Period-doubling bifurcations in the presence of colored noise

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(Received 8 December 1993)

We study the effects of colored noise on period-doubling bifurcations. Using the Feigenbaum map as a model, the technique of cumulant equations is applied to analyze the bifurcation behavior. We find that the universal properties of the period-doubling sequences are preserved in the case of colored noise. Moreover, the resonancelike response of the period-doubling cascade to the colored noise forcing is observed, while the noise correlation time is varied. This response is reflected in both power spectra and bifurcation diagrams.

PACS number(s): 02.50.Ey, 02.50.Ga, 05.40.+j, 05.50.+q

I. INTRODUCTION AND A MODEL

Period-doubling bifurcations (PDB's) are typical for a wide range of dissipative dynamical systems. There are a lot of examples of dynamical systems in physics, chemistry, biology, etc., which demonstrate PDB (see Refs. [1,2]). The infinite sequence of such bifurcations is one of the scenarios leading to chaos that have universal scaling properties [3].

Since in real physical systems noise already exists and is naturally inevitable due to dissipation [4], the investigation of the influence of noise on the bifurcation behavior is important. There are several basic studies that deal with the influence of noise on the PDB [5]. The main results of these works is that the sequence of PDB becomes finite under the influence of noise and that there exists a universal law which connects the maximal noise intensity σ_k and the possibility of resolving a cycle of period 2^k,

$$\sigma_k \propto \mu_{th}^{-k} , \qquad (1)$$

where μ_{th} is the universal constant $\mu_{th}=6.557\ldots$. However, in these papers only *white* noise was considered. The concept of white noise is a suitable mathematical abstraction which allows us to use the powerful techniques developed for Markovian processes [6]. This concept is suitable for internal fluctuations (for instance, thermal fluctuations). However, in real physics external noise from an environment of a system influences it stronger than an internal one. For external fluctuations, this restriction to white noise means that the interactions between a system and its environment are absolutely uncorrelated. This assumption, however, is far from reality. For real physical systems we typically find that external random perturbations have time scales comparable with those of the dynamical systems, i.e., finite correlation time of noise has to be taken into consideration. In view of colored noise in nonlinear systems it is possible to observe phenomena which do not occur in the case of white noise [7–9]. The purpose of this paper is to study the effects of colored noise on a rather simple system: the quadratic map.

A traditional model for the investigation of PDB is the family of discrete maps $x_{n+1}=f(x_n,a)$, where $f(x,a)$ is a function with a quadratic maximum and a is a bifurcation parameter. This map can be considered as a Poincaré map of a flow system. Despite its simplicity, such a one-dimensional map generates a rich dynamical behavior, as is often observed in real flow systems [1,10]. In the following we will use a special form of $f(x)$, i.e., the Feigenbaum map,

$$x_{n+1}=1-ax_n^2 . \qquad (2)$$

In Eq. (2), a is the bifurcation parameter. The bifurcation sequence of the fixed point with the period 2^k takes place for the parameter values a_k: $a_1=0.75$, $a_2=1.25$, $a_3=1.368\,099,\ldots$. The critical point $a_{cr}=1.401\,15\ldots$ corresponds to the accumulation of the PDB and to the transition to chaos.

To investigate noise effects we have to introduce a noise source into Eq. (2),

$$x_{n+1}=f(x_n)+g(x_n)\xi_n , \qquad (3)$$

where $g(x)$ is a function of the state variable and ξ_n is a random process with zero mean value symmetrically distributed on the region $[-\epsilon,+\epsilon]$, $\epsilon \ll 1$. (Note that the procedure of a correct introduction of the noise source into a Poincaré map of a real dynamical system is very

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complicated [11]. Here, we use only a model approximation to obtain qualitative results.) In the case of additive white noise $g(x)=\sigma$ (where σ is the noise intensity) with $\langle \xi_n \xi_{n+m} \rangle = \delta(m)$ [$\delta(m)$ is the δ function], the stochastic process x_n is a Markov one. Then, it is possible to use the technique based on the Frobenius-Perron equation [12]. In the case of colored noise, the situation is more difficult, since we cannot use the technique of Markov processes directly. A theory of the influence of colored noise has been developed recently [13]. A promising approach is based on the extension of the stochastic system (3) by introducing additional stochastic equations which describe the colored noise. For example, an exponentially correlated Gaussian noise can be modeled by the one-dimensional Ornstein-Uhlenbeck process [6]. For our discrete map we can use a discrete analogue of the Ornstein-Uhlenbeck process, i.e., the autoregressive process of first order

