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# Exact results of a solvable general spin-1 model

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We have introduced a type of universal transformation between spin variables $s_i$ and $\sigma_i$ to map the general spin-1 model onto the spin-$\frac{1}{2}$ Ising model and have employed the exact results of the latter to find the exactly solvable cases of the spin-1 model. Our transformations are not one-to-one correspondences between $s_i$ and $\sigma_i$ and they are applicable to all lattices, depending on the lattice structure only through the coordination number. On square, triangle, and honeycomb lattices the exact critical points are shown in certain subspaces of parameter space ($K$, $J$, $L$, $H$, and $\Delta$).

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## I. INTRODUCTION

Recently the study of the general spin-1 Ising model has received much attention [1–4]. This is because it exhibits rich phase transition phenomena from the first order to higher order and from usual critical points to multicritical points. As the special case of the general spin-1 Ising model, the Blume-Emery-Griffiths (BEG) model can be used to describe phase separation and superfluid ordering in $^3$He-$^4$He mixtures [5] or phase separation and ferromagnetism in binary alloys [6]. In the recent two decades the general spin-1 model has been extensively investigated by means of a variety of approximation methods such as mean-field theory [1,5,7,8], Bethe approximation [9], renormalization group techniques [2,3,10], series expansion methods [11], and Monte Carlo methods [12]. A few exact solutions have also been obtained. Griffiths [13] and Berker and Wortis [3] have got some exact solutions on square lattice. Very recently, Kolesík and Šamaj [4] have mapped the general spin-1 model onto three- and two-state vertex models and found its exactly solvable cases of the honeycomb lattice under the certain constrained conditions. Their results can cover the precious exact solutions [13–20]. Moreover, they argued their method can be extended to lattices with arbitrary coordination number. However, Lipowski and Suzuki [21] suggest that it is not suitable for square lattice. Maybe the structure of the honeycomb lattice (the coordination number $\gamma=3$) is essential to the method.

In this paper, we propose a kind of transformation between spin variables of the general spin-1 model and the spin-$\frac{1}{2}$ Ising model, and thus map the spin-1 model onto spin-$\frac{1}{2}$ Ising model with an external field under certain constrained conditions, which determines the corresponding subspaces of interaction parameters. Therefore we can employ the exact solutions of the latter to acquire the exact results of the former. The most important features of our method are that the transformations between spin variables are not one-to-one correspondences and they are applicable to all lattices and depend on the lattice structure only through the coordination number. We will show some exact results on square, triangle, and honeycomb lattices. Unfortunately, we cannot exhibit the three-dimensional solution of the spin-1 model because the exact solution of the three-dimensional spin-$\frac{1}{2}$ Ising model has not yet been found.

The paper is organized as follows. In Sec. II, we perform our transformation of spin variable and map the spin-1 Ising model onto the spin-$\frac{1}{2}$ Ising model with an external field. In Sec. III, we present exact critical points on two-dimensional square, triangle, and honeycomb lattices in terms of the known exact results of the spin-$\frac{1}{2}$ Ising model. Finally, in Sec. IV, we make a brief conclusion and give some discussion.

## II. MAPPING SPIN-1 MODEL ONTO SPIN-$\frac{1}{2}$ ISING MODEL IN A FIELD

The Hamiltonian of the general spin-1 Ising model is written as

$$-\beta\mathcal{H}=J\sum_{\langle i,j\rangle}s_is_j+L\sum_{\langle i,j\rangle}\tfrac{1}{2}(s_i+s_j)s_is_j+K\sum_{\langle i,j\rangle}s_i^2s_j^2+H\sum_{\langle i\rangle}s_i-\Delta\sum_{\langle i\rangle}s_i^2,\quad s_i=0,\pm1\,, \tag{1}$$

where $\beta=1/k_BT$ ($k_B$ is the Boltzmann constant) and $\langle i,j\rangle$ denotes the nearest-neighbor spin pairs, the sum-

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mations $\sum_{\langle i,j\rangle}$ and $\sum_{\langle i\rangle}$ are carried out over all nearest-neighbor pairs and spins, respectively. $J$, $L$, and $K$ are the reduced "interaction" parameters, $H$, the external field and $\Delta$ the crystal field. When $L=H=0$ in the Hamiltonian (1) the model reduces to the BEG model [5], and when $L=K=H=0$ to the Blume-Capel model [7].

