.7}$$

The Reynolds number is then precisely the measure of the minimum (long-time average) applied stress necessary for the maintenance of the boundary conditions.

In this example the solution of the Stokes equations satisfies both $\mathbf{U}\cdot\nabla\mathbf{U}=0$ and $\Delta\mathbf{U}=0$, so it is also a stationary solution of the Navier-Stokes equations for this geometry. At low Reynolds numbers this stationary linear velocity profile is actually nonlinearly stable [7], so that a unique long-time limit exists for all finite energy initial conditions. This situation is special because the solution of the Stokes equation for the lower bound on the energy dissipation rate is not generically a stationary solution of the Navier-Stokes equations. When it is, though, the minimizing flow may be realized and so the lower bound is sharp. When the minimizing solution of the Stokes equations is not a solution of the full nonlinear Navier-Stokes equations, the lower bound may never be obtained.

To produce an upper bound we must provide a divergence-free trial background flow satisfying the boundary and spectral conditions. The linear Couette flow profile in Eq. (3.2) satisfies the boundary conditions, and a natural place to start is to enquire whether the trial choice $\mathbf{U}(\mathbf{x})=\mathbf{i}Uz/h$ satisfies the spectral condition.

The usual Rayleigh-Ritz variational principle implies that the spectral constraint on $\mathbf{U}(\mathbf{x})$ is equivalent to the positivity of a certain functional. That is, the spectral constraint can be expressed as the condition

$$\int_0^{L_x}dx\int_0^{L_y}dy\int_0^{h}dz\{\tfrac{1}{2}\nu|\nabla\mathbf{v}|^2+\mathbf{v}\cdot(\nabla\mathbf{U})_{\text{sym}}\cdot\mathbf{v}\}>0 \tag{3.8}$$

for all divergence-free vector fields $\mathbf{v}(\mathbf{x})=\mathbf{i}v_x(x,y,z)+\mathbf{j}v_y(x,y,z)+\mathbf{k}v_z(x,y,z)$ periodic on, respectively, $[0,L_x]$ and $[0,L_y]$ in the x and y directions and satisfying the boundary conditions $\mathbf{v}(x,y,0)=0=\mathbf{v}(x,y,h)$. When $\mathbf{U}\cdot\nabla\mathbf{U}=0$, the $>$ sign in Eq. (3.8) can be replaced with a $\ge$ sign. For Couette flow,

$$(\nabla\mathbf{U})_{\text{sym}}=\frac{U}{2h}(\mathbf{ik}+\mathbf{ki}) \ , \tag{3.9}$$

so the condition is

$$\int_0^{L_x}dx\int_0^{L_y}dy\int_0^{h}dz\left[\tfrac{1}{2}\nu|\nabla\mathbf{v}|^2+\frac{U}{h}v_xv_z\right]\ge 0 \ . \tag{3.10}$$

The term $\sim|\nabla\mathbf{v}|^2$ in the integrand of the functional in Eq. (3.10) is manifestly positive, but the second term $\sim v_xv_z$ is of indefinite sign. So it will not be surprising that the positive term dominates at low Reynolds numbers, corresponding to large ν or small U. This expectation may be established formally by estimating the relative magnitudes of the two terms. According to the Schwarz inequality [8] and the relation $2ab\le a^2+b^2$,

$$\left|\int_0^{L_x}dx\int_0^{L_y}dy\int_0^{h}dz\, v_xv_z\right|\le\|v_x\|_2\|v_z\|_2\le\tfrac{1}{2}\|\mathbf{v}\|_2^2 \ . \tag{3.11}$$

And because each component of $\mathbf{v}$ is periodic in x and y and vanishes at $z=0$ and h, Poincarés inequality (Appendix B) implies

$$\|\mathbf{v}\|_2^2\le\frac{h^2}{\pi^2}\|\nabla\mathbf{v}\|_2^2 \ . \tag{3.12}$$

Then,

$$\int_0^{L_x} dx \int_0^{L_y} dy \int_0^h dz \left[\tfrac{1}{2}\nu |\nabla \mathbf{v}|^2 + \frac{U}{h} v_x v_z \right] \geq \frac{\nu}{2} \left[1 - \frac{R}{\pi^2} \right] \|\nabla \mathbf{v}\|_2^2 . \tag{3.13}$$

Thus if $R \leq \pi^2 \sim 10$, then the Couette profile satisfies the spectral condition and we can assert that its dissipation rate is an upper bound as well as a lower bound. When R is larger than this, then this argument no longer assures that the spectral constraint is satisfied by Couette flow and other profiles must be tried to derive an upper estimate. (We note that this crude bound, $R \leq \pi^2$, below which the laminar bound is realized is very conservative.)

The technical challenge for larger R is to choose a divergence-free background flow field that satisfies the boundary conditions at the plates, but which *also* satisfies the spectral constraint. Referring back to Eq. (3.8), it is apparent that this will be possible only if the $|\nabla \mathbf{v}|^2$ term dominates the $\mathbf{v}\cdot(\nabla \mathbf{U})_{\text{sym}}\cdot\mathbf{v}$ term for the relevant class of vector fields $\mathbf{v}(\mathbf{x})$. As suggested by the translation invariance of the problem in the x and y directions, we will restrict our attention to trial background flows of the form

$$\mathbf{U}(\mathbf{x}) = \mathbf{i}\phi(z) , \tag{3.14}$$

where the profile functions ϕ satisfies the boundary conditions $\phi(0)=0$ and $\phi(h)=U$. The spectral constraint is then expressed,

$$\int_0^{L_x} dx \int_0^{L_y} dy \int_0^h dz \left(\tfrac{1}{2}\nu |\nabla \mathbf{v}|^2 + \phi' v_x v_z \right) \geq 0 . \tag{3.15}$$

A profile with $\phi'=0$ would suffice, but then ϕ couldn't satisfy its boundary conditions. Keeping in mind the boundary conditions on $\mathbf{v}$, which demand v_x and v_z vanish at $z=0$ and $z=h$, we see that it may be possible to satisfy the spectral constraint with a choice of profile where the shear rate $\phi'(z)$ is small over most of the interval $[0,h]$, but where the necessarily nonvanishing values