$$\xi_{n+1}=\Gamma\xi_n+\eta_n , \tag{4}$$

where Γ is the parameter which determines the correlation time $\tau_{\rm cor}=-1/\ln|\Gamma|$, $|\Gamma|<1$, and η_n denotes white noise $\langle \eta_n \eta_{n+m} \rangle=(1-\Gamma^2)\sigma^2\delta(m)$. The intensity of the colored noise ξ_n is $\langle \xi_n^2 \rangle=\sigma^2$, i.e., it is independent of Γ. The power spectrum $S_\xi(\omega)$ of the colored noise has the form

$$S_\xi(\omega)=\frac{\sigma^2(1-\Gamma^2)}{1-2\Gamma\cos\omega+\Gamma^2} . \tag{5}$$

The power spectrum $S_\xi(\omega)$ takes its maximum at $\omega=0$ if $\Gamma>0$ and at $\omega=\pi$ if $\Gamma<0$.

Thus, we have a two-dimensional stochastic map,

$$x_{n+1}=f(x_n)+g(x_n)\xi_n, \quad \xi_{n+1}=\Gamma\xi_n+\eta_n . \tag{6}$$

The organization of the paper is as follows. In Sec. II we present the linear analysis. Section III is devoted to the bifurcation analysis of the cumulant map. The conclusions are presented in Sec. IV.

II. LINEAR ANALYSIS

To understand the properties of the map (6), we first perform a linear analysis in the vicinity of period-doubling bifurcations for additive noise [$g(x)=1$]. Let us consider the fixed point of period 1 [$x_0^{(0)}=f(x_0^{(0)})$] and linearize the map (6) near this point [14],

$$y_{n+1}=\rho_1 y_n+\xi_n , \tag{7}$$

where $y_{n+1}=x_n-x_0^{(0)}$. $\rho_1=f'(x_0^{(0)})$ is the characteristic multiplier at the fixed point $x_0^{(0)}$. For a fixed point of period 2^k one can analogously write

$$y_{n+2^k}=\rho_k y_n+\Xi_n^{(k)}, \quad \Xi_n^{(k)}=\sum_{l=0}^{2^k-1}\rho_k^{2^k-l-1}\xi_{n+l} , \tag{8}$$

where y_n now is a small disturbance around the fixed point $x_0^{(k)}=f^k(x_0^{(k)})$ of period 2^k and ρ_k is the characteristic multiplier of the k-iterated map at the fixed point $x_0^{(k)}$.

Note that in the noiseless linear map $y_{n+1}=\rho_1 y_n$, there is a bifurcation of the fixed point in the origin at the multiplier value $\rho_1=0$, if $\rho_1>0$ then the fixed point is a stable node and for $\rho_1<0$ it is a stable focus.

The power spectrum of the linearized map (7) can be easily obtained:

$$S_y^{(0)}(\omega)=S_\xi(\omega)/(1-2\rho_1\cos(\omega)+\rho_1^2) . \tag{9}$$

From Eq. (9) it follows that the power spectrum has a broadband peak at the basic frequency $\omega=0$ (or $\omega=2\pi$) if the multiplier ρ_1 is positive. For negative values of ρ_1 the power spectrum has a broadband peak at the subharmonic frequency $\omega=\pi$. Thus, the influence of noise gives rise to the fact that the power spectrum reflects a structure, which is typical for a fixed point of period 2, long before the occurrence of the bifurcation point in the appropriate deterministic system. It corresponds to noisy precursors of bifurcation as described by Wiesenfeld [15].