As we mentioned, Kolesík and Šamaj [4] have mapped the spin-1 model onto three- and two-state vertex models on honeycomb lattice and found the exactly solvable cases of the general spin-1 model under following constrained conditions:

$$e^{K}(e^{2J}-e^{-2J})-2(e^{J}\cosh L-e^{-J})=0 , \tag{2a}$$

$$A^3+e^{H-\Delta}[A-(e^{-J+K}-1)B]^3 +e^{-(H+\Delta)}[A-(e^{J+K-L}-1)B]^3=0 , \tag{2b}$$

where

$$A=e^{J+K-\Delta}[e^{L+H}(e^{-J+K}-1) +e^{-(L+H)}(e^{J+K-L}-1)] ,$$

$$B=1+e^{J+K-\Delta}[e^{L+H}+e^{-(L+H)}] ,$$

or

$$e^{K}(e^{2J}-e^{-2J})-2(e^{J}\cosh L-e^{-J})=0 , \tag{3a}$$

$$e^{\Delta}=-\frac{(e^{J+L}+e^{J-L}-2e^{-J})^3}{e^{J}(e^{2J}-e^{-2J})[e^{H}(e^{J+L}-e^{-J})^2+e^{-H}(e^{J-L}-e^{-J})^2]} . \tag{3b}$$

Equations (2a) and (2b) completely cover the known solvable cases [13–20] of the BEG model

$$e^{K}\cosh J-1=0 , \quad L=H=0 , \tag{4}$$

or

$$J=L=H=0 . \tag{5}$$

We now proceed to perform a way to get the exactly solvable cases. Let us first write down the reduced Hamiltonian of the spin-$\frac{1}{2}$ Ising model in the presence of a field.

$$-\beta\mathcal{H}'=J'\sum_{\langle i,j\rangle}\sigma_i\sigma_j+H'\sum_{\langle i\rangle}\sigma_i, \quad \sigma_i=\pm 1 , \tag{6}$$

where $\sigma_i$ $(i=1,2,\ldots,N)$ is the spin variables. Since the Ising model has been exactly solved on two-dimensional lattices, we expect a map between the spin-1 model and the spin-$\frac{1}{2}$ model to present the exact solution of the spin-1 model via the latter.

We introduce the following transformation between spin variables $s_i$ and $\sigma_i$:

$$\sigma_i=-s_i^2+s_i+1 . \tag{7}$$

Obviously, we attain

$$\sigma_i=\begin{cases}1 & \text{when } s_i=1,0\\ -1 & \text{when } s_i=-1,\end{cases} \tag{8}$$

which shows both spin states with $s_i=1,0$ correspond to a spin state $\sigma_i=1$. Therefore the transformation (7) gives a non-one-to-one correspondence relation between spin variables of the two models.

Substituting (7) into (6) produces

$$-\beta\mathcal{H}'=J'\sum_{\langle i,j\rangle}s_is_j-J'\sum_{\langle i,j\rangle}(s_i+s_j)s_is_j +J'\sum_{\langle i,j\rangle}s_i^2s_j^2+(\gamma J'+H')\sum_{\langle i\rangle}s_i -(\gamma J'+H')\sum_{\langle i\rangle}s_i^2+MJ'+NH' , \tag{9}$$

where $M$ indicates the total number of nearest-neighbor spin pairs, $N$ the total number of sites and $\gamma$ is the coordination number.

Comparing (9) with (1), it is easily seen that if one let

$$J=-L/2=K=J' , \quad H=\Delta=\gamma J'+H' , \tag{10}$$

it will lead to

$$-\beta\mathcal{H}=-\beta\mathcal{H}'-C_1 , \tag{11}$$

where

$$C_1=MJ'+NH' . \tag{12}$$

The Eq. (11) shows the map between both models via transformation (7).

Similarly, if we choose the following transformations:

$$\sigma_i=-s_i^2-s_i+1 \tag{13}$$

and

$$\sigma_i=2s_i^2-1 , \tag{14}$$

we will obtain

$$\sigma_i=\begin{cases}1 & \text{when } s_i=-1,0\\ -1 & \text{when } s_i=1,\end{cases} \tag{15}$$

$$\sigma_i=\begin{cases}1 & \text{when } s_i=\pm 1\\ -1 & \text{when } s_i=0,\end{cases} \tag{16}$$

respectively.