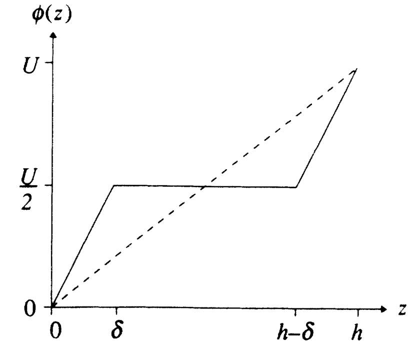

FIG. 4. Background flow profile $\phi(z)$. The straight line is the Couette flow profile.

of ϕ' are concentrated towards 0 and h where the components of $\mathbf{v}$ are relatively small. This hope may be realized in the background flow profile,

$$\phi(z) = \begin{cases} \dfrac{Uz}{2\delta} , & 0 \leq z \leq \delta \\ \dfrac{U}{2} , & \delta \leq z \leq h-\delta \\ \dfrac{U}{2\delta}(z-h+2\delta) , & h-\delta \leq z \leq h , \end{cases} \tag{3.16}$$

illustrated in Fig. 4. We refer to the parameter δ as the "boundary layer thickness." At a given value of R, δ will be adjusted to assure that the spectral constraint is satisfied. As will be shown below, the smaller δ is chosen, the more positive the functional in Eq. (3.15) will be, although at the expense of a poorer (larger) upper bound on the energy dissipation rate.

To see how this works, bound the $\phi' v_x v_z$ term in Eq. (3.15) as follows. Application of the fundamental theorem of calculus and the Schwarz inequality shows that the x-y integral of the product $v_x v_z$ is uniformly bounded in the interval $z \in [0,\delta]$ according to

$$\begin{aligned} \left| \int_0^{L_x} dx \int_0^{L_y} dy \; v_x(x,y,z) v_z(x,y,z) \right| &= \left| \int_0^{L_x} dx \int_0^{L_y} dy \left[\int_0^z dz' \frac{\partial v_x(x,y,z')}{\partial z} \right] \left[\int_0^z dz'' \frac{\partial v_z(x,y,z'')}{\partial z} \right] \right| \\ &\leq z \left[\int_0^{L_x} dx \int_0^{L_y} dy \int_0^\delta dz' \left[\frac{\partial v_x(x,y,z')}{\partial z} \right]^2 \right]^{1/2} \\ &\quad \times \left[\int_0^{L_x} dx \int_0^{L_y} dy \int_0^\delta dz' \left[\frac{\partial v_z(x,y,z')}{\partial z} \right]^2 \right]^{1/2} . \end{aligned} \tag{3.17}$$

An analogous estimate holds near the $z=h$ boundary. The $\int \phi' v_x v_z$ term is then simply estimated in terms of δ and the $\|\nabla \mathbf{v}\|_2^2$ term:

$$\begin{aligned} \left| \int_0^{L_x} dx \int_0^{L_y} dy \int_0^h dz \, \phi' v_x v_z \right| &= \frac{U}{2\delta} \left| \int_0^{L_x} dx \int_0^{L_y} dy \int_0^\delta dz \, v_x v_z + \int_0^{L_x} dx \int_0^{L_y} dy \int_{h-\delta}^h dz \, v_x v_z \right| \\ &\leq \frac{U\delta}{4} \left\{ \frac{1}{2\sqrt{2}} \left\| \frac{\partial v_x}{\partial z} \right\|_2^2 + \frac{\sqrt{2}}{2} \left\| \frac{\partial v_z}{\partial z} \right\|_2^2 \right\} \leq \frac{U\delta}{8\sqrt{2}} \|\nabla \mathbf{v}\|_2^2 . \end{aligned} \tag{3.18}$$

The incompressibility constraint on $\mathbf{v}$ was used in the last step above [9]. Thus,

$$\int_0^{L_x} dx \int_0^{L_y} dy \int_0^h dz \{ \tfrac{1}{2}\nu|\nabla \mathbf{v}|^2 + \phi' v_x v_z \} \geq \frac{\nu}{2}\left[1 - \frac{U\delta}{4\sqrt{2}\nu}\right] \|\nabla \mathbf{v}\|_2^2 \,, \tag{3.19}$$

and the boundary layer thickness may be adjusted so that the spectral condition is fulfilled by choosing

$$\delta = 4\sqrt{2}\frac{\nu}{U} = 4\sqrt{2}hR^{-1} \,. \tag{3.20}$$

For background flows of the form in Eq. (3.14), $\mathbf{U}\cdot\nabla\mathbf{U} \equiv 0$, and the solution for the auxiliary field $\mathbf{V}$ in Theorem 1 is $\mathbf{V} \equiv 0$. Thus the upper bound on the energy dissipation rate corresponding to the restriction in Eq. (3.20) is

$$\limsup_{T\to\infty} \langle \nu \|\nabla \mathbf{u}\|_2^2 \rangle_T \leq \nu \|\nabla \mathbf{U}\|_2^2 = \nu L_x L_y \int_0^h \phi'(z)^2 dz = \frac{1}{8\sqrt{2}} \frac{U^3}{h} L_x L_y h \,. \tag{3.21}$$

This is a rigorous upper bound on the energy dissipation, valid when $R \geq 8\sqrt{2}$ so that $\delta \leq h/2$. It is interesting for a number of reasons. First, it is *independent of the viscosity* in accord with Kolmogorov's scaling view of turbulent energy dissipation (this will be discussed further in Sec. IV). In addition, the prefactor $(8\sqrt{2})^{-1} \approx 0.088$ is substantially less than $O(1)$ hinting that this result is substantially more than a formalized dimensional analysis argument. Furthermore, our analysis, and in particular the background profile in Eq. (3.14), hints at the generic boundary layer form of high Reynolds numbers flows (see Appendix A). The spectral constraint compels us to consider the boundary layer structure for the trial background flow configurations.

