Fluctuations in a decentralized agent-resource system

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In this paper, a general formulation for a model of market-based agent-resource systems is proposed, building on the computational ecosystem model of Huberman and Hogg [in *The Ecology of Computation*, edited by B. A. Huberman (North-Holland, Amsterdam, 1988)]. Our approach, based on one-step Markov processes and Van Kampen's large-system-size expansion, allows the effects of fluctuations to be explored, particularly in finite systems where stochastic processes assume increasing importance.

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I. INTRODUCTION

With the increasing complexity of telecommunication and computational systems, an urgent requirement is developing for theoretical frameworks for addressing basic principles of distributed systems [1]. At present there is insufficient understanding of principles required to predict performance, to explain behavior, and to establish design methodologies. There are also insufficient frameworks to describe such *distributed* systems [2]. The substantial vacuum in theoretical bases for distributed communication and computational systems stems largely from the historical preoccupation of computer and telecommunication science with uniprocessor systems [3].

The emergence of large decentralized systems is giving rise to the need for a general theoretic guide to the behavior of large collections of locally controlled, asynchronous, and concurrent processes interacting with an unpredictable environment. In particular this requires understanding the relation between the overall behavior of the computational system and that of its constituents, whose decisions are based upon local, imperfect, delayed, and conflicting information. In many other systems, particularly in nature and societies, distributed systems with very complex behavior and modes of operation have evolved. There is a growing awareness that many of the theoretical tools which have been developed with considerable success to describe physical systems [4–7], particularly in condensed matter physics, may be exploited in other fields, such as biology and economics [8–11].

An economic approach to open communications and computational systems [1] has some intuitive appeal since market mechanisms have evolved to facilitate resource management in social systems and appear on the whole to work reasonably well under certain conditions. In the case of converging communication, computational, broadcasting, and financial systems, it has been proposed that market-consistent platforms may have implications well beyond merely architectural and processing considerations [12].

A theoretical framework has been formulated for describing a self-organizing open computational system with resources, free agents, and payoff mediated interactions; the model overcomes the limitations inherent in previous work [13,14]. The theory, encompassing the large-system-size expansion due to Van Kampen [15,16], allows a master equation for the evolution of the system to be written in which deterministic and stochastic components are clearly separable. The dominant contribution of the fluctuations is given by the first-order term in the expansion and corresponds to a linear Fokker-Planck equation (FPE). The theory allows for the effects of fluctuations to be explored, particularly in finite systems where such processes assume increasing importance.

II. MODEL

A central feature of open systems is the nonlinear nature of their dynamics, which gives rise to a rich repertoire of behavioral regimes ranging from stable equilibrium to oscillations and chaotic states. The possible behavioral regimes in an open system are called attractors. Nonlinear systems allow in general for multiple solutions and phenomena such as instability and transition from one attractor of the system to another are expected to happen when the system is driven away from equilibrium. A crucial element in understanding how these phenomena take place is the role of fluctuations, generated by the many degrees of freedom constituting the system. One has therefore to construct a theory which integrates both the deterministic evolution equation, describing the macroscopic behavior of the system, and the stochastic part which deals with fluctuations within the system.

Huberman and Hogg [13] considered a model for a computational ecosystem which incorporates the basic features of open nonhierarchical systems. The model is marketlike and consists of a set of computational agents capable of choosing among a given number of possible resources or strategies in order to carry out various computational tasks. Because of the lack of central control in such systems, each computational agent makes freely its choice according to its perceived payoffs of the available resources. Various features such as asynchrony in execution, competition and/or cooperation among agents, incomplete knowledge and delayed information can be modeled through the payoff function. These features generally lead to nonlinear dynamics, characteristic of in-

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teracting systems, resulting in a multitude of possible behaviors in the computational ecosystem. One would like to be able to identify the different regimes in order to avoid nonoptimal strategies and chaotic behavior.

Kephart, Hogg, and Huberman [14] studied this model, including some of the above-mentioned features and carried out detailed calculations in the case of a computational system with two resources. Their approach in solving the dynamical equations of the system is based mainly on computer simulations, compared with some theoretical results. Their theoretical analysis is essentially to treat the system in the mean-field approximation. As argued by Kephart, Hogg, and Huberman, the mean-field approach is certainly justifiable in systems consisting of a large number of agents and indeed becomes exact when this number is infinite. For systems with a relatively small number of agents, however, the mean-field approximation cannot be relied on for an adequate description of the system's behavior and fluctuations have to be taken into account.

In this paper we propose to build on the model for the computational ecosystem of Huberman *et al.* and formulate a general theory of a market-based agent-resource system consisting of free agents sharing the utilization of available resources. Our approach allows the effects of fluctuations to be investigated systematically in the form of a large-system size expansion due to Van Kampen [15,16].

In this section we write down a master equation for a general agent-resource system, in analogy to Huberman and Hogg's probabilistic evolution equation for a computational ecosystem with two resources. We look at the master equation obtained as describing a Markovian jump process, originally studied in physics and chemistry. We then go on to apply Van Kampen's system size expansion in the case of an agent-resource system with two resources. The deterministic equation for the behavior of the system arises as the lowest-order term in the expansion and is seen to coincide with the mean-field equation which Kephart, Hogg, and Huberman derived. The main contribution of the fluctuations comes in the form of a linear Fokker-Planck equation. Up to this order the noise in the system is linear and the solution of the master equation is given by a Gaussian (normal) distribution. Nonlinear effects of fluctuations are then included as small perturbations to the linear noise approximation.

In Sec. III some results for a two resource system in a time-independent state are presented and discussed. Finally, Sec. IV summarizes the main points and suggests directions for further work.

A. The master equation

In its simplest form, the model consists of N agents sharing M resources in a distributed environment with no central controller. The way this is achieved is by allowing agents to bid for the available resources, based on the evaluation of the payoff that each agent performs at any given time in the operation. This evaluation is made from the agent's point of view and may not necessarily be correct or accurate, if incomplete knowledge or delayed information are taken into account. In this paper, however, we are not concerned with giving well-defined payoff functions as the exact form of these functions is irrelevant to the general formalism which is being presented here [1]. For example, in the case of a computational system the payoff may be related to computational measures of system performance such as memory allocation, execution time, computing accuracy, operating cost, etc.