The transformations (13) and (14) also give non-one-to-one correspondence relations between spin variables. Following the same procedure as before, we reach

$$-\beta\mathcal{H}'=J'\sum_{\langle i,j\rangle}s_is_j+J'\sum_{\langle i,j\rangle}(s_i+s_j)s_is_j +J'\sum_{\langle i,j\rangle}s_i^2s_j^2-(\gamma J'+H')\sum_{\langle i\rangle}s_i -(\gamma J'+H')\sum_{\langle i\rangle}s_i^2+MJ'+NH', \tag{17}$$

for the transformation (13), and

$$-\beta\mathcal{H}'=4J'\sum_{\langle i,j\rangle}s_i^2s_j^2-(2\gamma J'-2H')\sum_{\langle i\rangle}s_i^2+MJ'-NH' , \tag{18}$$

for the transformation (14). In the Hamiltonians (17) and (18) if we set

$$\begin{aligned}&J=L/2=K=J' ,\\&-H=\Delta=\gamma J'+H' ,\end{aligned} \tag{19}$$

and

$$\begin{aligned}&K=4J' ,\\&J=L=H=0 ,\\&\Delta=2\gamma J'-2H' ,\end{aligned} \tag{20}$$

respectively, we will derive

$$-\beta\mathcal{H}=-\beta\mathcal{H}'-C_1 \tag{21}$$

and

$$-\beta\mathcal{H}=-\beta\mathcal{H}'-C_2 , \tag{22}$$

where $C_1$ is determined by (12), and

$$C_2=MJ'-NH' . \tag{23}$$

The Eqs. (21) and (22) also show the connection between both models via transformations (13) and (14).

In order to study the characteristic of phase transition, we first write down the partition function of model (1)

$$Z=\sum_{\{s_i=0,\pm1\}}\exp(-\beta\mathcal{H}\{s_i\}) . \tag{24}$$

Let us suppose the constrained conditions (10), (19), and (20), are satisfied, thus we can employ the Eqs. (11), (21), and (22) to rewrite (24) as follows:

$$\begin{aligned}Z&=e^{-C}\sum_{\{\sigma_i=\pm1\}}P(\sigma_1)P(\sigma_2)\cdots P(\sigma_N)\exp(-\beta\mathcal{H}'\{\sigma_i\})\\&=e^{-C}\sum_{\{\sigma_i=\pm1\}}\exp[-\beta\mathcal{H}'\{\sigma_i\}+\sum_i\ln P(\sigma_i)]\\&=e^{-C}\sum_{\{\sigma_i=\pm1\}}\exp\left[J'\sum_{\langle i,j\rangle}\sigma_i\sigma_j+H'\sum_{\langle i\rangle}\sigma_i+\sum_{\langle i\rangle}\ln P(\sigma_i)\right] ,\end{aligned} \tag{25}$$

where $C=C_1$ or $C_2$ depending on the use of (10), (19), or (20). $P(\sigma_i)$ denotes the degeneracy arising from the transformation between $s_i$ and $\sigma_i$. Obviously, we here have to choose $P(1)=2$ and $P(-1)=1$. When $P(1)=P(-1)=1$, the expression (25) reduces to the usual case.

For the purpose of comparison, we prefer to change (25) into a usual standard form through a further transformation. We write the partition function (25) as

$$Z=e^{-C}\sum_{\{\sigma_i=\pm1\}}\exp\left[-\beta\sum_{\langle i,j\rangle}\mathcal{H}'_{ij}\right] , \tag{26}$$

where

$$-\beta\mathcal{H}'_{ij}=J'\sigma_i\sigma_j+\frac{H'}{\gamma}(\sigma_i+\sigma_j)+\frac{1}{\gamma}[\ln P(\sigma_i)+\ln P(\sigma_j)] . \tag{27}$$

The above expression can be transformed into

$$-\beta\mathcal{H}'_{ij}=\tilde{J}\sigma_i\sigma_j+\frac{\tilde{H}}{\gamma}(\sigma_i+\sigma_j)+A , \tag{28}$$

in which $\tilde{J}$, $\tilde{H}$, and $A$ are determined by the set of equations:

$$J'+\frac{2H'}{\gamma}+\frac{2}{\gamma}\ln P(+1)=\tilde{J}+\frac{2\tilde{H}}{\gamma}+A , \tag{29}$$

$$J'-\frac{2H'}{\gamma}+\frac{2}{\gamma}\ln P(-1)=\tilde{J}-\frac{2\tilde{H}}{\gamma}+A , \tag{30}$$

$$-J'+\frac{1}{\gamma}[\ln P(+1)+\ln P(-1)]=-\tilde{J}+A . \tag{31}$$

The solution is

$$\begin{aligned}&\tilde{J}=J' ,\\&A=\frac{1}{\gamma}[\ln P(+1)+\ln P(-1)]=\frac{1}{\gamma}\ln 2 ,\\&\tilde{H}=H'+\tfrac{1}{2}[\ln P(+1)-\ln P(-1)]=H'+\tfrac{1}{2}\ln 2 .\end{aligned} \tag{32}$$