The upper bound on the average drag force implied by Eq. (3.21) is $F \leq F_{\max}$, where

$$F_{\max} = \tau_{\max} L_x L_y = \frac{1}{8\sqrt{2}} U^2 L_x L_y \,, \tag{3.22}$$

and in nondimensional terms we have

$$\frac{h^2 \tau_{\max}}{\nu^2} = \frac{1}{8\sqrt{2}} \frac{U^2 h^2}{\nu^2} = \frac{1}{8\sqrt{2}} R^2 \,. \tag{3.23}$$

Summarizing the results of this section, we have proved that for the boundary-driven shear flow considered, the viscous drag $(h^2\tau/\nu^2)$ is bounded from above and below in terms of the Reynolds number according to

$$R \leq \frac{h^2\tau}{\nu^2} \leq \frac{1}{8\sqrt{2}} R^2 \,. \tag{3.24}$$

IV. DISCUSSION

It is of interest to see how the rigorous analytical bounds derived in the previous section compare with approximate theories and/or with experiments. A conventional statistical turbulence theory of high Reynolds number boundary-driven shear flow is briefly developed in Appendix A, yielding detailed predictions for the drag as a function of Reynolds number (modulo one fitting parameter that must be determined phenomenologically). In Fig. 5, we plot the rigorous upper and lower bounds derived in Sec. II along with the results of the closure approximation of Appendix A, with the customary value of the fitting parameter. The theory predicts the so-called "logarithmic friction law" for the drag with asymptotic behavior as $R \to \infty$,

$$\frac{h^2\tau}{\nu^2} \to \frac{\kappa^2}{4} \frac{R^2}{(\ln R)^2} \approx 0.04 \frac{R^2}{(\ln R)^2} \,, \tag{4.1}$$

where the von Kármán constant (the fitting parameter κ) has nominal value $\kappa \sim \frac{4}{10}$. This is to be compared with the upper bound derived here,

$$\frac{h^2\tau}{\nu^2} \leq \frac{1}{8\sqrt{2}} R^2 \approx 0.088 R^2 \,. \tag{4.2}$$

At high R the theory and the rigorous upper estimate have the same Reynolds number exponent and a comparable prefactor, but the approximate theory has additional logarithmic factors.

Both the prediction of the theory and the results of our rigorous analysis can be compared with recent high Reynolds number experiments on turbulent flow between concentric cylinders [10,11], at least in the limit of large aspect ratio and a narrow gap. (A calculation similar to that in Sec. III can be carried out in cylindrical coordinates appropriate for the concentric cylinder geometry, yielding the same asymptotic bound as in Eq. (4.2) [12].) The fit to the experimental data from Ref. [11] is also plotted in Fig. 5, showing that the logarithmic friction law fits the experimental data very well in the

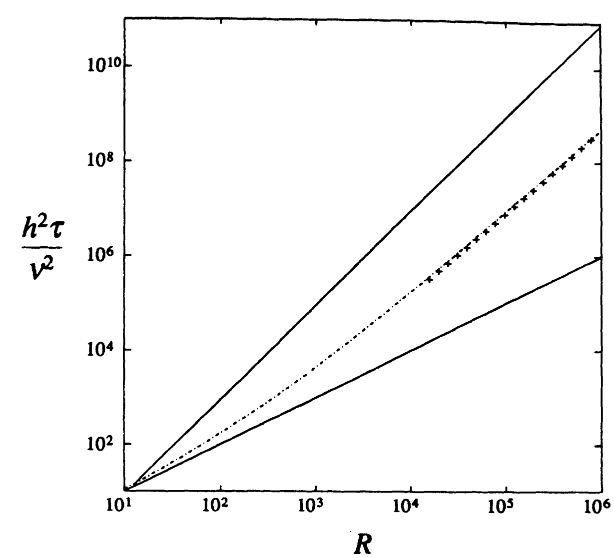

FIG. 5. Drag force vs Reynolds number. Solid lines are the upper and lower bounds from Eq. (3.24). The dashed line is the approximate theory prediction from Eqs. (A23) and (A24). The discrete points (+) are the high R fit to experimental data from Ref. [11].

$R \sim 10^4 - 10^6$ range. This leads to the reasonable conjecture that the empirical-approximate logarithmic friction law may be quantitatively capturing the asymptotic behavior. If this is the case, then the upper estimate in Eq. (4.2)—at least so far as the Reynolds number dependence is concerned—is sharp to within logarithms. Such results are encouraging, but it is natural to ask where there might be room for improvement in the estimates.

Ideally, the variational problem for the optimal background flow field in Theorem 2 will be solved exactly, yielding the best estimates that this method has to offer. In the application in Sec. III, however, two short cuts were taken in order to obtain explicit results. First, trial profiles were sampled from a very restricted class of functions, namely, simple piecewise linear functions of one variable as in Fig. 4. This was done primarily for analytical convenience. Second, the spectral condition on the profile was not explicitly verified. Rather, elementary and somewhat crude estimates were employed to ensure that the constraint was satisfied. In all likelihood the constraint is over satisfied resulting in an over estimate of the best bound. What is being neglected in the verification of the spectral constraint for the piecewise linear profiles in Sec. III is the divergence-free restriction on the functions in the domain of the operator in Eqs. (2.15) and (2.16) or, equivalently, in the domain of the functional in Eq. (3.15). To optimally check the spectral constraint for a given test background profile, the eigenvalue problem in Eqs. (2.15) and (2.16) should really be solved exactly, and the lowest eigenvalue should be determined as a functional of the test background profile ϕ. In the case of piecewise linear $\phi(z)$, the eigenvalue problem is a set of linear, piecewise-constant coefficient differential equations. These equations have been solved exactly in the two-dimensional situation ($u_y = 0 = v_y$, $\partial/\partial y = 0$) for the piecewise linear profiles, yielding the same R^2 power-law bound on the drag but with the prefactor reduced by more than an order of magnitude [13]. The three-dimensional case cannot be better than this, indicating that such a simple profile may not be so close to optimal as to yield logarithmic modifications to scaling.