At various times, agents will evaluate the perceived payoffs corresponding to the available resources and a fraction of these agents will switch to the resource with the highest payoff. Because the process of payoff evaluation does not happen continuously for all agents, a probabilistic description seems suited to model the dynamics of the system. One therefore defines the joint probability distribution

$$P(\mathbf{n},t)=P(n_1,n_2,\ldots,n_M,t) \tag{1}$$

representing the probability that n_1 agents are using resource 1, n_2 agents are using resource 2, etc. at time t. Conservation of the number of agents in the system implies

$$\sum_{i=1}^{M} n_i = N \ . \tag{2}$$

Moreover, the normalization condition requires

$$\sum_{\mathbf{n}} P(\mathbf{n},t)=1 \ , \tag{3}$$

where the sum extends over all vectors $\mathbf{n}$ which satisfy relation (2). Following Huberman and Hogg [13], we now suppose that the probability distribution P at time $t+\Delta t$ is related to the probability at time t through

$$P(\mathbf{n},t+\Delta t)=\sum_{\mathbf{n}'} P(\mathbf{n}{:}\mathbf{n}')P(\mathbf{n}',t) \ , \tag{4}$$

where $P(\mathbf{n}{:}\mathbf{n}')$ is the probability for a transition from a distribution $\mathbf{n}'$ to a distribution $\mathbf{n}$ occurring in the interval Δt. The sum is over all initial distributions $\mathbf{n}'$ which are compatible with the constraint (2). We make the further assumption that, for a small enough time interval Δt, either no change occurs at all or there is a single change from some resource j to some resource i performed by an agent. This means that all the components of the vector $\mathbf{n}$ are identical with the components of the vector $\mathbf{n}'$ except when there is a single change in which case we have

$$n_i'=n_i-1, \quad n_j'=n_j+1, \quad n_k'=n_k \quad \text{for } k\neq i,j \ , \tag{5}$$

meaning that resource i has one more user and resource j one less at time $t+\Delta t$. Separating off the term corresponding to $\mathbf{n}'=\mathbf{n}$ (no change) in Eq. (4), we have

$$P(\mathbf{n},t+\Delta t)=P(\mathbf{n}{:}\mathbf{n})P(\mathbf{n},t)+\sum_{\mathbf{n}'\neq\mathbf{n}} P(\mathbf{n}{:}\mathbf{n}')P(\mathbf{n}',t) \ . \tag{6}$$

The transition probability $P(\mathbf{n}{:}\mathbf{n}')$ is proportional to Δt, the initial number of agents using resource j, n_j', and the probability that resource i is perceived to be better. This latter probability is denoted by $\rho_i(\mathbf{n}')$ where, in the general case, it depends on the initial distribution $\mathbf{n}'$. For

$\mathbf{n}'\neq\mathbf{n}$ we thus have

$$P(\mathbf{n}{:}\mathbf{n}')=\alpha n_j'\rho_i(\mathbf{n}')\Delta t \;, \tag{7}$$

where the constant α represents the average number of choices made by an agent in the time interval Δt. The probability for there to be no change ($\mathbf{n}'=\mathbf{n}$), $P(\mathbf{n}{:}\mathbf{n})$, is given by one minus all the single changes with the initial distribution $\mathbf{n}$. Equation (6) is now written as

$$P(\mathbf{n},t+\Delta t)-P(\mathbf{n},t) = \alpha\Delta t\left[-\sum_i\sum_{j\neq i} n_j\rho_i(\mathbf{n})P(\mathbf{n},t) + \sum_i\sum_{j\neq i} n_j'\rho_i(\mathbf{n}')P(\mathbf{n}',t)\right] \tag{8}$$

and the dynamical equation for the probability distribution $P(\mathbf{n},t)$ is obtained by taking the limit $\Delta t\to 0$

$$\frac{\partial}{\partial t}P(\mathbf{n},t)=\alpha\sum_i\sum_{j\neq i}[n_j'\rho_i(\mathbf{n}')P(\mathbf{n}',t)-n_j\rho_i(\mathbf{n})P(\mathbf{n},t)] \;, \tag{9}$$

which is interpreted as the master equation governing the dynamics of the distributed agent-resource system with M resources and N agents, the components of the vector $\mathbf{n}$ constituting the dynamic variables.

The two assumptions made above [(4) and (5)] confer two important properties to systems whose dynamics is governed by a master equation of the form given in (9). The first one is the Markovian property which states that the probability describing the system at time t is uniquely determined by the state of the system at the previous time $t-\Delta t$, if Δt represents a unit time interval. The second assumption restricts the master equation (9) to a subclass of Markov processes known as one-step processes. Such processes, going under a variety of names (birth and death, nearest-neighbor interaction, generation and recombination) are very common in physics and chemistry and have also been used more recently in fields such as evolutionary biology and economics (eg., see [11]).

B. The large-system-size expansion

For simplicity, from now on we shall restrict the number of resources in the agent-resource system to two. This will also allow us to compare some of our results with those of Kephart, Hogg, and Huberman. With $M=2$, the number of independent variables is reduced to one by virtue of Eq. (2). Let us then consider $n=n_1$ as the independent variable and scale it as $f=n/N$, so that f now refers to the fraction of agents using resource 1. By taking $M=2$ in Eq. (9) and $N^{-1}=\varepsilon^2$, we obtain the following master equation for a system with two resources (Fig. 1):

$$\frac{\partial}{\partial t}P(f,t)=\alpha\varepsilon^{-2}\{r(f+\varepsilon^2)P(f+\varepsilon^2,t) + g(f-\varepsilon^2)P(f-\varepsilon^2,t) - [r(f)+g(f)]P(f,t)\} \;, \tag{10}$$

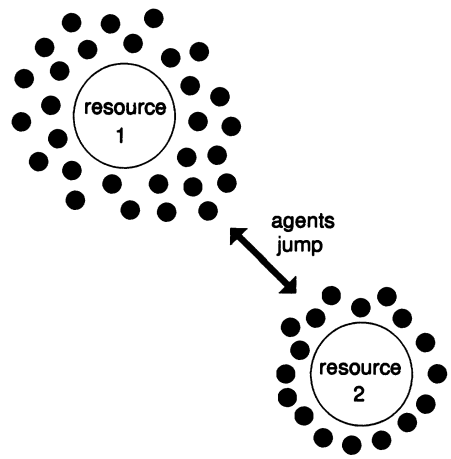

FIG. 1. Schematic diagram of a decentralized agent-resource system made of computational agents sharing two resources. The system's behavior is determined by the agents' evaluation of the payoff associated with each resource.