Using Eqs. (27)–(32), we finally obtain the following form as usual:

$$\begin{aligned}Z&=e^{-C}\sum_{\{\sigma_i=\pm1\}}\exp\left\{\tilde{J}\sum_{\langle i,j\rangle}\sigma_i\sigma_j+\frac{\tilde{H}}{\gamma}\sum_{\langle i,j\rangle}(\sigma_i+\sigma_j)+\sum_{\langle i,j\rangle}A\right\}\\&=e^{-C+AM}\sum_{\{\sigma_i=\pm1\}}\exp\left\{\tilde{J}\sum_{\langle i,j\rangle}\sigma_i\sigma_j+\tilde{H}\sum_{\langle i\rangle}\sigma_i\right\} .\end{aligned} \tag{33}$$

Now, the general spin-1 model is completely equivalent to the Ising model with nearest-neighbor interaction parameter $\tilde{J}$ and an external field $\tilde{H}$. We would like to emphasize the equivalence occurs only in the subspaces of parameter space determined by (10), (19), or (20) and the periodic boundary condition should also be considered. As we see once again our transformations depend on the lattice structure only through the coordination number.

## III. EXACT RESULTS

We now employ the equivalence of both models to yield the exact results of the spin-1 model. In order to obtain the specific results, we first consider the case in which Eq. (10) is satisfied (case 1). Thus, Eq. (33) takes the form

$$Z=e^{-C_1+AM}\sum_{\{\sigma_i=\pm1\}}\exp\left\{\tilde{J}\sum_{\langle i,j\rangle}\sigma_i\sigma_j+\tilde{H}\sum_{\langle i\rangle}\sigma_i\right\} , \tag{34}$$

where $-C_1+AM=MJ-NH+(N/2)\ln 2$, $\tilde{J}=J'=J$, and $\tilde{H}=H'+\frac{1}{2}\ln 2=H-\gamma J+\frac{1}{2}\ln 2$. As we have known, the critical point $(\tilde{J}_s^*,\tilde{H}_s^*)$ of the spin-$\frac{1}{2}$ Ising model on square lattice is determined by [22]

$$\tanh\tilde{J}_s^*=\sqrt{2}-1\ ,\qquad \tilde{H}_s^*=0\ . \tag{35}$$

Using Eqs. (10), (32), and (35) yields the critical point $P_{1S}(J_{1S}^*,L_{1S}^*,K_{1S}^*,H_{1S}^*,\Delta_{1S}^*)$ of the general spin-1 Ising model, where

$$J_{1S}^*=\tilde{J}_S^*,\quad L_{1S}^*=-2\tilde{J}_S^*,\quad K_{1S}^*=\tilde{J}_S^*\ ,\qquad H_{1S}^*=4\tilde{J}_S^*-\frac{1}{2}\ln 2,\quad \Delta_{1S}^*=4\tilde{J}_s^*-\frac{1}{2}\ln 2\ . \tag{36}$$

Similarly, on triangle and honeycomb lattices, the critical points $(\tilde{J}_T^*,\tilde{H}_T^*)$ and $(\tilde{J}_H^*,\tilde{H}_H^*)$ of the spin-$\frac{1}{2}$ Ising model are determined by [23]

$$\tanh\tilde{J}_T^*=1-\sqrt{3}\ ,\qquad \tilde{H}_T^*=0 \tag{37}$$

and

$$\tanh\tilde{J}_H^*=1/\sqrt{3}\ ,\qquad \tilde{H}_H^*=0\ , \tag{38}$$

respectively. Therefore, we get the critical points of the spin-1 model $P_{1T}(J_{1T}^*,L_{1T}^*,K_{1T}^*,H_{1T}^*,\Delta_{1T}^*)$ (for triangle lattice) and $P_{1H}(J_{1H}^*,L_{1H}^*,K_{1H}^*,H_{1H}^*,\Delta_{1H}^*)$ (for honeycomb lattice), where