The next step will be to derive the optimal profile from the variational principle. The spectral constraint leads to an interesting kind of variational problem for the optimal background flow. A simpler variational problem with such a spectral constraint (a scalar problem resulting from the relaxation of the divergence-free condition on the test functions for the spectral condition) has recently been solved exactly [14], yielding Euler-Lagrange equations of the form of the nonlinear Schrödinger equation. That "optimal" profile function has been computed analytically, and the resulting upper bound scales the same as in Eq. (4.2) with a slightly improved prefactor. We want to stress that *the extremum flow of the variational problem will not be, nor is it meant to be, a mean flow* in the usual sense of statistical turbulence theory.

The lesson learned from (i) enforcing the divergence-free condition in the spectral constraint exactly for a nonoptimal profile on the one hand, and (ii) finding the optimal profile while relaxing the divergence-free condition in the spectral constraint on the other hand, is that either approach yields the Kolmogorov-type scaling in Eq. (4.2). This indicates that qualitative improvement in the bounds, in the form of corrections to scaling, will require that we solve both problems simultaneously.

The Euler-Lagrange equations for the optimal profile are nonlinear and as yet unsolved. We illustrate their derivation now by considering (for simplicity) the example of two-dimensional flows in the x-z plane ($u_y = 0 = v_y$, $\partial/\partial y = 0$). Restricting attention to plane parallel background flows, the upper bound variational principle can be stated,

$$\limsup_{T\to\infty} \frac{1}{L_x h}\langle \nu \|\nabla \mathbf{u}\|_2^2\rangle_T \le \inf\left\{ \nu\frac{U^2}{h^2} + \frac{\nu}{h}\int_0^h \Phi(z)^2 dz \,\middle|\, \int_0^h \Phi(z)dz = 0\ ,\ \lambda_0\{\Phi\} \ge 0 \right\} , \tag{4.3}$$

where $\Phi(z)$ is the deviation of the background shear from the Couette profile and $\lambda_0\{\Phi\}$ is the lowest eigenvalue of the boundary value problem,

$$\lambda u = -\nu\Delta u + \frac{\partial p}{\partial x} + \left[\frac{U}{h} + \Phi(z)\right] w\ , \tag{4.4a}$$

$$\lambda w = -\nu\Delta w + \frac{\partial p}{\partial z} + \left[\frac{U}{h} + \Phi(z)\right] u\ , \tag{4.4b}$$

$$0 = \frac{\partial u}{dx} + \frac{\partial w}{\partial z}\ , \tag{4.4c}$$

where the components $u(x,z)$ and $w(x,z)$ vanish at $z=0$ and $z=h$, and are periodic in x. It is easy to see that the set of square-integrable mean-zero functions $\Phi(z)$ satisfying the spectral constraint $\lambda_0\{\Phi\} \ge 0$ is convex, so nonvanishing extrema will occur on the boundary of this set. Hence the inequality may be replaced by an equality, $\lambda_0\{\Phi\} = 0$, and the constraint may be imposed by the usual method of Lagrange multipliers. That is, we seek critical points of the functional,

$$F\{\Phi\} = \frac{\nu}{h}\int_0^h \Phi(z)^2 dz + \alpha\int_0^h \Phi(z)dz + \beta\lambda_0\{\Phi\}\ , \tag{4.5}$$

where α and β are the Lagrange multipliers. The Euler-Lagrange equations are simply

$$0 = \frac{\delta F}{\delta\Phi} = 2\frac{\nu}{h}\Phi + \alpha + \beta\frac{\delta\lambda_0}{\delta\Phi}\ . \tag{4.6}$$

Although the functional $\lambda_0\{\Phi\}$ cannot be expressed in any explicit form, the variation appearing in the Euler-Lagrange equations is the result of elementary (regular, nondegenerate) first-order perturbation theory. With an

e^{ikx} x dependence in u, w, and p, we may replace $\partial/\partial x$ with ik, eliminate u and p, and express everything in terms of $w(z)=e^{-ikx}w(x,z)$ and $w'=dw/dz$:

$$\frac{1}{k}\frac{\delta\lambda_0}{\delta\Phi(z)}=\frac{i[w(z)^*w'(z)-w(z)w'(z)^*]}{\int_0^h[|w'(z')|^2+k^2|w(z')|^2]dz'} . \qquad (4.7)$$

Hence the Euler-Lagrange equations in (4.6) express the shear profile Φ in terms of the $\lambda_0=0$ eigenfunctions, and the problem becomes one of solving the nonlinear boundary value problem

$$0=\nu\left[\frac{d^2}{dz^2}-k^2\right]^2 w+ik\frac{d}{dz}\left[\left[\frac{U}{h}+\Phi(z)\right]w\right]$$
$$+ik\left[\frac{U}{h}+\Phi(z)\right]\frac{dw}{dz} , \qquad (4.8a)$$

$$\Phi(z)=\alpha'\{1-i[w(z)^*w'(z)-w(z)w'(z)^*]\} , \qquad (4.8b)$$

where $w(0)=0=w(h)$, $w'(0)=0=w'(h)$, and α' is adjusted so that Φ has mean zero—that is, w is normalized according to

$$1=\int_0^h i[w(z)^*w'(z)-w(z)w'(z)^*]dz . \qquad (4.8c)$$

This is a nonlinear Orr-Sommerfield type equation (as arises in linear hydrodynamic stability; see Ref. [7]) for which a numerical approach has been developed [15], and which might be successfully tackled by the tools of matched asymptotic analysis as $R\to\infty$.

Technical considerations aside, a basic open question for the shear-flow problem is this: can we deduce the functional form of the logarithmic friction law rigorously, as an upper bound, directly from the incompressible Navier-Stokes equations? We hope that further development of the results in this paper will contribute to the resolution of this question.

It also remains a challenge to develop similar background flow decompositions and variational upper bounds for open systems like grid generated turbulence or for flow past a solid object. In the case of flow past a sphere, the solution of the Stokes equations for the lower bound yields the classical low Reynolds number formula for the "Stokes drag,"

$$F_{\min}=6\pi\nu Ur , \qquad (4.9)$$

where r is the radius of the sphere. At high Reynolds numbers we anticipate scaling of the turbulent drag similar to that found in Sec. III, i.e., $F\sim U^2r^2$, but *without* the apparent logarithmic corrections that arise in wall-bounded shear flows. This expectation has never yet been shown to follow rigorously from the incompressible Navier-Stokes equations.

