Two-spin models with classical chaos and different quantum universality classes

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The local fluctuations in the quantum energy spectrum of a classically completely chaotic autonomous dynamical system are expected to be the same as those in the eigenvalues of Gaussian random Hermitian matrices. That relationship between a dynamical system and the random matrix theory is examined here by introducing a model of an autonomous system of two nonlinearly coupled spins. The proposed model can realize all three universality classes—the orthogonal, the unitary, and the symplectic—of Gaussian random matrices depending upon the nature of the nonlinearity. The proposed system evolves in a finite-dimensional Hilbert space which is in contrast with the existing models of autonomous systems requiring an infinite-dimensional Hilbert space for their description. The model is, therefore, not only free from the unpleasant problem of the truncation of the Hilbert space required for numerical work in the case of an infinite-dimensional Hilbert space but can also be used to examine for a dynamical system those aspects of the random matrix theory that are dependent on the dimension of the matrices. Those aspects are of particular interest in the Brownian motion theory of the transition from a Gaussian orthogonal ensemble to a Gaussian unitary ensemble.

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The characteristics of the local fluctuations in the quantum energy spectrum of generic autonomous dynamical systems which exhibit global chaos in their classical phase space are found to be the same as those of the eigenvalues of the Gaussian random Hermitian matrices (see for example Refs. [1,2], and references therein). A similar relationship is observed between the behavior of the local fluctuations in the quasienergy spectrum of periodically driven systems and the ensembles of circular random matrices [1,2]. The characteristics of the spectral fluctuations of the random matrices of sufficiently large dimensions having the same group of canonical transformations are also the same. The ensemble of random matrices is accordingly classified in three universality classes: the orthogonal ensemble (OE), the unitary ensemble (UE), and the symplectic ensemble (SE). The OE is an ensemble of real orthogonal matrices, the UE is that of complex unitary matrices, whereas the SE is an ensemble of quaternion real matrices. In particular, the distribution $P(s)$ of the nearest-neighbor spacing s for the three universality classes is very closely approximated by the Wigner surmise [1,2]

$$\begin{aligned} P(s)&=(\pi s/2)\exp(-\pi s^2/4) \quad (\mathrm{OE})\,, \\ P(s)&=(32s^2/\pi^2)\exp(-4s^2/\pi) \quad (\mathrm{UE})\,, \\ P(s)&=(2^{18}s^4/3^6\pi^3)\exp(-64s^2/9\pi) \quad (\mathrm{SE})\,. \end{aligned} \tag{1}$$

For small spacings $s, P(s)\sim s^{\beta}$, where $\beta=1$, 2, and 4, respectively, for the OE, UE, and SE, i.e., the neighboring levels repel each other linearly, quadratically, or quartically depending upon whether the ensemble is orthogonal, unitary, or symplectic. To which universality class a dynamical system belongs is determined only by the symmetry properties of the Hamiltonian. If the Hamiltonian is invariant under an antiunitary transformation then the system is described by the orthogonal ensemble, whereas a system without any antiunitary symmetry is described by a unitary ensemble. The systems with a half-integral spin possessing an antiunitary but no other symmetry belong to the symplectic ensemble. The eigenvalue spectrum of those Hamiltonians exhibits Kramers' degeneracy. The levels of regular dynamical systems, on the other hand, generically follow Poissonian distribution, i.e., for those systems, the nearest-neighbor spacing distribution is exponential: $P(s)=\exp(-s)$. The distribution function $P(s)$ is then maximum at $s=0$ or, in other words, the levels of a regular dynamical system exhibit clustering.

The dynamical system theory [3] is able to provide some but not a complete treatment of the relationship between the local fluctuations in the eigenvalue spectrum of random matrices and those in the quantum energy or quasienergy spectrum of classically chaotic systems. The evidence of that relationship is provided largely by studies on the model systems. Those studies—both on autonomous and periodically driven systems—show that the energy or quasienergy fluctuations of the dynamical systems are indeed described by the random matrix theory (see Refs. [1,2] and the references therein). There is, however, a basic difference in the models studied for the two kinds of dynamical systems, in that whereas the Hilbert space of quantized periodically driven models investigated so far is finitely dimensional; that for autonomous systems is infinitely dimensional. For autonomous systems there is, consequently, the unpleasant problem of the cutting off of the Hilbert space for numerical calculations. Moreover, those models of autonomous systems cannot test those aspects of the random matrix theory that depend on the dimensions of the matrices. Such aspects are of particular interest for studies of the transition from one universality class to the other [1,4]. Here we propose a model of an autonomous dynamical system capable of realizing all three universality classes in a finite-dimensional Hilbert space. We discuss also the results of our attempts to test some predictions of the Brownian motion theory [1] of transition from linear to quadratic repulsion. To our knowledge, the model of Ref. [5] is the only example reported in the literature of an autonomous dynamical system capable of realizing all three universality classes.