In the same way we can obtain the power spectrum of a 2^k period fixed point,

$$S_y^{(k)}(\omega)=S_\Xi^{(k)}(\omega)/(1-2\rho_k\cos(2^k\omega)+\rho_k^2) , \tag{10}$$

where

$$S_\Xi^{(k)}(\omega)=S_\xi(\omega)\left\{\left[\sum_{l=0}^{2^k-1}\rho_k^{2^k-l-1}\cos(l\omega)\right]^2+\left[\sum_{l=0}^{2^k-1}\rho_k^{2^k-l-1}\sin(l\omega)\right]^2\right\} .$$

The shapes of the power spectra for $k=2$ are shown in Fig. 1 for $\rho_2=0.1$ and for $\rho_2=-0.1$. If $\rho_k<0$ then the power spectrum $S_y^{(k)}(\omega)$ has broadband peaks at the

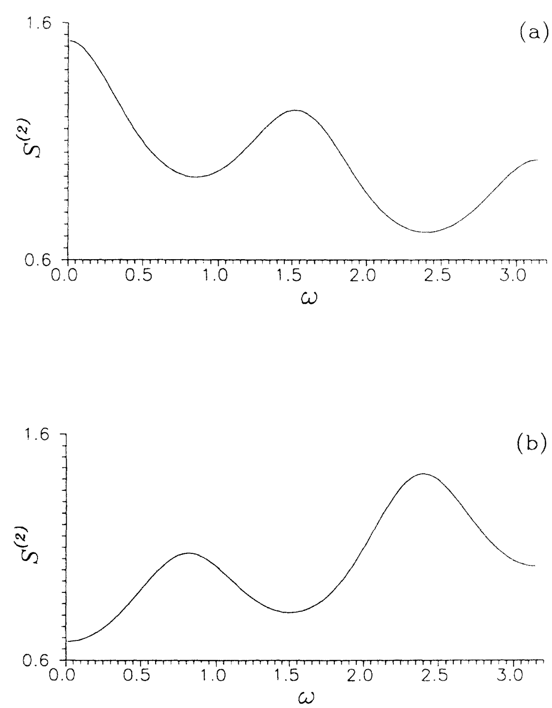

FIG. 1. Power spectrum (10) of the linearized map (8) for $k=2$ (period 4 fixed point): (a) $\rho_2=0.1$; (b) $\rho_2=-0.1$.

subharmonics $\omega_k = (2n+1)\pi/2^{k-1}$, $(n = 0,1,2,\ldots)$.

Let us consider the dependence of the intensity of these subharmonics $I^{(k)}(\Gamma) = S_y^{(k)}(\omega = \omega_k, \Gamma)$ of the 2^k fixed point on the parameter Γ of colored noise. It is easy to see that such a dependence takes its maximal value at the parameter $\Gamma = \Gamma_{\max}^{(k)}$, determined by the expression

$$\Gamma_{\max}^{(k)} = [1 - \sin(\omega_k)]/\cos(\omega_k) \; . \tag{11}$$

Especially, it gives for $k=1$ (period 2 fixed point), $\omega_1 = \pi$ and $\Gamma_{\max}^{(1)} = -1$; for $k=2$, $\omega_2 = \pi/2$ and $\Gamma_{\max}^{(2)} = 0$; for $k=3$ (period 8 fixed point) $\omega_3 = \pi/4$ (or $3\pi/4$) and $\Gamma_{\max}^{(3)} \approx -0.41$ (or 0.41) (see Fig. 2). Thus, the dependence of the subharmonics intensity of 2^k cycles on the colored noise parameter Γ exhibits a resonancelike shape. In other words, the linear response of the system demonstrates resonancelike sensitivity to the variations of the characteristic time scale of noise.