There are a number of technical complications associated with application of Theorem 2's upper bound variational principle to this problem. First, it is not so simple to construct continuous divergence-free vector fields that satisfy the no-slip boundary conditions on the surface of the sphere *and* the spectral constraint. The well-known solution of the Stokes equations is one such field, but since it is not an exact solution of the Navier-Stokes equations unless $R=0$, it cannot generally produce an upper bound as well. We also know the relatively simple irrotational divergence-free vector field corresponding to an exact stationary solution of the Euler equations for this geometry, i.e., the Navier-Stokes equations with $R=\infty$. But the irrotational field doesn't satisfy the no-slip boundary conditions and so cannot be used at finite R. An appropriate candidate test field might be manufactured by some means, but then there remains the problem of checking the spectral constraint in an unbounded domain where the spectrum of the operator in Eqs. (2.15) and (2.16) might well be continuous. Furthermore, it will likely be the case that $\mathbf{U}\cdot\nabla\mathbf{U}$ is not a gradient so the auxiliary field $\mathbf{V}$ will have to be computed and included in the bound. These problems will be the object of future research.

The spectral constraint leads to consideration of the complete set of eigenfunctions of the associated eigenvalue problem. When the background profile is near optimal, this approach provides a novel way to generate a basis that is "adapted" to turbulent flow problems. It will be interesting to look at the structure of these flow fields, with the hope that elements of the turbulent dynamics may be illuminated in these coordinates.

Another variational approach to bounds on flow quantities, based on a decomposition into mean and fluctuation flows, was developed earlier this century [16]. The predictions of that method [17], both the Reynolds number scaling and the magnitudes of prefactors, are generally the same as those derived in this paper directly from the Navier-Stokes equations.

We derived the variational principle in Theorem 2 by manipulating the equations of motion and solutions without regard to questions of existence or regularity of solutions. Although it is not known if unique solutions exist, so-called weak solutions are available, which satisfy integrated versions of the energy evolution equation in the form of an inequality (see, for example, Ref [2]). This is sufficient to ensure that our analysis carries through, so we can assert that the variational upper bounds hold for the weak solutions of the Navier-Stokes equations.

A general observation about the variational principle for the upper bounds is that the spectral constraint can be considered a nonlinear stability condition on the background flow field. That is, $\mathbf{U}(\mathbf{x})$ satisfies the spectral constraint if and only if it is an appropriately nonlinearly stable stationary solution of the Navier-Stokes equations with the same boundary conditions, half the viscosity, and some applied body force. By "appropriately" nonlinearly stable we mean that the kinetic energy in any deviation decays in time at a rate uniform in the initial perturbation. We are thus hopeful that, via the background flow approach developed here, methods and results from nonlinear stability theory [7] can be taken over into more general studies of nonstationary and turbulent flows.

The upper bound variational principle derived in this paper can also be rephrased with an eye toward making a connection with an old argument in turbulence theory, namely a "marginal stability" hypothesis. The idea, which goes back at least to Malkus' work in convection [18], is that mean turbulent flows organize themselves into marginally stable configurations. The hypothesis that a well mixed turbulent core is bounded by thin laminar boundary layers whose thicknesses are determined by the condition of marginal stability leads to predictions for the global transport properties of the flow. (For the cylindrical Couette geometry, the prediction is a drag $\sim R^{5/3}$ dependence on the Reynolds number [19].) This is a completely heuristic principle, though, which has never been derived from the equations of motion. The following corollary of the upper bound variational principle, however, is suggestively similar in spirit to this marginal stability hypothesis.

Corollary (upper bound principle). Suppose $\mathbf{U}(\mathbf{x})$ is a stationary solution of the Euler equations [Eqs. (2.1) and (2.2) with vanishing viscosity] which (i) satisfies the boundary conditions for the Navier-Stokes problem, and (ii) is marginally nonlinearly stable *as if* it was a solution of the Navier-Stokes equations. Then, $\nu\|\nabla\mathbf{U}\|_2^2$ is an upper bound on the largest possible time-averaged energy dissipation rate for solutions of the Navier-Stokes equations.

Proof. When $\mathbf{U}$ is a solution of the Euler equations, $\mathbf{U}\cdot\nabla\mathbf{U}$ is a gradient. Then the auxiliary field $\mathbf{V}=0$ and the spectral constraint is precisely $\mathbf{U}$'s marginal stability, in the language of the energy method for nonlinear stability. ■

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APPENDIX A: STATISTICAL TURBULENCE THEORY AND THE LOGARITHMIC FRICTION LAW

Because of the effectively random behavior of turbulent flows it is very natural to attempt a statistical formulation, and this is a classical approach to turbulence theory. The idea is to decompose a turbulent velocity vector field into its mean and fluctuating parts (the "Reynolds decomposition") in an attempt to isolate and extract relevant averaged physical quantities. The "mean" in this approach may be a time average—appropriate for a steady configuration which, although fluctuating at all times, has well behaved time averaged characteristics—or an ensemble average where the average is over initial conditions in some class. For the purposes of this discussion we will consider a steady state turbulent flow and assume that time averages of all quantities exist.