$$J_{1T}^*=\tilde{J}_T^*,\quad L_{1T}^*=-2\tilde{J}_T^*,\quad K_{1T}^*=\tilde{J}_T^*\ ,\qquad H_{1T}^*=6\tilde{J}_T^*-\frac{1}{2}\ln 2,\quad \Delta_{1T}^*=6\tilde{J}_T^*-\frac{1}{2}\ln 2\ , \tag{39}$$

$$J_{1H}^*=\tilde{J}_H^*,\quad L_{1H}^*=-2\tilde{J}_h^*,\quad K_{1H}^*=\tilde{J}_H^*\ ,\qquad H_{1H}^*=3\tilde{J}_h^*-\frac{1}{2}\ln 2,\quad \Delta_{1H}^*=3\tilde{J}_H^*-\frac{1}{2}\ln 2\ . \tag{40}$$

Next, we consider the other two cases (case 2 and case 3) given by (19) and (20), respectively. Using the similar procedure, we will respectively attain the following results:

$$\begin{aligned}
&J_{2S}^*=\tilde{J}_S^*,\quad L_{2s}^*=2\tilde{J}_S^*,\quad K_{2S}^*=\tilde{J}_S^*,\\
&J_{2T}^*=\tilde{J}_T^*,\quad L_{2T}^*=2\tilde{J}_T^*,\quad K_{2T}^*=\tilde{J}_T^*,\\
&J_{2H}^*=\tilde{J}_H^*,\quad L_{2H}^*=2\tilde{J}_h^*,\quad K_{2H}^*=J_H^*,\\
&H_{2S}^*=-4\tilde{J}_S^*+\tfrac{1}{2}\ln 2,\quad \Delta_{2S}^*=4\tilde{J}_S^*=\tfrac{1}{2}\ln 2,\\
&H_{2T}^*=-6\tilde{J}_T^*=\tfrac{1}{2}\ln 2,\quad \Delta_{2T}^*=6\tilde{J}_T^*-\tfrac{1}{2}\ln 2,\\
&H_{2H}^*=-3\tilde{J}_H^*+\tfrac{1}{2}\ln 2,\quad \Delta_{2H}^*=3\tilde{J}_H^*-\tfrac{1}{2}\ln 2,
\end{aligned} \tag{41}$$

for case 2, and

$$\begin{aligned}
&J_{3S}^*=0,\quad L_{3S}^*=0,\quad K_{3S}^*=4\tilde{J}_S^*,\\
&J_{3T}^*=0,\quad L_{3T}^*=0,\quad K_{3T}^*=4\tilde{J}_T^*,\\
&J_{3H}^*=0,\quad L_{3H}^*=0,\quad K_{3H}^*=4\tilde{J}_H^*,\\
&H_{3S}^*=0,\quad \Delta_{3S}^*=8\tilde{J}_S^*+\ln 2,\\
&H_{3T}^*=0,\quad \Delta_{3T}^*=12\tilde{J}_T^*+\ln 2,\\
&H_{3H}^*=0,\quad \Delta_{3H}^*=6\tilde{J}_H^*+\ln 2,
\end{aligned} \tag{42}$$

TABLE I. Summary of results for the square, triangle, and honeycomb lattices where $\tanh\tilde{J}_S^*=\sqrt{2}-1$ ($\tilde{J}_S^*\cong 0.4407$), $\tanh\tilde{J}_T^*=2-\sqrt{3}$ ($\tilde{J}_T^*\cong 0.2747$), and $\tanh\tilde{J}_H^*=1/\sqrt{3}$ ($\tilde{J}_H^*\cong 0.6585$).