where

$$r(f)=f[1-\rho(f)],\quad g(f)=(1-f)\rho(f) \;. \tag{11}$$

The coefficients r and g are called, respectively, the recombination and generation coefficients. Although solutions of Eq. (10) are in general very hard or even impossible to find in the nonlinear case, the stationary (time-independent) probability distribution (if it exists) is readily obtained [16] in terms of the generation and recombination coefficients r and g, as a recurrent relation:

$$P^s(f)=\frac{g(f-\varepsilon^2)g(f-2\varepsilon^2)\cdots g(0)}{r(f)r(f-\varepsilon^2)\cdots r(\varepsilon^2)}P^s(0) \;, \tag{12}$$

where $P^s(f)$ is the time-independent solution of Eq. (10). Relation (12) determines all the $P^s(n)$ in terms of $P^s(0)$, which itself can be determined with the help of the normalization condition (3). The one-step formulation allows us therefore to find the exact stationary state of the system simply by knowing the explicit expression of the coefficients r and g. Since the latter are functions of $\rho(f)$, the knowledge of the transition probability function ρ is all that is needed to determine the time-independent solution, provided a stationary state exists.

In general, however, the master equation (10) can be solved exactly only when the r and g functions are linear. For nonlinear processes, which are relevant to open systems, there is a need for approximation methods for dealing with fluctuations. On the other hand, any approximation scheme must be based on an expansion of relevant quantities in powers of a small parameter. Among the various methods which have been developed for dealing with stochasticity in the physical sciences, we shall apply one such scheme due to Van Kampen [15,16] to the agent-resource system described by the master equation (10). Van Kampen devised a general method for deriving such an expansion for Markov processes where the appropriate expansion parameter is the inverse of the size of the system. The basic idea is that fluctuations are caused by the discrete nature of systems which are made of a large number of individuals (particles in physical systems and agents in distributed agent-resource systems). Contrasting with the microscopic origin of fluctuations, the deterministic features of the system depend on all the individual agents together and are therefore macroscopic in nature. One then expects that fluctuations become less important as the size of the system increases. In fact they turn out to be proportional to the square root of the size and are thus of order $N^{-1/2}$ with respect to macroscopic variables (the number of agents N taken as a measure for the size of the system).

In order to proceed with the expansion, the initial probability distribution (at $t=0$) is normally taken as a sharp distribution peaked at some macroscopic value of f, which is formally given by

$$P(f,t=0)=\delta(f-f_0) \ . \tag{13}$$

At later times the distribution $P(f,t)$ is expected to remain approximately sharply peaked at some macroscopic position $\phi(t)$ while its width, defined by the standard deviation, will be of order $N^{-1/2}=\varepsilon$. The stochastic variable f can then be expressed as

$$f=\phi(t)+\varepsilon\xi \ , \tag{14}$$

where the deterministic part $\phi(t)$ is a function to be fixed and ξ is the new (purely stochastic) variable, representing fluctuations in the system. As a result of the time-dependent transformation (14), $P(f,t)$ will transform into a new distribution depending on ξ,

$$P(f,t)=\Pi(\xi,t) \ , \tag{15}$$

while the transformation of the derivatives gives

$$\frac{\partial P}{\partial t}=\frac{\partial \Pi}{\partial t}-\frac{1}{\varepsilon}\frac{d\phi}{dt}\frac{\partial \Pi}{\partial \xi} \ . \tag{16}$$

Similarly, the functions $r(f)$ and $g(f)$ will transform into the functions $\bar{r}(\xi)$ and $\bar{g}(\xi)$, respectively. We can now recast the master equation (10) into the new form

$$\frac{\partial \Pi}{\partial t}-\frac{1}{\varepsilon}\frac{d\phi}{dt}\frac{\partial \Pi}{\partial \xi}=\alpha\varepsilon^{-2}\{\bar{r}(\xi+\varepsilon)\Pi(\xi+\varepsilon)+\bar{g}(\xi-\varepsilon)\Pi(\xi-\varepsilon)-[\bar{r}(\xi)+\bar{g}(\xi)]\Pi(\xi)\} \ . \tag{17}$$

Now by expanding the functions $\bar{r}$, $\bar{g}$, and Π in powers of the small parameter ε, we arrive at the following expression:

$$\frac{\partial \Pi}{\partial t}-\frac{1}{\varepsilon}\frac{d\phi}{dt}\frac{\partial \Pi}{\partial \xi}=\alpha\sum_{n=1}^{\infty}\frac{\varepsilon^{n-2}}{n!}\left[-\frac{\partial}{\partial \xi}\right]^2[(-1)^{-n}\bar{r}(\xi)+\bar{g}(\xi)]\Pi(\xi,t) \ . \tag{18}$$

We further expand $\bar{r}$ and $\bar{g}$ around the macroscopic value ϕ and obtain

$$\frac{\partial \Pi}{\partial t}-\frac{1}{\varepsilon}\frac{d\phi}{dt}\frac{\partial \Pi}{\partial \xi}=\alpha\sum_{k=1}^{\infty}\varepsilon^{k-2}\sum_{n=1}^{k}\frac{1}{n!(k-n)!}a_n^{(k-n)}(\phi)\left[-\frac{\partial}{\partial \xi}\right]^n[\xi^{(k-n)}\Pi(\xi,t)] \ , \tag{19}$$

where

$$a_n^{(i)}(\phi)=\frac{\partial^i}{\partial \phi^i}[(-1)^n r(\phi+g(\phi)] \ . \tag{20}$$

There is one divergent term (proportional to ε^{-1}) on either side of Eq. (20). We can make these terms cancel each other by choosing the function $\phi(t)$ (so far arbitrary) such that it obeys the equation

$$\alpha^{-1}\frac{d\phi}{dt}=a_1(\phi)=\rho(\phi)-\phi \ , \tag{21}$$

which is a nonlinear differential equation independent of fluctuations. This is how the macroscopic or deterministic law for the evolution of the system emerges from the master equation. Its solution $\phi(t)$ with initial condition ϕ_0 gives the deterministic component of the original variable f. It also provides the function to be used in the change of variable (14). As expected, Eq. (21) is identical with the mean-field equation given by Kephart, Hogg, and Huberman for a computational ecosystem with two resources [14]. The time-independent solutions of Eq. (21) are given by the roots of

$$\rho(\phi)=\phi \ . \tag{22}$$

If we assume that there is only one such solution which is a global minimum then all time-dependent solutions of Eq. (21), regardless of the initial value ϕ_0, will tend towards the stable solution asymptotically.