Suppose that $\mathbf{u}(\mathbf{x},t)$ is a turbulent solution to the Navier-Stokes equations

$$\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\nu\Delta\mathbf{u}+\mathbf{f}\ , \tag{A1}$$

$$\nabla\cdot\mathbf{u}=0\ , \tag{A2}$$

with a time independent body force $\mathbf{f}(\mathbf{x})$, and for $\mathbf{x}\in\Omega$ with some specified time independent boundary conditions. We decompose $\mathbf{u}(\mathbf{x},t)$ as

$$\mathbf{u}(\mathbf{x},t)=\mathbf{U}(\mathbf{x})+\mathbf{v}(\mathbf{x},t)\ , \tag{A3}$$

where $\mathbf{U}$ is the time average of the velocity field,

$$\mathbf{U}(\mathbf{x})=\langle\mathbf{u}(\mathbf{x},\cdot)\rangle=\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau\mathbf{u}(\mathbf{x},t)dt\ , \tag{A4}$$

and $\mathbf{v}(\mathbf{x},t)$ is the time dependent fluctuating component satisfying

$$\langle\mathbf{v}(\mathbf{x},\cdot)\rangle=0\ . \tag{A5}$$

The mean flow $\mathbf{U}$ satisfies the boundary conditions that $\mathbf{u}$ satisfies, while the fluctuation field $\mathbf{v}$ satisfies the homogeneous version of the boundary conditions. That is, if a component of $\mathbf{u}$ is specified on the boundary then the corresponding component of $\mathbf{v}$ vanishes there, if the normal derivative of a component of $\mathbf{u}$ is given then the normal derivative of that component of $\mathbf{v}$ vanishes on the boundary, and if $\mathbf{u}$ is periodic in some direction then so is $\mathbf{v}$.

Suppose that time averages of time derivatives vanish and that time averaging commutes with spatial derivative operations. Then, taking the time average of Eqs. (A1) and (A2) we find

$$\mathbf{U}\cdot\nabla\mathbf{U}+\langle\mathbf{v}\cdot\nabla\mathbf{v}\rangle+\nabla P=\nu\Delta\mathbf{U}+\mathbf{f}\ , \tag{A6}$$

$$\nabla\cdot\mathbf{U}=0\ , \tag{A7}$$

where the mean pressure is

$$P(\mathbf{x})=\langle p(\mathbf{x},\cdot)\rangle\ . \tag{A8}$$

Subtracting Eq. (A6) from Eq. (A1) and Eq. (A7) from Eq. (A2) leads to the equations of motion for the fluctuations:

$$\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}+\mathbf{U}\cdot\nabla\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{U}-\langle\mathbf{v}\cdot\nabla\mathbf{v}\rangle+\nabla(p-P)=\nu\Delta\mathbf{v}\ , \tag{A9}$$

$$\nabla\cdot\mathbf{v}=0\ . \tag{A10}$$

The stationary Navier-Stokes-like equations satisfied by the mean flow in Eqs. (A6) and (A7) include an additional effective force ($\langle\mathbf{v}\cdot\nabla\mathbf{v}\rangle$) from the turbulent fluctuations. Noting $\mathbf{v}$'s vanishing divergence, the fluctuation force may be written as the divergence of a tensor, so the full stress tensor balancing the body force in Eq. (A6) is

$$S_{ij}^{(\text{total})}=-\delta_{ij}P+\nu\left\{\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right\}+\langle v_iv_j\rangle\ . \tag{A11}$$

Turbulence gives rise to an additional stress driving the mean flow $\langle v_iv_j\rangle$, known as the *Reynolds stress.* In Eqs. (A9) and (A10) we observe that the fluctuations themselves are driven by the mean flow and pressure fields, rather than being driven directly by the body force or by inhomogeneous boundary conditions.

A hierarchy of equations relating the mean flow to various time averaged moments of the fluctuations may be derived. In order to solve the stationary (albeit nonlinear) problem for **U**, the Reynolds stress $\langle v_i v_j \rangle$ must be supplied. A stationary equation for the Reynolds stress may be derived, but it will involve higher correlation functions such as $\langle v_i v_j v_k \rangle$. This problem continues and the hierarchy never closes: at each stage another function from a higher stage is required. This is the essence of the so-called *closure problem* in turbulence theory, a familiar quandry in nonlinear statistical physics. To make any progress some truncation of the hierarchy must be introduced by hand, and such closure approximations can be considered analogous to mean field theories in statistical mechanics.

Most efforts have concentrated on developing closures at the level of the Reynolds stress because, just as a Newtonian fluid is characterized by a simple relationship between the shear stress and the rate of strain, so too would one hope that nature strives to realize a simple relationship between the Reynolds stress and the mean velocity's strain rate. This is even more compelling considering that it is the mean's rate of strain tensor which, on average, supplies the energy to the turbulent fluctuations. To see this, note that for either periodic or rigid boundary conditions the turbulent kinetic energy evolves according to

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{v}\|_2^2 = -\nu\|\nabla\mathbf{v}\|_2^2 - \frac{1}{2}\int_\Omega v_i \left\{\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right\} v_j \, d^d x + \int_\Omega \frac{\partial v_i}{\partial x_j}\langle v_i v_j \rangle d^d x \ . \qquad \text{(A12)}$$

On average in a steady state the first and last terms vanish, the remaining balance being between the rate of energy supply to the turbulent fluctuations and the rate of viscous energy dissipation by the turbulent fluctuations. Viscosity dissipates turbulent kinetic energy, so there must be a significant (negative, in fact) spatial correlation between $\langle v_i v_j \rangle$ and the mean's shear rate:

$$\int_\Omega \left\{\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right\} \langle v_i v_j \rangle d^d x = -2\nu \langle \|\nabla\mathbf{v}\|_2^2 \rangle < 0 \ . \qquad \text{(A13)}$$

A particular closure scheme can be applied to the problem of shear-driven flow between parallel plates, i.e., for the setup in Fig. 3. At high Reynolds numbers a turbulent state is presumed to be realized, accompanied by a nonlinear mean flow profile and a Reynolds number dependence for the drag distinct from those relevant to the laminar Couette flow. The equations for the mean flow $\mathbf{U}(\mathbf{x})$ are Eqs. (A6) and (A7) with the boundary conditions

$$\mathbf{U}(x,y,0) = 0 \ , \quad \mathbf{U}(x,y,h) = \mathbf{i}U \ . \qquad \text{(A14)}$$

In order to proceed, some assumptions must be made.