| | | Square | Triangle | Honeycomb |
|---|---|---|---|---|
| $P_1$ | $J_1^*$ | $\tilde{J}_S^*$ | $\tilde{J}_T^*$ | $\tilde{J}_H^*$ |
| | $L_1^*$ | $-2\tilde{J}_S^*$ | $-2\tilde{J}_T^*$ | $-2\tilde{J}_H^*$ |
| | $K_1^*$ | $\tilde{J}_S^*$ | $\tilde{J}_T^*$ | $\tilde{J}_H^*$ |
| | $H_1^*$ | $4\tilde{J}_s^*-\frac{1}{2}\ln 2$ | $6\tilde{J}_T^*-\frac{1}{2}\ln 2$ | $3\tilde{J}_H^*-\frac{1}{2}\ln 2$ |
| | $\Delta_1^*$ | $4\tilde{J}_S^*-\frac{1}{2}\ln 2$ | $6\tilde{J}_t^*-\frac{1}{2}\ln 2$ | $3\tilde{J}_H^*-\frac{1}{2}\ln 2$ |
| $P_2$ | $J_2^*$ | $\tilde{J}_S^*$ | $\tilde{J}_T^*$ | $\tilde{J}_H^*$ |
| | $L_2^*$ | $2\tilde{J}_S^*$ | $2\tilde{J}_T^*$ | $2\tilde{J}_H^*$ |
| | $K_2^*$ | $\tilde{J}_S^*$ | $\tilde{J}_T^*$ | $\tilde{J}_H^*$ |
| | $H_2^*$ | $-4\tilde{J}_S^*+\frac{1}{2}\ln 2$ | $-6\tilde{J}_T^*+\frac{1}{2}\ln 2$ | $-3\tilde{J}_H^*+\frac{1}{2}\ln 2$ |
| | $\Delta_2^*$ | $4\tilde{J}_S^*-\frac{1}{2}\ln 2$ | $6\tilde{J}_T^*-\frac{1}{2}\ln 2$ | $3\tilde{J}_H^*-\frac{1}{2}\ln 2$ |
| $P_3$ | $J_3^*$ | 0 | 0 | 0 |
| | $L_3^*$ | 0 | 0 | 0 |
| | $K_3^*$ | $4\tilde{J}_S^*$ | $4\tilde{J}_T^*$ | $4\tilde{J}_H^*$ |
| | $H_3^*$ | 0 | 0 | 0 |
| | $\Delta_3^*$ | $8\tilde{J}_S^*+\ln 2$ | $6\tilde{J}_T^*+\ln 2$ | $12\tilde{J}_H^*+\ln 2$ |

for case 3. The results are all summarized in Table I. We have noticed that Griffiths [13] and Berker and Wortis [3] have presented the transformation (14) through a symmetric consideration and got the critical points $P_{3S}$, $P_{3T}$, and $P_{3H}$. However, the other two transformations (7) and (13) and thus the corresponding critical points $P_1$ and $P_2$ have not yet been mentioned. Berker [24] has also introduced a similar transformation that maps higher spin Ising models onto spin half for arbitrary lattices, and used it to obtain accurate (but not exact) critical temperature for all high spins. Guided by the same idea, we again find two transformations between spin variables. One is

$$\sigma_i=-\tfrac{2}{3}t_i^3+\tfrac{13}{6}t_i\ , \tag{43}$$

the other is

$$\sigma_i=-t_i^2+\tfrac{5}{4}\ , \tag{44}$$

where $t_i=\pm\frac{1}{2},\pm\frac{3}{2}$ is the spin variable of the spin-$\frac{3}{2}$ model. They result in

$$\sigma_i=\begin{cases}1 & \text{when } t_i=\frac{1}{2},\frac{3}{2}\\ -1 & \text{when } t_i=-\frac{1}{2},-\frac{3}{2},\end{cases} \tag{45}$$

and

$$\sigma_i=\begin{cases}1 & \text{when } t_i=\pm\frac{1}{2}\\ -1 & \text{when } t_i=\pm\frac{3}{2},\end{cases} \tag{46}$$

respectively, and map the spin-$\frac{3}{2}$ model onto the spin-$\frac{1}{2}$ Ising model under certain constrained conditions. Of course, we can attain the exact critical points of the spin-$\frac{3}{2}$ model on special subspaces of parameter space.

## IV. CONCLUSION

We have mapped the general spin-1 model onto the spin-$\frac{1}{2}$ model through transformations (7), (13), and (14).

Since the transformations between spin variables are not one-to-one correspondence, it requires to introduce the transformation degeneracy to the calculation of the partition function. We have also found the critical points on square, triangle, and honeycomb lattices in terms of the known exact results of the spin-$\frac{1}{2}$ Ising model.

The important features of our transformations are that they are not one-to-one correspondences $s_i$ and $\sigma_i$ and they are applicable to all lattices, depending on the lattice structure only through the coordination number. That means that the transformations are universal. In detail, once we know the exact solution of the spin-$\frac{1}{2}$ Ising model on arbitrary dimensionality and structure of lattices, we can obtain the exact results of the spin-1 Ising model through for example (10), (32), and (35). The method can be extended to higher spin models.

As we see, our results are worked out on a "smaller" subspace of parameter space $(J,L,K,H,\Delta)$ and, thus, they are certain special solvable cases of the general spin-1 model. Especially, when we only consider the honeycomb lattice, our subspaces are the special solutions of (2). In addition, in our method presented here, we have not found a tricritical point.

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