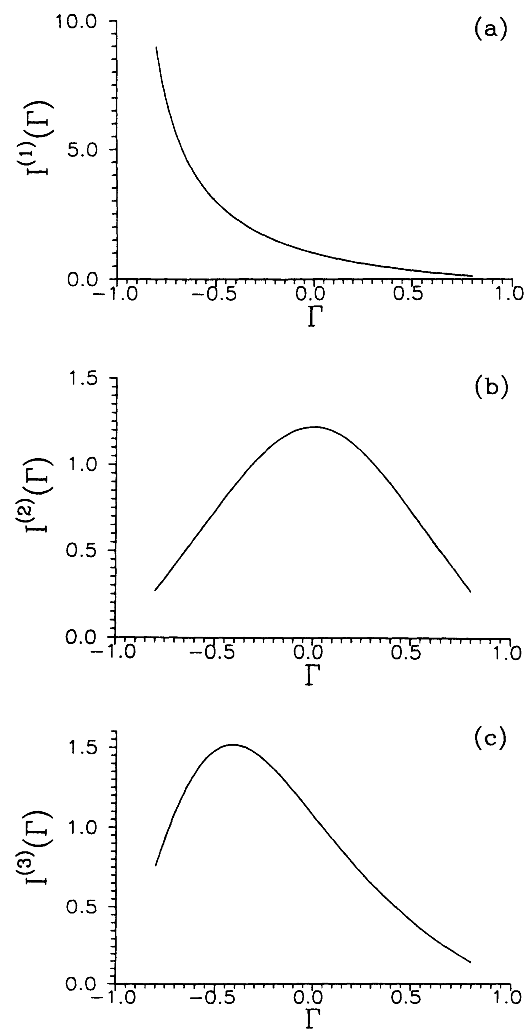

FIG. 2. The dependence of the intensity of subharmonics I^k at the frequency $\omega_k = \pi/2^{k-1}$ versus Γ: (a) $k=1$; (b) $k=2$; (c) $k=3$.

III. BIFURCATION ANALYSIS OF CUMULANT EQUATIONS

We now pass over to the nonlinear bifurcation analysis of the two-dimensional stochastic map (6). When the noise is added to a nonlinear dynamical system, we are forced to consider an averaged stationary characteristics of the appropriate stochastic process instead of limit sets of a dynamical system. Such characteristics may be stationary probability density, power spectrum, correlation function, etc. [16]. One approach to the bifurcation analysis of stochastic systems is using the cumulant expansion of a probability density [17,18,9]. As is well known, a stochastic process can be determined both in terms of probability densities and in terms of cumulants. The evolution of probability density is described by kinetic equations (in our case, the Frobenius-Perron equation). Such equations are of integral or partial-differential (or both) types. The bifurcation analysis of such a type of equations is a very difficult problem. The evolution of cumulants is generally described by the ordinary equations (ordinary differential equations in the case of flow systems or difference equations in the case of systems with discrete time) [19]. Thus it is possible to make a transition from stochastic equations (or from a corresponding kinetic equation) to dynamical ones which describe the evolution of cumulants of a stochastic process. However, due to the nonlinearity of the system, the chain of cumulant equations is unclosed. To close it one can use approximations which take into account only a finite number of cumulants [19]. Thus, we can carry out an ordinary bifurcation analysis [20] of the dynamical systems which describes the evolution of cumulants. The bifurcations of the equilibrium states of a system of cumulant equations correspond to the qualitative changes in the stationary probability density of the appropriate stochastic system. A rather simple procedure is the one-moment approximation which involves only the first-order cumulants (i.e., mean values). In our case this approximation corresponds to the noiseless system. The second-order approximation is the Gaussian one which includes first- and second-order cumulants. This Gaussian approximation describes correctly the behavior in the limit of weak noise, as shown for the bifurcation analysis of a phase transition induced by colored noise [9] and for the analysis of period-doubling bifurcations in the presence of *white* noise [21]. In addition to the last paper, we will use here a two parametrical bifurcation analysis and take into account colored noise.

First, the notions for cumulants are introduced,

$$\begin{aligned} X_n &\equiv \langle x_n \rangle \; , \\ Y_n &\equiv \langle x_n^2 \rangle - \langle x_n \rangle^2 \; , \\ Z_n &\equiv \langle x_n \xi_n \rangle \; , \end{aligned} \tag{12}$$

where the brackets $\langle \; \rangle$ mean the averaging over the realizations of noise η_n. X_n is the mean value of the state variable x_n, Y_n is the mean square displacement of x_n, and Z_n is the mutual moment of x_n and ξ_n.

Consider at first the case of additive noise $[g(x)=1]$. We derive the equations for the cumulants in the Gaussian approximation directly from the stochastic map (6). Especially for the Feigenbaum map (2) it follows a three-dimensional map,

$$X_{n+1}=1-a(X_n^2+Y_n)\ ,$$
$$Y_{n+1}=4aX_n(aX_nY_n-Z_n)+\sigma^2\ ,$$
$$Z_{n+1}=\Gamma(\sigma^2-2aX_nZ_n)\ . \tag{13}$$

Here, the terms of an order higher than Y_n^2 are neglected, since we will analyze the case of weak noise. The initial conditions for the cumulant map (13) are $X_0=x_0$, $Y_0=0$, $Z_0=0$, where x_0 is the fixed point of the corresponding deterministic map.