First we impose the symmetry of the geometry on the mean flow. Translation invariance in the x and y directions imply that the mean flow field, the mean pressure, and the Reynolds stress are functions of z alone. The divergence-free condition combined with the boundary conditions then immediately give $U_z = 0$. Homogeneity and isotropy in the y direction imply $U_y = 0$. The remaining component is $U_x(z)$, which when translated by $U/2$ should be antisymmetric about the middle of the gap at $z = h/2$. That is, alternative boundary conditions for the mean profile between $z = 0$ and the middle of the gap at $z = h/2$ are $U(0) = 0$ and $U_x(h/2) = U/2$. Using these symmetry assumptions, the x component of Eq. (A6) becomes

$$\frac{d}{dz}\langle v_x v_z \rangle = \nu \frac{d^2 U_x(z)}{dz^2} \ . \qquad \text{(A15)}$$

Integrating up from $z = 0$ and using the vanishing boundary conditions on the fluctuations $[v_x(x,y,0,t) = 0 = v_z(x,y,0,t)]$ we obtain

$$\langle v_x v_z \rangle(z) = \nu U_x'(z) - \nu U_x'(0) \ , \qquad \text{(A16)}$$

where the prime denotes derivative with respect to z. The last term above is proportional to the mean wall shear stress,

$$\langle \tau \rangle = \nu U_x'(0) \ , \qquad \text{(A17)}$$

which is to be determined.

Next, a closure must be introduced. Along the general lines discussed earlier we inject a functional relationship between the Reynolds stress and the mean shear. By analogy with viscous shear stresses we assume a relation

$$\langle v_x v_z \rangle(z) = -\mu(z) U_x'(z) \ , \qquad \text{(A18)}$$

introducing the proportionality factor $\mu(z)$, with units of viscosity $[(\text{length})^2 \times (\text{time})^{-1}]$, known as the *eddy viscosity*. The eddy viscosity should be positive: the minus sign in Eq. (A18) is dictated by Eq. (A13) which requires that $U_x' \langle v_x v_z \rangle$ be negative on average.

The eddy viscosity is supposed to be a property of the flow field and not the fluid itself, so we should construct it out of local quantities determined by the mean flow and the geometry, *not* out of material parameters. We choose the natural inverse time scale given by the mean shear rate $U_x'(z)$, and the natural length scale—known as the *mixing length*— given by z, the distance to the rigid wall. The eddy viscosity is then defined,

$$\mu(z) = \kappa^2 z^2 U_x'(z) \ , \qquad \text{(A19)}$$

where the dimensionless absolute constant κ is a fitting parameter in this theory, known as the *von Kármán constant*. The von Kármán constant must be determined empirically, and it has a nominal value $\kappa \approx 0.40$. The closure that we make is thus [20]

$$\langle v_x v_z \rangle(z) = -\kappa^2 z^2 U_x'(z)^2 \ . \qquad \text{(A20)}$$

Note that this closure assumption is to be used in the lower half of the flow where $0 \le z \le h/2$. For z between $h/2$ and h, the mixing length z should be replaced by $h - z$.

For $0 \le z \le h/2$, Eq. (A16) becomes

$$0 = \kappa^2 z^2 U_x'(z)^2 + \nu U_x'(z) - u_*^2 \ , \qquad \text{(A21)}$$

where u_* is the velocity scale defined by the (as yet unknown) wall shear stress:

$$u_* = \sqrt{\langle\tau\rangle} = \sqrt{\nu U_x'(0)}\ . \tag{A22}$$

Solving the quadratic equation in Eq. (A21) for U_x' as a function of z and integrating up from $z=0$, we find the flow profile

$$\frac{\kappa}{u_*}U_x(z) = \ln\left\{\frac{2\kappa u_* z}{\nu} + \left[1+\left(\frac{2\kappa u_* z}{\nu}\right)^2\right]^{1/2}\right\} + \frac{1-\left[1+\left(\frac{2\kappa u_* z}{\nu}\right)^2\right]^{1/2}}{\frac{2\kappa u_* z}{\nu}}\ . \tag{A23}$$

The velocity scale u_* is fixed by the boundary condition $U_x(h/2)=U/2$:

$$\frac{\kappa}{2}\frac{U}{u_*} = \ln\left\{\kappa\frac{u_*}{U}R + \left[1+\left(\kappa\frac{u_*}{U}R\right)^2\right]^{1/2}\right\} + \frac{1-\left[1+\left(\kappa\frac{u_*}{U}R\right)^2\right]^{1/2}}{\kappa\frac{u_*}{U}R}\ . \tag{A24}$$

For a given Reynolds number, Eq. (A24) is to be solved for u_* as a function of U and R. Then Eqs. (A23) and (A22) yield explicit predictions for the mean flow profile and the wall shear stress. A typical profile at moderately high Reynolds number is sketched in Fig. 6. The shear in the mean flow is concentrated in thin layers near the rigid boundaries, and at large Reynolds numbers these boundary layers have thickness $\delta \sim \nu/u_*$.

In the limit $R \to 0$, the solution of Eq. (A24) is

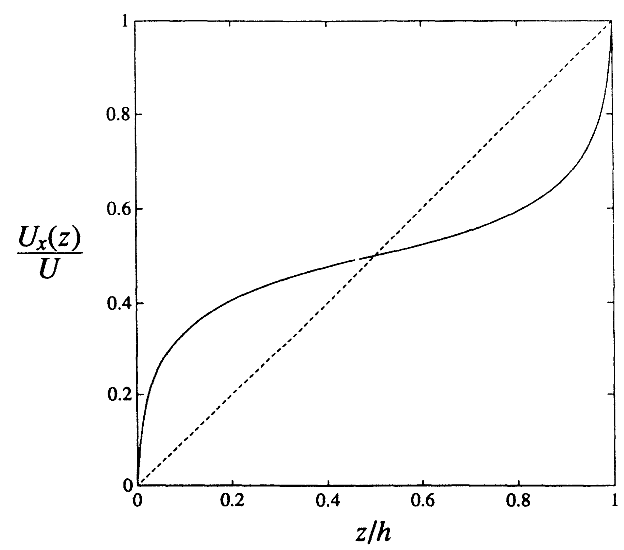

FIG. 6. Mean flow profile from the closure theory (solid line) for $R=10^4$. Note the boundary layer structure as compared to the background flow profile in Fig. 4. The dashed line is the laminar Couette flow profile.