C. The fluctuations equation

1. The linear Fokker-Planck equation

Once the deterministic part $\phi(t)$ of the variable f has been fixed through Eq. (21), we insert this function into the master equation (19), so that the coefficients a_n are now determined, and end up with an equation governing the fluctuations in the system,

$$\frac{\partial}{\partial t}\Pi(\xi,t)=\alpha\sum_{k=2}^{\infty}\varepsilon^{k-2}\sum_{n=1}^{k}\frac{1}{n!(k-n)!}\times a_n^{(k-n)}(\phi)\left[-\frac{\partial}{\partial \xi}\right]^n\times[\xi^{(k-n)}\Pi(\xi,t)] \ . \tag{23}$$

Note that Eq. (23) does not contain any divergent terms (in ε). As the right-hand side in Eq. (23) is effectively an expansion in powers of ε, the effects of fluctuations around the macroscopic value ϕ can be determined perturbatively round $\varepsilon=0$. The lowest-order approximation is given by taking the limit $\varepsilon=0$ in (23) and results in a linear Fokker-Planck equation with time-dependent coefficients [through $\phi(t)$]

$$\alpha^{-1}\frac{\partial}{\partial t}\Pi(\xi,t)=-a_1'(\phi)\frac{\partial}{\partial\xi}(\xi\Pi)+\frac{1}{2}a_2(\phi)\frac{\partial^2}{\partial\xi^2}\Pi\ . \tag{24}$$

The solutions to this well-known equation are given by normal or Gaussian distributions, whose only nonzero moments are the first two moments: $\langle\xi\rangle$ and $\langle\xi^2\rangle$. Instead of dealing directly with Eq. (24), it is easier to solve the associated moment equations

$$\begin{aligned}\alpha^{-1}\frac{\partial}{\partial t}\langle\xi\rangle&=a_1'(\phi)\langle\xi\rangle\ ,\\ \alpha^{-1}\frac{\partial}{\partial t}\langle\xi^2\rangle&=2a_1'(\phi)\langle\xi^2\rangle+a_2(\phi)\ ,\end{aligned}\tag{25}$$

subject to the appropriate initial conditions for the fluctuation variable ξ. In this case, the initial fluctuation (at $t=0$) is taken to be zero in accordance with Eq. (13), leading to $\langle\xi\rangle_0=\langle\xi^2\rangle_0=0$. The original probability distribution $P(f,t)$ is also a normal distribution by virtue of Eq. (15) with mean and variance given by [taking account of the transformation (14)]

$$\begin{aligned}\langle f(t)\rangle&=\phi(t)+\varepsilon\langle\xi(t)\rangle\ ,\\ \langle\langle f^2(t)\rangle\rangle&=\varepsilon^2\langle\langle\xi^2(t)\rangle\rangle\ ,\end{aligned}\tag{26}$$

where the variance $\langle\langle f^2\rangle\rangle=\langle f^2\rangle-\langle f\rangle^2$. This gives us the solution to the master equation (10) in the Fokker-Planck equation of linear-noise approximation.

2. *Higher-order corrections to the FPE: nonlinear effects*

In general, the linear-noise approximation is adequate for dealing with fluctuations in most systems provided these have a large number of agents since, as we saw above, the FPE (24) does not depend on the size parameter ε. There is one situation, however, where higher-order terms in the fluctuation equation (23) should be included. This corresponds to "discrete" systems or generally systems with a relatively small number of agents. There is further reason for including these terms when the coefficients r and g are themselves nonlinear in f.

In this section we calculate the nonlinear effects of fluctuations beyond the FPE approximation by keeping terms of order ε and ε^2 in Eq. (23) and ignoring higher-order terms. These terms add fluctuation effects of order N^{-1}, that is, the dynamics (to this order) is made sensitive to random events due to a single agent in the system. We shall see that the average value of f will also be affected by a term of the same order. In order to solve the fluctuation equation (23) including higher-order terms in ε, we treat the additional terms as perturbations to the FPE (24) (which is independent of ε). Therefore, and keeping in the spirit of the large N expansion, the solution of (23) will be approximated by a Gaussian distribution except that it no longer obeys the FPE since it is modified by the higher-order terms added to Eq. (24). In view of this, it is easier to solve the first and second moment equations (25) by adding the appropriate correction terms, rather than attempting to tackle Eq. (23) directly. In Appendix A we derive the general moment equation from Eq. (23).

Expanding the moment equation (A4) for the first two moments $\langle\xi\rangle$ and $\langle\xi^2\rangle$ up to order ε^2 we find

$$\begin{aligned}\alpha^{-1}\frac{\partial}{\partial t}\langle\xi\rangle&=a_1'\langle\xi\rangle+\frac{\varepsilon}{2}a_1''\langle\xi^2\rangle+\frac{\varepsilon^2}{6}a_1'''\langle\xi^3\rangle\ ,\\ \alpha^{-1}\frac{\partial}{\partial t}\langle\xi^2\rangle&=\varepsilon a_2'\langle\xi\rangle+\left[2a_1'+\frac{\varepsilon^2}{2}a_2''\right]\langle\xi^2\rangle\\ &\quad+\varepsilon a_1''\langle\xi^3\rangle+\frac{\varepsilon^2}{3}a_1'''\langle\xi^4\rangle+a_2\ .\end{aligned}\tag{27}$$

These equations involve the third and the fourth moments as well and it is therefore necessary to take into account the equations for $\langle\xi^2\rangle$ and $\langle\xi^4\rangle$, up to order ε and 1, respectively. We thus have to solve the resulting system of four coupled linear differential equations

$$\alpha^{-1}\frac{\partial}{\partial t}\begin{bmatrix}\langle\xi\rangle\\ \langle\xi^2\rangle\\ \langle\xi^3\rangle\\ \langle\xi^4\rangle\end{bmatrix}=\begin{bmatrix}a_1' & \frac{\varepsilon}{2}a_1'' & \frac{\varepsilon^2}{6}a_1''' & 0\\ \varepsilon a_2' & \left[2a_1'+\frac{\varepsilon^2}{2}a_2''\right] & \varepsilon a_1'' & \frac{\varepsilon^2}{3}a_1'''\\ 3a_2 & 3\varepsilon a_2' & 3a_1' & \frac{3}{2}\varepsilon a_1''\\ 0 & 6a_2 & 0 & 4a_1'\end{bmatrix}\begin{bmatrix}\langle\xi\rangle\\ \langle\xi^2\rangle\\ \langle\xi^3\rangle\\ \langle\xi^4\rangle\end{bmatrix}+\begin{bmatrix}0\\ a_2\\ -\varepsilon a_3\\ 0\end{bmatrix}\ . \tag{28}$$

The results obtained for $\langle\xi\rangle$ and $\langle\xi^2\rangle$ are then used in (26) to give the solution to the master equation (10) with the leading or first-order nonlinear contribution to fluctuations included.