The study of classically chaotic autonomous models is also important because the description of the corresponding quantized Hamiltonians as random matrices has no *a priori* justification in the sense that (a) most of the elements of the Hamiltonians in standard representations are zero due to various selection rules so that the assumption of the matrix elements being random variables does not hold. In fact the matrices of the usual dynamical systems look more like sparse matrices (for a discussion of the applicability of the Wigner surmise in those cases, see Ref. [6]). (b) The energy-level density of the dynamical Hamiltonians do not follow Wigner's semicircle law, which is the law obeyed by the energy levels of a Gaussian random matrix. The model introduced here also suffers from the aforementioned noncompliance with the basic requirements of a Gaussian random matrix, and yet gives nearest-neighbor spacing statistics in agreement with the Wigner surmise. However, it has not been possible to confirm the predictions of the theory of Brownian motion of random matrices for the transition from Gaussian orthogonal ensemble (GOE) to Gaussian unitary ensemble (GUE) in the proposed model. Note that the quasienergies of periodically driven systems are uniformly distributed on a circle of unit radius which is in accordance with the requirements of a circular ensemble. The periodically driven systems are therefore better placed compared to the autonomous systems as regards their relationship with random matrix theory. For example, the model of a kicked top, which is an example of a periodically driven system, has been shown [4] to undergo the transition from circular orthogonal ensemble (COE) to circular unitary ensemble (CUE) in accordance with the predictions of Brownian motion theory.

Our system consists of two nonlinearly coupled spins denoted below by $\mathbf{L}$ and $\mathbf{M}$. In the case of classical motion, the spins $\mathbf{L}$ and $\mathbf{M}$ obey Poisson bracket relations, whereas in the quantum version $\mathbf{L}\rightarrow\hbar\hat{\mathbf{L}}$ and $\mathbf{M}\rightarrow\hbar\hat{\mathbf{M}}$, where the components of the operators $\hat{\mathbf{L}}$ and $\hat{\mathbf{M}}$ obey the angular momentum commutation relations

$$[\hat{\mathbf{L}}_x,\hat{\mathbf{L}}_y]=i\hat{\mathbf{L}}_z,\quad [\hat{\mathbf{M}}_x,\hat{\mathbf{M}}_y]=i\hat{\mathbf{M}}_z,\quad [\hat{\mathbf{L}}_i,\hat{\mathbf{M}}_j]=0,\ \text{etc}\ . \tag{2}$$

Besides being coupled nonlinearly to each other, the spins are coupled linearly to external magnetic fields denoted below by the vectors $\mathbf{a}$ and $\mathbf{b}$. The nonlinear coupling is chosen so as to make the system acquire the desired symmetry. The symmetry operations can act independently or jointly on the two spins. First, we construct a Hamiltonian possessing an antiunitary symmetry. The simplest Hamiltonian with antiunitary symmetry for the coupled spins interacting with external magnetic fields is obtained by coupling a component of one spin with a component of the other. Thus if λ_x is the coupling between the x components of the two spins, then the Hamiltonian

$$H_{\mathrm{GOE}}=\mathbf{a}\cdot\mathbf{L}+\mathbf{b}\cdot\mathbf{M}+\lambda_x L_x M_x\ , \tag{3}$$

is invariant under the antiunitary transformation $\mathbf{L}\rightarrow-\mathbf{L}$ and $M\rightarrow-\mathbf{M}$ followed by the unitary transformation consisting of the reflection in the plane formed by $\mathbf{a}$ and the x axis for the spin $\mathbf{L}$, and that in the plane formed by the vector $\mathbf{b}$ and the x axis for the spin $\mathbf{M}$. The antiunitary transformation changes the sign of the linear terms in (3). That change in the sign is restored by the aforementioned unitary transformation because the components of the angular momentum vectors in the plane of reflection change sign, whereas those perpendicular to it remain unchanged. The Hamiltonian (3) is a generalization of the model introduced by Feingold and Peres [7].