$$u_* \to \frac{U}{\sqrt{R}}\ , \tag{A25}$$

yielding precisely the laminar stress in Eq. (3.7). The fitting parameter κ does not enter into this limit of the model. In the opposite limit of $R \to \infty$, Eq. (A24) predicts

$$u_* \to \frac{\kappa}{2}\frac{U}{\ln R}\ , \tag{A26}$$

and the nondimensional drag force

$$\frac{h^2\tau}{\rho\nu^2} \to \frac{\kappa^2}{4}\frac{R^2}{(\ln R)^2}\ . \tag{A27}$$

Asymptotically at high Reynolds numbers, this theory predicts a turbulent drag force proportional to the square of the speed U with logarithmic corrections. The energy dissipation rate is not strictly independent of the viscosity in this theory. The prediction is that it vanishes proportional to $[\ln(1/\nu)]^{-2}$ as $\nu \to 0$, all other parameters being held fixed.

This example has been developed for illustrative purposes with a minimum number of fitting parameters (just one), but it reproduces the essential features of the conventional wisdom, both theoretical and experimental, concerning the structure of the mean flow and the turbulent drag in wall-bounded shear flows. The introduction of an eddy viscosity, a mixing length, and a closure as in Eq. (A20), though, constitute completely uncontrolled approximations which do not follow systematically from the Navier-Stokes equations.

APPENDIX B: POINCARÉ'S INEQUALITY

Suppose Ω is a set for which the negative Laplacian $-\Delta$, along with boundary conditions, is a strictly positive self-adjoint operator with a discrete spectrum and smallest eigenvalue $\lambda_1 > 0$. Suppose further than $f(\mathbf{x})$ and its gradient $\nabla f(\mathbf{x})$ are square integrable on a set Ω and that $f(\mathbf{x})$ satisfies boundary conditions (typically Dirichlet or Neumann conditions) that allow for the integration by parts,

$$\int_\Omega f(\mathbf{x})^*[-\Delta f(\mathbf{x})]d^dx = \|\nabla f(\mathbf{x})\|_2^2\ . \tag{B1}$$

Poincaré's inequality asserts that

$$\|f(\mathbf{x})\|_2^2 \le \frac{1}{\lambda_1}\|\nabla f(\mathbf{x})\|_2^2\ . \tag{B2}$$

To see this, let $\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots$ be the complete orthonormal basis of eigenfunctions of $-\Delta$ with the boundary conditions on Ω, and $0<\lambda_1 \le \lambda_2 \le \ldots$ be the corresponding set of eigenvalues. Then the spectral decomposition of f is

$$f(\mathbf{x}) = \sum_{n=1}^{\infty} f_n \phi_n(\mathbf{x})\ ,\quad f_n = \int_\Omega \phi_n(\mathbf{x})^* f(\mathbf{x}) d^dx\ . \tag{B3}$$

According to Parseval's theorem,

$$\|f(\mathbf{x})\|_2^2 = \sum_{n=1}^{\infty} |f_n|^2 \tag{B4}$$

and

$$\|\nabla f(\mathbf{x})\|_2^2 = \sum_{n=1}^{\infty} \lambda_n |f_n|^2 . \tag{B5}$$

Thus, because $\lambda_n/\lambda_1 \geq 1$ for all n,

$$\|f(\mathbf{x})\|_2^2 = \sum_{n=1}^{\infty} |f_n|^2 \leq \sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda_1} |f_n|^2 = \frac{1}{\lambda_1} \|\nabla f(\mathbf{x})\|_2^2 . \tag{B6}$$

[1] H. Tennekes and J. L. Lumley, *A First Course in Turbulence* (MIT Press, Cambridge, MA, 1972).

[2] P. Constantin and C. Foias, *The Navier-Stokes Equations* (University of Chicago Press, Chicago, 1988).

[3] C. Bardos, F. Golse, and C. D. Levermore, J. Stat. Phys. **63**, 323 (1991).

[4] C. R. Doering and P. Constantin, Phys. Rev. Lett. **69**, 1648 (1992).

[5] P. Constantin and C. R. Doering (unpublished).

[6] P. Constantin and C. R. Doering (unpublished).

[7] P. G. Drazin and W. H. Reid, *Hydrodynamic Stability* (Cambridge University Press, Cambridge, England, 1981); B. Straughn, *The Energy Method, Stability and Nonlinear Convection* (Springer-Verlag, Berlin, 1992).

[8] The Schwarz inequality states that $|\int fg| \leq \|f\|_2 \|g\|_2$.

[9] $\nabla\cdot\mathbf{v}=0$ implies that $\|\partial v_z/\partial z\|_2^2 \leq \frac{1}{2}\{\|\partial v_z/\partial z\|_2^2 + \|\partial v_x/\partial x\|_2^2 + \|\partial v_y/\partial y\|_2^2 + \|\partial v_x/\partial y\|_2^2 + \|\partial v_y/\partial x\|_2^2\}$.

[10] D. P. Lathrop, J. Fineberg, and H. L. Swinney, Phys. Rev. Lett. **68**, 1515 (1992).

[11] D. P. Lathrop, J. Fineberg, and H. L. Swinney, Phys. Rev. A **46**, 6390 (1992).

[12] P. Constantin, SIAM Rev. **36**, 73 (1994).

[13] C. R. Doering and M. Orwoll (unpublished).

[14] P. Constantin and C. R. Doering (unpublished).

[15] P. Constantin, C. R. Doering, and D. Lohse (unpublished).

[16] For a review, see L. N. Howard, Annu. Rev. Fluid Dyn. **4**, 473 (1972).

[17] E. C. Nickerson, J. Fluid Mech. **38**, 807 (1969); F. H. Busse, *ibid.* **41**, 219 (1970).

[18] W. V. R. Malkus, Proc. R. Soc. London Ser. A **225**, 185 (1954).

[19] P. S. Marcus, J. Fluid Mech. **146**, 65 (1984).

[20] L. Prandtl, Angew. Math. Mech. **5**, 136 (1925).