At the next level of the approximation (the second-order nonlinear effects), one goes further and keeps all terms in Eq. (23) up to order ε^4. In this case fluctuation effects due to a fraction ($1/N$th) of an agent are taken into account. This only makes sense if agents themselves are seen not as elementary processes but rather as groups of agents or collections of tasks to be performed. As in the previous case, the solution of $\langle\xi\rangle$ and $\langle\xi^2\rangle$ to order

ε^4 will involve two additional moment equations (for $\langle \xi^5 \rangle$ and $\langle \xi^6 \rangle$) besides those for $\langle \xi^3 \rangle$ and $\langle \xi^4 \rangle$. Consequently, the corresponding system to be solved will consist of six coupled linear differential equations, which can be derived in a similar way by expanding the first six moment equations to the required order. We shall also calculate the effects due to the ε^3 and ε^4 terms in the following section.

III. NUMERICAL RESULTS AND DISCUSSION

In this section we present some numerical results based on the theoretical analysis given in the preceding section. Our main objective is to investigate the approximation scheme for fluctuations based on the large-system-size expansion of Van Kampen. We already saw that the one-step Markovian formulation of the problem allowed us, in particular, to calculate the exact probability distribution for time-independent solutions [see Eq. (12)]. We shall therefore restrict our numerical calculations to stationary solutions of the system in order to make a direct comparison with exact results and test the validity of the approximation over a range of parameter values. As a result of considering time-independent solutions only, the equations for the deterministic law (21) and the linear FPE fluctuations (25) reduce to static nonlinear equations, while the differential system (28) giving the first nonlinear corrections from fluctuations reduces to a simple linear system of equations [where the left-hand side of (28) is zero].

Numerical values for the input parameters (coming into the transition probability function ρ) will be taken as in Ref. [14] to provide a further comparison with some results that these authors obtained in their numerical simulations. In general, the exact form of ρ is not known and will depend on several features of the problem at hand, like incomplete, uncertain, or delayed information on the available resources, as well as other factors influencing the choice that agents make. We will follow the authors in Ref. [14] and make ρ a function of the fractional number of agents f using resource 1, through the payoffs G_1 and G_2 for using resources 1 and 2, respectively,

$$G_1 = 7 - f_1, \quad G_2 = 7 - 3f_2 . \tag{29}$$

Figure 2 shows the payoffs G_1 and G_2 as a function of f. They model a simple competitive behavior between agents so that the payoff for using each resource decreases with the number of agents already using the same resource. An agent will therefore choose to switch to the other resource if its payoff is larger. The system reaches a stability point when the two payoffs are equal so agents will prefer staying with the resource they are using. For G_1 and G_2 given in (29) the optimal behavior of the system occurs for $f = 0.75$, that is, 75% of all agents using resource 1. The decision region can be made less sharply defined by introducing an uncertainty element in the payoff evaluation of agents. This can be achieved by introducing Gaussian noise with standard deviation σ round the true value of the payoff. The resulting transition probability ρ is given by [13]

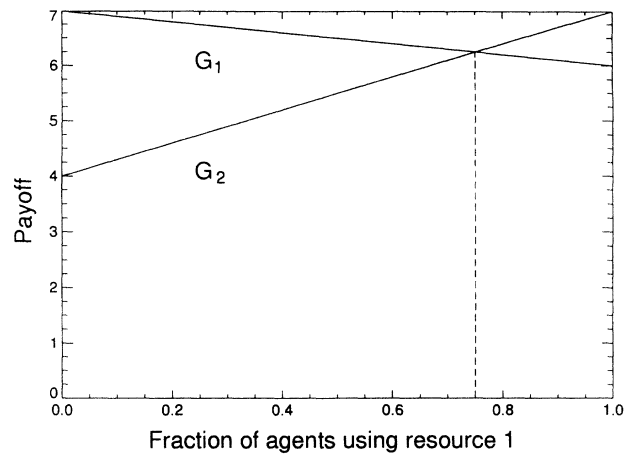

FIG. 2. The linear payoff functions $G_1 = 7 - f_1$ and $G_2 = 7 - 3f_2$ associated with resource 1 and resource 2, respectively. The system is in equilibrium when the two payoffs are equal, i.e., at the crossing point between the two lines.

$$\rho = \frac{1}{2}\left[1 + \text{erf}\left[\frac{G_1 - G_2}{2\sigma}\right]\right] \tag{30}$$

and shown in Fig. 3 for a value $\sigma = 0.125$. The two limiting cases of $\sigma = 0$ and $\sigma = \infty$ correspond respectively to perfect knowledge ($f = 0.75$) and complete lack of information on payoffs, leading to the uniform distribution of agents ($f = 0.5$). In fact, by approximating f with its deterministic contribution ϕ, we obtain a graphical solution of the macroscopic equation (20) represented in Fig. 3 by the crossing point between the curves $\rho(\phi)$ and ϕ. This point gives the equilibrium solution which now, due to a nonzero value of σ, is slightly offset from the optimal value ($f = 0.75$). The macroscopic value of f for $\sigma = 0.125$ is $\phi = 0.724$.

In Fig. 4 we show the mean or average (fractional)

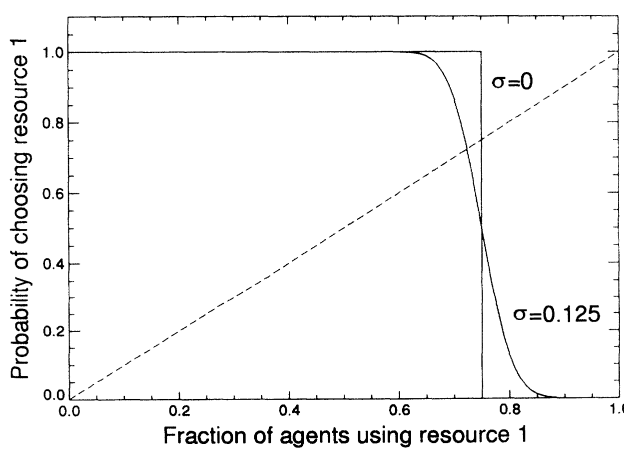

FIG. 3. The transition probability (from resource 2 to resource 1), $\rho(f)$, corresponding to the payoffs shown in Fig. 2, for two values of the uncertainty parameter σ. The intersection with the line $\rho = f$ gives the solution to the time-independent macroscopic equation (22).