Next we construct a Hamiltonian which breaks the antiunitary symmetry possessed by (3). The simplest such Hamiltonian is constructed by adding to (3) another quadratic interaction term like L_zM_z. However, we found better statistics working with a Hamiltonian with cubic nonlinearity:

$$\begin{aligned}H_{\mathrm{GUE}}=&\,\mathbf{a}\cdot\mathbf{L}+\mathbf{b}\cdot\mathbf{M}+\lambda_x L_x M_x+\lambda_{xz}L_z(M_xM_z+M_zM_x)+\lambda_{yz}L_z(M_yM_z+M_zM_y)+\mu_{xz}M_z(L_xL_z+L_zL_x)\\&+\mu_{yz}M_z(L_yL_z+L_zL_y)\ .\end{aligned} \tag{4}$$

Here the λ's and μ's are the coupling constants. The reason why the antiunitary symmetry of the quadratic Hamiltonian could be broken more easily with cubic rather than quadratic nonlinearity may be a reflection of the intuitive expectation that a nonlinearity of order higher than that of the antiunitarily symmetric part is perhaps more effective in breaking that symmetry.

Finally, the Hamiltonian whose quantum energies exhibit Kramers' degeneracy is obtained by allowing it to have an antiunitary symmetry but no other geometric symmetry, along with restricting the total spin only to half-integral values. We have obtained the Kramers' degeneracy by working with a Hamiltonian which is symmetric under the antiunitary symmetry $L\rightarrow-L, M\rightarrow-M$ and has a quartic nonlinearity:

$$\begin{aligned}H_{\mathrm{GSE}}=&\,\lambda_x L_x M_x+\mu_z L_z^2M_z^2+\alpha_{1xz}L_xM_z+\alpha_{2xz}L_zM_x+\alpha_{1yz}L_yM_z\\&+\alpha_{2yz}L_zM_y+\beta_{1xy}(L_xL_z+L_zL_x)+\beta_{1yz}(L_yL_z+L_zL_y)+\beta_{2xz}(M_zM_x+M_xM_z)+\beta_{2yz}(M_zM_y+M_yM_z)\ .\end{aligned} \tag{5}$$

The numerical values of the coupling constants in (5) are chosen so as to avoid the symmetry of the Hamiltonian under the exchange of the spins.

Note that for the Hamiltonians (3)–(5), besides the energy E, the magnitudes $|\mathbf{L}|$ and $|\mathbf{M}|$ of the individual spins are the constants of motion. Feingold and co-workers [7] have studied classical motion corresponding to (3) with $\mathbf{a}=\mathbf{b}=(0,0,1)$ and $\lambda_x=1$ and have identified the regions of regular, predominantly chaotic, and mixed motions in the space of the values of E, $|\mathbf{L}|$, and $|\mathbf{M}|$. They have found, in particular, that for $|\mathbf{L}|=|\mathbf{M}|=3.5$ the classical motion is predominantly chaotic for $|E|<6.6$, regular for $|E|>9.1$, and mixed in between.

In the quantum version, $\hat{\mathbf{L}}^2$ and $\hat{\mathbf{M}}^2$ commute with the Hamiltonians (3)–(5) and hence the eigenstates of the Hamiltonians can be labeled by the eigenvalues of $\hat{\mathbf{L}}^2$ and $\hat{\mathbf{M}}^2$. The eigenvalues of $\hat{\mathbf{L}}^2$ and $\hat{\mathbf{M}}^2$ are $\hbar^2L(L+1)$ and $\hbar^2M(M+1)$, respectively, where the total angular momentum quantum numbers L and M are integers or half-integers. The classical magnitude of the spins is related to quantum numbers L and M as $|\mathbf{L}|^2=\hbar^2L(L+1)$ and $|\mathbf{M}|^2=\hbar^2M(M+1)$. For given values of the classical spins and quantum numbers L and M, the value of $\hbar$ is thus determined. The lower the value of $\hbar$, the better the agreement expected between local fluctuations in the quantum energies and the random matrix theory [1]. The value of $\hbar$ can be decreased by increasing the value of L and M, i.e., by increasing the dimension of the Hilbert space because, for a given L and M, the spins $\mathbf{L}$ and $\mathbf{M}$ span the Hilbert space of dimensions $(2L+1)$ and $(2M+1)$, respectively, so that the coupled spin system evolves in a space of dimension $(2L+1)(2M+1)$. However, if the Hamiltonians are symmetric under the exchange of $\mathbf{L}$ and $\mathbf{M}$, then the Hilbert space reduces to two decoupled spaces. The state of the system is symmetric under the exchange of the spins in one space, and antisymmetric in the other. It follows that if $|\mathbf{L}|=|\mathbf{M}|=L$, then the dimension of the symmetric space is $(L+1)(2L+1)$, and that of the antisymmetric one is $L(2L+1)$ [6]. Since it is the smallness of $\hbar$ that matters in achieving agreement between the dynamical systems and the random matrix theory, the reduction of the Hilbert space proves very useful in numerical work because for a given total spin, i.e., for a given $\hbar$ the diagonalization of a spin-exchange symmetric system is required to be carried in a space whose dimension is almost half of the dimension of the full space spanned by a nonsymmetric system.