First in the case of white noise ($\Gamma=0$), the cumulant map (13) is reduced to a two-dimensional one [21]. In the corresponding bifurcation diagram in the parameter plane (σ^2,a) the bifurcation lines of the birth of fixed point of periods 2^1, 2^1, 2^3, 2^4, and 2^5 are shown (Fig. 3). These lines correspond to the condition $\rho=+1$, where ρ is the characteristic multiplier of the fixed point (other multipliers are less than unity). From this diagram, we first conclude that the sequence of period doubling is bounded. There exists boundary values of noise intensity σ_k after which we cannot observe the fixed point of period 2^k (these points are marked by squares). These values refer to the bifurcations of codimension 2 at which the second multiplier of the fixed points becomes equal to "-1". It is essential that the sequence of σ_k satisfies the scaling law $\sigma_k \propto \mu^{-k}$, $\mu=6.592$ (cf. Fig. 4, in which the dependence of σ_k versus k is shown). The theoretical value is $\mu_{\text{th}}=6.557\ldots$. Thus, the technique of cumulant equations allows the correct description of the qualitative picture of the noise influence on the PDB and, in addition, a quantitative description of the universal scaling behavior.

Next, let us take into account the more general case of colored noise. We first consider the bifurcation lines in the parameter plane (Γ,a) of system (13). The results are shown in Fig. 5, where the bifurcation lines correspond-

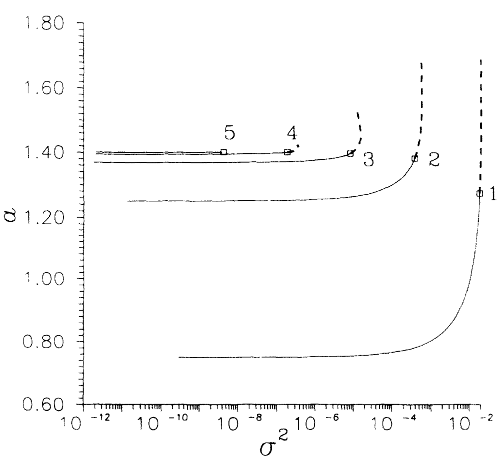

FIG. 3. Bifurcation diagram of the cumulant map (13) in the case of white noise. The bifurcation lines of birth of the fixed points of period 2^k for $k=1,2,3,4,5$ are shown.

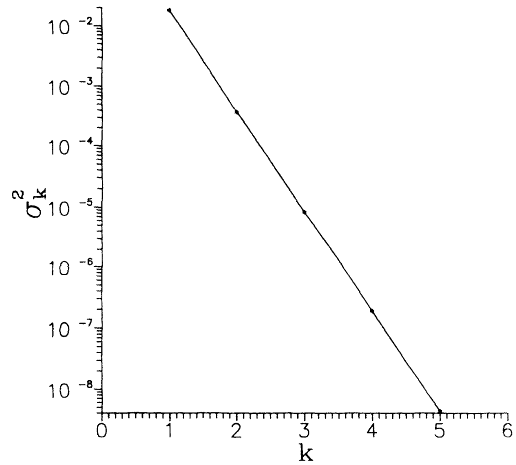

FIG. 4. The dependence of σ_k^2 versus k (asterisk) and a fit by the law μ^k (line), $\mu=6.592$.