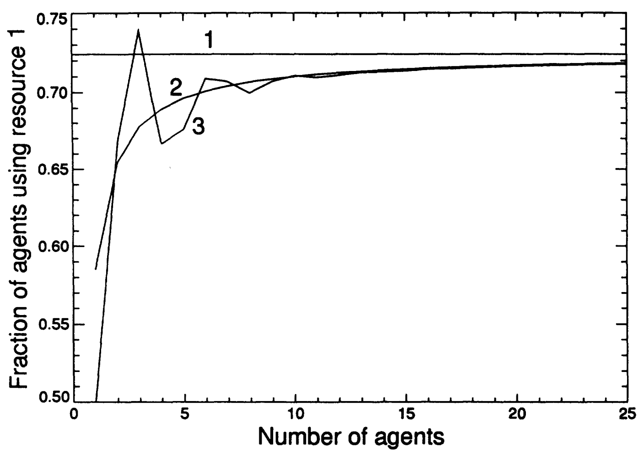

FIG. 4. Average (fractional) number of agents using resource 1 as a function of the total number of agents in the system N for $\sigma=0.125$, in three orders of the large-system-size expansion. Curve 1 represents the mean-field value (unaltered by the linear noise), curve 2 includes the first nonlinear corrections due to fluctuations, and curve 3 represents the exact result (full effects of fluctuations included).

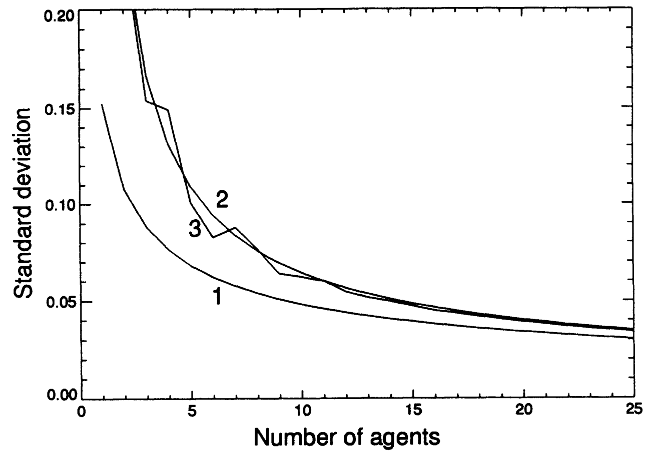

FIG. 5. Standard deviation of the time-independent distribution (for the number of agents using resource 1) as a function of the total number of agents in the system N for $\sigma=0.125$, in three orders of the large-system-size expansion. Curve 1 represents the FPE (linear-noise) approximation, curve 2 includes the first nonlinear corrections due to fluctuations, and curve 3 represents the exact result (full effects of fluctuations included).

number of agents f using resource 1 for various orders in the Van Kampen approximation, as well as the exact value calculated from (12), as a function of the total number of agents N, for $\sigma=0.125$. Curve 1 represents the mean-field value (unaltered by the linear noise), curve 2 includes the first nonlinear corrections due to fluctuations, and curve 3 represents the exact result (full effects of fluctuations included). The exact curve shows a rather strong dependence on N in the region of small number of agents, increasing from $f=0.5$ for $N=1$ towards the macroscopic value $\phi=0.724$ asymptotically. As can be seen in the figure, the macroscopic approximation (no fluctuations) and the FPE approximation show no dependence of the average on N. One needs to include the first nonlinear corrections from fluctuations [obtained by solving the time-independent linear system from (25)] to observe a nontrivial dependence. We see that the first-order nonlinear corrections approximate well the exact result, not only for large values of N, as expected since the approximation becomes exact in the limit of large N, but also down to the low values of N characterizing systems with a small number of agents. The same conclusions are reached by considering the standard deviation as a function of N for different orders of the approximation in Fig. 5. We should also note at this point that our exact results in Figs. 4 and 5 [obtained from the theoretical prediction of the probability distribution (12)] confirm the Monte Carlo simulations of Kephart, Hogg, and Huberman [14].

In order to see how the Van Kampen approximation depends on the uncertainty parameter σ, we have plotted the time-independent probability distributions for different orders in the approximation as well as the exact distribution, for three values of σ: 0.125, 0.5, and 1 (vertically) and three values of N: 3, 5, and 10 (horizontally) in Fig. 6. As expected the approximation works better for larger N, as we see all approximating distributions converging towards the exact curve. However, for a fixed value of N we notice that the approximation somewhat worsens with decreasing σ. The second-order nonlinear corrections apparently become larger for small σ, resulting in a shift in the peak of the distribution of the same order as the one caused by the first-order corrections. This shift gradually disappears when σ increases. Moreover, for increasing σ it seems that the role of nonlinear fluctuations (in shifting the mean and spreading slightly the distribution) is suppressed. These observations can be explained by noting the relationship between internal noise (modeled by the stochastic variable ξ), inherent in systems with a finite number of agents, and the noise deliberately introduced by adding a random element in the agents' decision making process (modeled by σ). The result is that, in systems with agents with (almost) perfect knowledge (small σ), nonlinear fluctuations are prominent (especially in the region of small N); a larger uncertainty in this knowledge (large σ) blurs the region of optimal decisions made by the agents and suppresses the nonlinear effects of internal fluctuations. This latter fact was noticed by Kephart, Hogg, and Huberman [14] and used in systems with delayed information to reduce the effects of persistent oscillations and chaos, which are, after all, manifestations of nonlinearities in the fluctuations.