To compute local fluctuations in the quantum energies of (3) and (4), we have taken $|\mathbf{L}|=|\mathbf{M}|=3.5$, which are the values for which Feingold and co-workers [7] investigated the chaotic properties of (3) in its classical phase space for $\mathbf{a}=\mathbf{b}=(0,0,1)$ and $\lambda_x=1$. We let $L=M$ and $\mathbf{a}=\mathbf{b}$ in (3) and $\lambda_{xz}=\mu_{xz}$ and $\lambda_{yz}=\mu_{yz}$ in (4), so that both (3) and (4) are symmetric under the exchange of the two spins. We take $L=M=16$ and diagonalize the Hamiltonian in the symmetric space of dimension 561. For better statistics, we choose ten different closely spaced values of the parameters for each Hamiltonian, and compute the corresponding set of eigenvalues. The values of the parameters for which we observed the GOE behavior using H_{GOE} are

$$\mathbf{a}=\mathbf{b}=(1.1+g,0.1+g,1.0),\quad \lambda_x=1.1+g\ , \tag{6}$$

where $g=0$, 0.05, 0.25, 0.3, 0.35, 0.4, 0.45, 0.6, 0.9, and 1.05. The values of the parameters in (4) for which we observed GUE behavior are $\lambda_{xz}=\mu_{xz}=0.025$, $\lambda_{yz}=\mu_{yz}=0.05$,

$$\begin{aligned}&\mathbf{a}=\mathbf{b}=(1.154+g,0.105+g,1.0)\ ,\\&\lambda_x=1.098+g\ ,\end{aligned} \tag{7}$$

$g=0$, 0.524, 0.262, 0.314, 0.367, 0.419, 0.472, 0.629, 0.944, and 1.101.

To realize Kramers' degeneracy by using (5), one of the spins is required to be half-integral, and the other to be integral. We should also avoid symmetries other than the antiunitary one. We therefore took $L=10.5$ and $M=10$, and found good agreement with GSE statistics for the level spacings of (5) with $\lambda_z=1.0+g$, ($g=0$, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, and 0.45), $\mu_z=0.005\lambda_x$, $\alpha_{1xz}=0.1$, $\alpha_{2xz}=0.2$, $\alpha_{1yz}=0.2$, $\alpha_{2yz}=0.1$, $\beta_{1xz}=\beta_{2xz}$