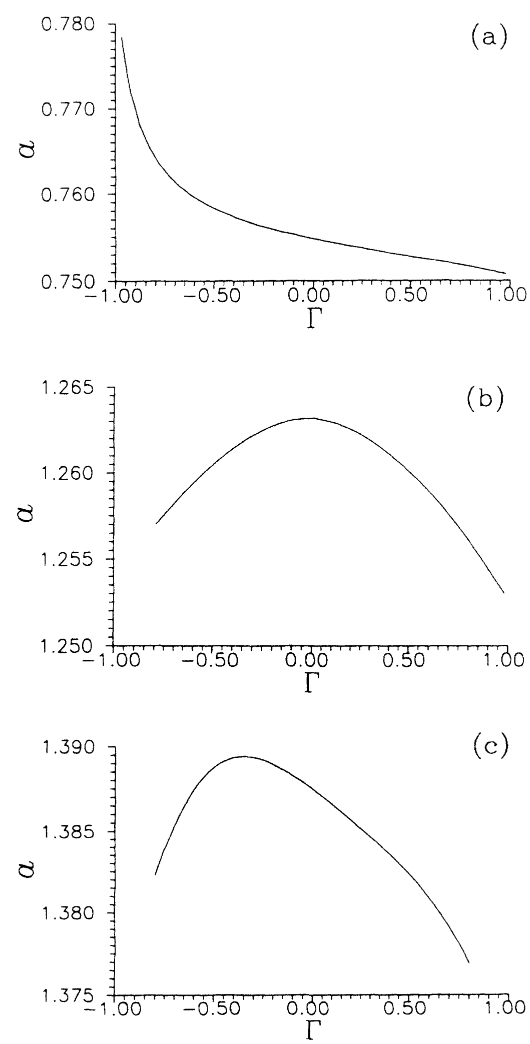

FIG. 5. Bifurcation diagram of the cumulant map (13) in the case of colored noise for the fixed point of period: (a) $k=1$; (b) $k=2$; (c) $k=3$.

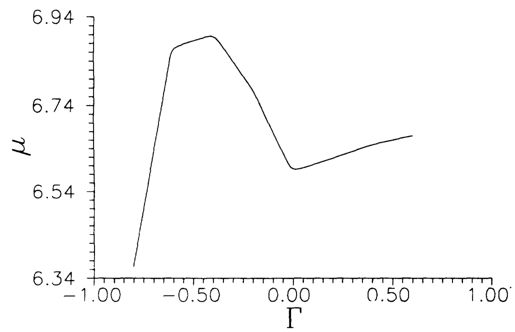

FIG. 6. The dependence of the scaling constant μ versus colored noise parameter Γ.

ing to the condition $\rho=+1$ are plotted for fixed points of periods 2^1 [Fig. 5(a)], 2^2 [Fig. 5(b)], and 2^3 [Fig. 5(c)]. These figures show that the bifurcation lines have maximal values at the parameter Γ strictly corresponding to those obtained from the linear analysis of the power spectrum [cf. Fig. 2 and Eq. (11)]. The bifurcation diagram in the parameter plane (σ^2,a) looks similar to that of white noise. Although the scaling dependence (1) is preserved, we find that the scaling parameter μ develops depending on the colored noise parameter Γ (Fig. 6). Observe from this figure that this dependence $\mu(\Gamma)$ is maximal at the parameter Γ value, which corresponds to the fixed point of high period ($k \geq 3$), and is minimal at $\Gamma=0$, which is related to the case of white noise. (In our calculations we are able to analyze fixed points up to the period 2^5. Probably in the limit $k \to \infty$ we cannot observe such a dependence, since the period of the fixed point becomes greater than the noise correlation time.)

Therefore, the response of PDB to the influence of colored noise demonstrates a resonancelike behavior when the noise correlation time is varied. These results testify that there exists an optimal value of the noise correlation time at which the strength of the interaction between the fluctuating environment and the system achieves a maximal value. Such a situation takes place when the characteristic time scale of the noise coincides with that of the dynamical system.

IV. CONCLUSIONS

In this study we have shown that the universal scaling is preserved in the case of colored noise forcing and that there exists a resonancelike response of the system to the colored noise influence both in the linear sense and in the bifurcation behavior. This effect can be interpreted as another type of the phenomenon of stochastic resonance which has been intensively studied recently [22]. The phenomenon of stochastic resonance takes place in nonlinear stochastic systems excited by external periodic force. When the intensity of external noise provides a characteristic time scale of the system, which coincides with the period of external signal, then the stochastic resonance occurs and it is possibly an amplification of signal at the output of the system. In our case we have quite another situation. The system has characteristic time scales, which are present in the absent of noise, and resonancelike phenomena take place when the characteristic time scale of the external noise is varied. We expect that such behavior is typical for a wide class of dynamical systems that exhibit PDB sequences. For example, in a previous study [23] we have investigated the influence of colored noise on the chaotic regimes of the Lorenz system. We have shown that in the case of a Feigenbaum-type attractor the Lorenz system becomes sensitive to the variation of the noise correlation time. Moreover, the response (we calculated the maximal Lyapunov exponent) demonstrated the same resonancelike behavior as in our case here.

ACKNOWLEDGMENTS

We are grateful to A. Pikovsky, U. Feudel, Yu. L. Klimontovich, L. Schimansky-Geier, H. Herzel, and W. Ebeling for valuable discussions. Two of us (A.N. and V. A.) acknowledge support from Max-Planck-Gesellschaft.

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