We can draw the following conclusions: the approximation works reasonably well for all values of σ considered and the first-order nonlinear corrections are sufficient for correctly estimating fluctuation effects in the system, especially if the uncertainty parameter is not too small. Furthermore it seems that the approximation is best suited for systems with a moderate value of the global uncertainty parameter σ (≈ 0.5), where nonlinear effects of fluctuations, although significant, converge rapidly in the expansion. This may be the range of σ to look for in realistic systems, where agents are neither expected

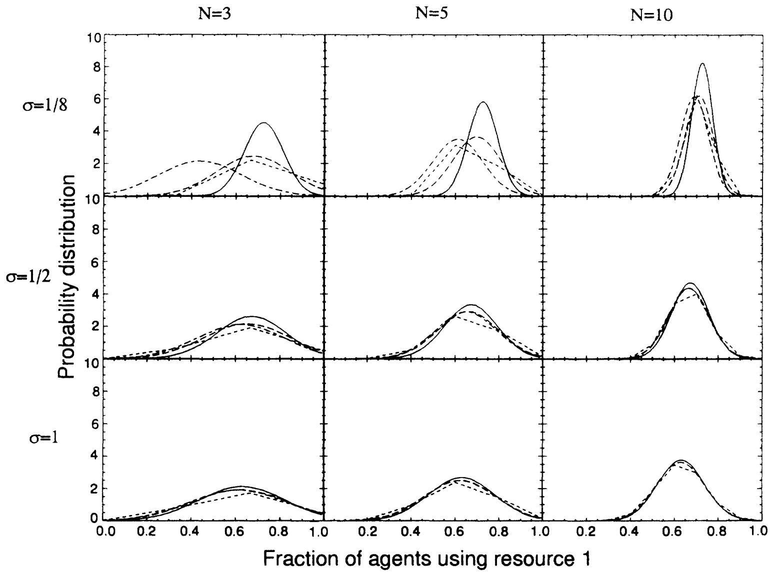

FIG. 6. The time-independent probability distribution P for three different values of N and three different values of σ, in three orders of the large-system-size expansion; full line: mean-field (including linear-noise) result, long dash: first-order nonlinear corrections included, long dash–short dash: second-order nonlinear corrections included, short dash: exact solution. For a discussion of these figures see Sec. III.

to have perfect knowledge nor be completely ignorant about the payoffs of their transactions. In more complex systems, uncertainty in the system may be represented by more than one σ.

So far in our analysis we have only looked at systems with a single macroscopic stable behavior, a consequence of the unique (stable) fixed point occurring at the intersection between the linear payoff functions G_1 and G_2 in Fig. 2. The simple competitive behavior displayed in Fig. 2 can be changed by making the payoff functions nonlinear, i.e., introducing cooperation as well as competition between agents in the system. Whereas competition meant that agents would favor a resource if it had less agents using it, cooperation is expressed by an increased payoff when a resource is used by more agents. The interplay of these two tendencies through nonlinear payoffs leads to a richer range of possible behaviors in the system. The example of a nonlinear (cubic) payoff for resource 1, $G_1=0.5-10(f-0.1)(f-0.5)(f-0.9)$, and a constant payoff for resource 2, $G_2=0.5$, in Fig. 7(a) illustrates the case of a bistable system where the two possible macroscopic states are equally probable. The roots of the time-independent macroscopic equation are represented by the intersection points between the transition probability ρ and the bisectrice, as seen in Fig. 7(b) (for $\sigma=0.35$). The outer points are stable equilibria whereas the middle point leads to an unstable state. Depending on which side of this point the initial resource market share is, the system will eventually settle in one of the macroscopic states characterized by the two peaks in the time-independent probability distribution (see Fig. 8). However, this situation will not stay unchanged forever; internal fluctuations will cause the system to (macroscopically) flip into the other macroscopic state given enough time. Here again, the one-step Markov formulation allows us to find an explicit expression for the average time for such a transition to happen [16]

$$\tau_{AB}=\alpha^{-1}\sum_{n=n_A+1}^{n_B}\frac{1}{r(n)P^s(n)}\sum_{m=0}^{n-1}P^s(m)\ , \qquad (31)$$

where n_A and n_B represent the sites of the peaks in the distribution of the bistable system. This expression depends on the total number of agents N and the uncertainty parameter σ through P^s and ρ. It is valid for any number of agents (since it includes full fluctuation effects) and in the thermodynamic limit we recover the exponential growth of τ_{AB} in N and $1/\sigma$ (e.g., Ref. [17]).

As can also be seen from Fig. 8, the system's dynamics depends notably on the uncertainty parameter σ. By in-

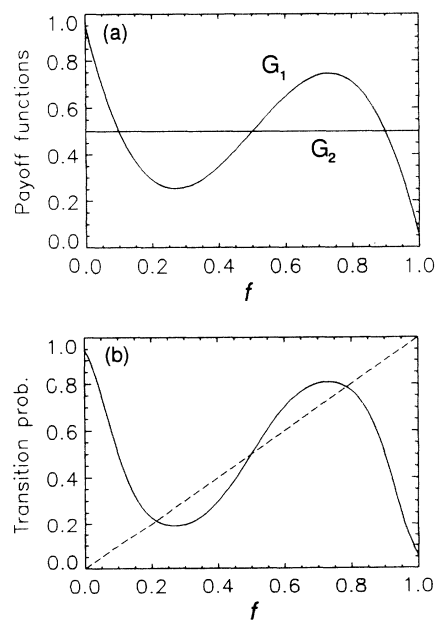

FIG. 7. (a) System with nonlinear payoff functions. In this example resource 1 incorporates both competitive and cooperative behaviors, $G_1=0.5-10(f-0.1)(f-0.5)(f-0.9)$, and resource 2 keeps a constant payoff, $G_2=0.5$, leading to a bistable system. (b) The corresponding transition probability $\rho(f)$ (full line), for $\sigma=0.35$. The intersection with the line $\rho=f$ gives the equilibrium points (two stable points and one unstable).

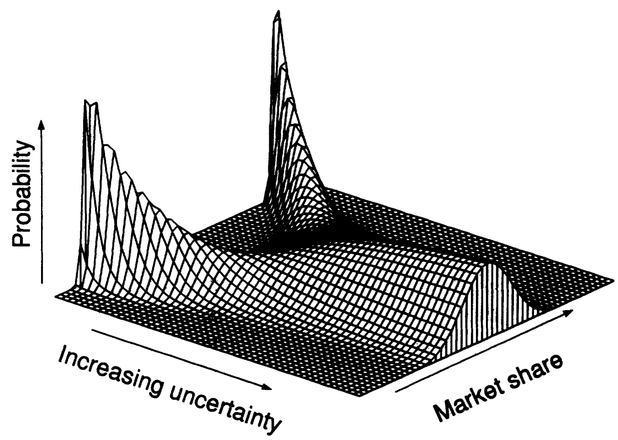

FIG. 8. The (exact) time-independent probability distribution for the bistable system arising from the payoff functions shown in Fig. 7(a), as a function of the market share of resource 1, f, and the uncertainty parameter, σ, both varying from 0 to 1.

creasing σ the two peaks are seen to gradually get closer to each other and merge into a single (symmetric) peak for a critical value of σ. The critical value of σ can be found by inspection of the time-independent macroscopic equation. Thus σ plays the role of a control parameter which can change qualitatively the dynamical phase space of the system, in this case from a system with two attractors to a system with a single one. This is reminiscent to phase transitions in physical systems such as the spontaneous magnetization of a ferromagnetic system which happens by lowering the temperature below a critical value (Curie temperature). Above this value the overall magnetization is zero and symmetric while below it there are two possible states of opposite magnetization. By choosing one state or the other, the system breaks its spatial symmetry, just like by decreasing σ below its critical value in the agent-resource system we see a sudden transition from an equal distribution of agents on the two resources to a definite bias towards one or the other.