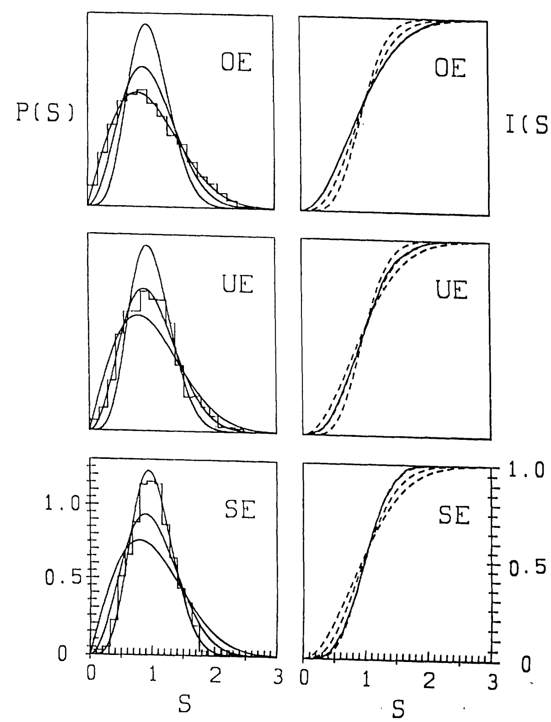

FIG. 1. The histogram on each of the plots in the left-hand column of the figure is the numerically computed distribution $P(S)$ for the nearest-neighbor spacing S of the eigenvalues of the Hamiltonian corresponding to the ensemble marked on the plot. The solid curve with maximum height on each of the plots for the histograms is the $P(S)$ for the Wigner surmise for the SE followed by that for the UE and OE. The solid curve on each of the plots in the right-hand column is the numerically computed staircase function $I(S)$ as a function of S for the Hamiltonian corresponding to the ensemble marked on the plot. The dashed curves on each of those plots are the $I(S)$ for the Wigner surmise. The dashed curve that is uppermost near $S=0$ in each of those plots is the Wigner surmise for the OE, followed by those for the UE and SE.

$=0.3$, and $\beta_{1yz}=\beta_{2yz}=0.4$.

To determine the statistical properties of the local fluctuations in the quantum energies, we unfold the spectrum in each case to a constant density, and rescale it so as to have the mean spacing equal to unity. The histogram of the probability density $P(s)$ and the staircase function

$$I(s)=\int_0^s P(s)ds \;, \tag{8}$$

obtained after averaging over ten sets of eigenvalues for each universality class are plotted in Fig. 1 as a function of s. Plotted along with the histogram are the curves for the Wigner surmise (1). The curve with the maximum height on each of the plots of $P(s)$ corresponds to the Wigner surmise for SE followed by that for UE and OE, respectively. The numerically obtained histogram in each case clearly justifies the identification of the Hamiltonians as belonging to a particular ensemble. That identification is further confirmed by the plots of the staircase function $I(s)$ as a function of s in Fig. 1. The solid curves in those plots are the results of the numerical computation whereas the dashed curves represent (8) for each of the three universality classes. The solid curve in each case almost overlaps with the one corresponding to the appropriate universality class.

The model Hamiltonians introduced here were used also to study the transition from GOE to GUE by determining the statistics of the nearest-neighbor spacing of the Hamiltonian $H_{\mathrm{GOE}}+\lambda H_{\mathrm{GUE}}$ as λ is varied from zero onwards. Transition from GOE at $\lambda=0$ to GUE for large λ is indeed observed. We determined the value λ_c of λ at which the transition takes place for different values of L, i.e., for different dimensions of the Hilbert space. However, the transition does not follow the law $\lambda_C \sim 1/\sqrt{N}$ that is predicted for an ensemble of random matrices of dimension N on the basis of Brownian motion theory [1]. In fact the plot of $\ln(\lambda_c)$ as a function of $\ln(N)$ is found to be just an irregular scatter of points. Increasing the dimension of the Hilbert space also does not show any tendency of agreement with predictions of the random matrix theory. The same negative result also has been observed by Lenz, Wiedemann, and Saher [8]. The observed disagreement can be due to the fact that, strictly speaking, the model Hamiltonian is more like a sparse matrix because most of its elements are zero, and the sparsity evidently continues to increase with the dimension of the Hilbert space. Since the random matrix theory is for an ensemble of matrices whose elements are random variables, it need not be applicable to an ensemble of sparse matrices, as most of their elements are always zero, i.e., deterministic. Our results indicate that although the Wigner surmise may hold for matrices as sparse as encountered here, some other characteristics of the local fluctuations in Gaussian random matrices may not hold. The question of the applicability of the Wigner surmise to sparse matrices has also been discussed in Ref. [6]. However, the reduction of sparsity by increasing the nonlinearity also did not help in improving the agreement of the observed transition from GOE to GUE with random matrix theory [8]. The disagreement between the dynamical system behavior and random matrix theory also could be due to the possible presence of large regions of regular motion for the intermediate values of the transition parameter λ. However, exploring the nature of the classical motion for all intermediate values of λ is a formidable task. The issue of the reasons for the noncompliance of the transition from GOE to GUE of the dynamical system proposed here with the predictions of the random matrix theory is therefore left open.

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