IV. CONCLUSION

In this paper, we have taken and further developed Huberman and Hogg's model for computational ecosystems [13], and formulated a framework for a general marketlike agent-resource system in terms of a one-step Markovian master equation. Our approach enables us to analyze fluctuation effects within the system by making use of the large system-size expansion. A deterministic equation governing the dynamics of the system in the limit of large numbers of agents arises as the lowest-order contribution in the expansion, and coincides with the equation obtained in the mean-field approach [14]. The next order term gives the main contribution of the fluctuations and turns out to be a linear Fokker-Planck equation. The probability distribution describing the dynamics of the system is therefore a Gaussian distribution to this order in the expansion, providing a linear-noise approximation. Higher-order terms are included to provide nonlinear corrections to the FPE in two stages: the first-order nonlinear corrections representing fluctuations due to individual agents in the system and the second-order ones which are proportional to a fraction of an agent. These higher-order corrections are crucial when the number of agents is relatively small and the mean-field theory inadequate.

To test the approximation in the case of our agent-resource system, we have taken a system with two resources and considered time-independent states for which an analytical expression giving the exact probability distribution is available. The payoff functions associated with the two resources were taken to model a simple competitive strategy between agents with an uncertainty parameter monitoring the accuracy of information available. Our exact theoretical results agree with the corresponding numerical simulations in Ref. [14], using the same values for input parameters. We find that the inclusion of nonlinear fluctuation terms is necessary to observe the shift in the market share value of resource 1 which appears for a system with a small number of agents. Furthermore, we find that the first nonlinear corrections are generally sufficient for mimicking the full effects of fluctuations, for practically any number of agents. We also studied the sensitivity to accuracy of the information available to agents and the main observation is that higher uncertainty leads to the suppressing of nonlinear noise effects. This is compatible with the conclusion in [14] that an increase in the uncertainty parameter lowers the threshold for persistent oscillations and chaos in systems with time delay, since these nonoptimal behaviors are the result of nonlinearities taking over in the dynamical equations. In view of the results obtained we conclude that the approximation works generally well and can therefore be reliably used for time-dependent solutions, for which no exact analytical treatment of fluctuations is available, as well as generalizations to systems with more than two resources and with more marketlike features (time-delay, inhomogeneous agents, etc.) included.

Finally, we saw how the one-step Markov formulation also enables us to find the exact time-independent distribution in the case of a bistable system, which results from nonlinear payoff functions. This allows us in particular to calculate the average time that a system takes for leaving the stable state it is in and "flipping" into the other one. This effect is essentially due to fluctuations within the system.

We should note, however, that Van Kampen's approximation scheme is not suited for the treatment of fluctuations in situations involving instabilities or critical behavior. In other words, the large-system-size expansion is only valid when there is one globally stable macroscopic solution (like the simple competitive system considered above) or in the immediate vicinity of any locally stable solution (near each peak of the bistable system in Fig. 8). Information on the multiplicity of solutions and their stability is obtained by studying the macroscopic equation (21). In general, however, a system with multiple minima, such as the bistable system considered in the preceding section, requires a different treatment of fluctuations near instability points. This interesting issue will be addressed in a separate work. Bearing in mind these limitations, we can conclude that Van Kampen's approximation method provides a powerful way to sort out the main properties of fluctuations in distributed agent-resource systems.

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APPENDIX: THE GENERAL MOMENT EQUATION

An alternative and usually more practical way for solving the master equation (23) is to solve the moment equations associated with it, since knowledge of the moments uniquely define the probability distribution. The ith moment of $\Pi(\xi,t)$ is given by

$$\langle \xi^i(t) \rangle = \int d\xi\, \xi^i \Pi(\xi,t) \quad \text{(A1)}$$

and its evolution equation is obtained by multiplying both sides of Eq. (23) by ξ^i and integrating over the stochastic variable ξ,

$$\frac{\partial}{\partial t}\langle \xi^i(t)\rangle = \alpha \sum_{k=2}^{\infty}\sum_{n=1}^{k} \frac{\varepsilon^{k-2}}{n!(k-n)!} a_n^{(k-n)} \times \int d\xi\, \xi^i \left[-\frac{\partial}{\partial \xi}\right]^n \times [\xi^{(k-n)}\Pi(\xi,t)] \; . \quad \text{(A2)}$$

We can easily show by induction that the integral appearing in the right-hand side is equal to

$$\int d\xi\, \xi^i \left[-\frac{\partial}{\partial \xi}\right]^n [\xi^{(k-n)}\Pi(\xi,t)] = \frac{i!}{(i-n)!}\langle \xi^{i+k-2n}\rangle \; . \quad \text{(A3)}$$

This expression is zero if the power i is strictly smaller than the index n. We then obtain an evolution equation for $\langle \xi^i \rangle$ as a power series in the size parameter ε,

$$\frac{\partial}{\partial t}\langle \xi^i(t)\rangle = \alpha \sum_{k=2}^{\infty}\sum_{n=1}^{k} \varepsilon^{k-2} \frac{i!}{n!(k-n)!(i-n)!} a_n^{(k-n)} \times \langle \xi^{i+k-2n}\rangle \; , \quad \text{(A4)}$$

with all terms satisfying $i<n$ identically zero. It is now easy to see that, if the series is truncated at a given order k in the expansion, the equation for the moment $\langle \xi^i \rangle$ will involve all moments up to the $(i+k-2)$th moment. In other words, moments of order $i+k-1$ and above will not be required for solving the evolution equation. As a special case, the Fokker-Planck equation approximation is obtained by taking $k=2$, which eliminates all moments of third and higher order.

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