Modulational instabilities in the discrete deformable nonlinear Schrödinger equation

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We study analytically and numerically modulational instability for the discrete deformable nonlinear Schrödinger (NLS) equation which represents a natural link between the properties of the integrable Ablowitz-Ladik model and the nonintegrable discrete NLS equation. We show how different discretizations of the nonlinear interaction change modulational instability in the lattice and, correspondingly, conditions for localized modes to exist.

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Many nonlinear systems exhibit an instability that leads to a self-induced modulation of an input plane wave with the subsequent generation of localized pulses. This phenomenon is known as modulational instability and it is responsible for many physically interesting effects such as the filamentation of laser beams, the formation of envelope solitons in nonlinear optical fibers, and cavitons in plasmas as well as the breakup of monochromatic ocean waves. Although the modulational instability problem has been deeply studied in the case of continuous nonlinear models such as the sine-Gordon and nonlinear Schrödinger (NLS) equations (see, e.g., Refs. [1–5] to cite a few), only a few papers exist for the case of discrete models [6–10]. One of the main effects of modulational instability is the creation of localized pulses [11], so that modulational instability may be considered as the leading mechanism for energy localization in homogeneous nonlinear systems (see, e.g., [12,9,13]). In lattice systems, however, different discretizations of the same continuous system may have different effects on modulational instability leading to different conditions for the existence of localized modes.

The aim of the present paper is just to investigate the modulational instability problem for the discrete NLS models by studying interplay between the nonlinear on-site and intersite interactions as a mechanism for changing modulational instability and conditions for the localized modes to exist in a lattice. As a particular example, we consider the following discrete version of the NLS equation:

$$i\frac{d\psi_n}{dt} + D(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + \epsilon\,|\psi_n|^2\psi_n + \lambda|\psi_n|^2(\psi_{n+1} + \psi_{n-1}) = 0, \quad (1)$$

with positive ϵ and λ. Equation (1) is a Hamiltonian system with the following noncanonical Poisson brackets:

$$\{f,g\} = \frac{i}{2}\sum_{k=1}^{N}\left(1+\lambda|\psi_k|^2\right)\left(\frac{\partial f}{\partial\psi_k^*}\frac{\partial g}{\partial\psi_k} - \frac{\partial f}{\partial\psi_k}\frac{\partial g}{\partial\psi_k^*}\right) \quad (2)$$

and the Hamiltonian

$$H = \sum_{k=1}^{N}\{[\,(2\epsilon+\lambda)/\epsilon\,\lambda]\ln(1+\lambda|\psi_k|^2) - \psi_k^*[\psi_{k+1} - (\lambda/\epsilon)\psi_k + \psi_{k-1})]\}. \quad (3)$$

This model was introduced by one of us (M.S.) in Ref. [14] (see also [15]) and it can be viewed as a deformation of the standard discrete NLS (DNLS) equation

$$i\frac{d\psi_n}{dt} + D(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + \gamma\,|\psi_n|^2\psi_n = 0. \quad (4)$$

Indeed, by using the parametrization $\lambda = (\gamma - \epsilon)/2$ one can see that for $\epsilon = \gamma$ Eq. (1) reduces to the standard DNLS equation (4) with the canonical Poisson brackets, while for $\epsilon = 0$ it gives the Ablowitz-Ladik (AL) model [16]

$$i\frac{d\psi_n}{dt} + D(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + \frac{\gamma}{2}|\psi_n|^2(\psi_{n+1} + \psi_{n-1}) = 0, \quad (5)$$

which is an integrable discrete version of the NLS equation with a noncanonical ("deformable") Poisson structure. The fact that the AL system appears as a deformation of the standard discretization of the NLS equation holds true also at the quantum level and this fact can be used to continuously deform the energy levels of one system into the other. (For a detailed analysis of the quantum problem of these lattice models see Refs. [14, 15, 17].) Besides these mathematical aspects, there is a deep physical motivation for the inclusion in Eq. (1) of both the on-site nonlinearity of the standard DNLS equation and the intersite (off-diagonal) nonlinearity characterizing the AL model [14, 15]. This can be seen by considering the simplest quasiclassical equation which describes the propagation of molecular excitations (see, e.g., Ref. [18] and references therein),

$$i(d\psi_n/dt) + \omega_n\psi_n + J_n\,(\psi_{n+1} + \psi_{n-1}) = 0, \quad (6)$$

where ψ_n is the complex mode amplitude of a particular molecular vibration, ω_n is the on-site frequency of this vibration, and J_n is the next-neighbor resonance interaction energy. Taking into account on-site nonlinearities in Eq. (6), first we should modify the local frequency ω_n as

$$\omega_n \to \omega_0 + \omega_1 |\psi_n|^2, \tag{7}$$

and this kind of anharmonicity corresponds to a standard polaron model (i.e., in fact, to the DNLS model). Similarly, coupling of the resonance interaction to low-frequency (on-site) vibrations leads to [cf. Eq. (7)]

$$J_n \to J_0 + J_1 |\psi_n|^2, \tag{8}$$

and the combination of Eqs. (7) and (8) with Eq. (6) gives exactly the generalized (deformable) NLS equation (1). Another application of the model (1) may be found in nonlinear optics: The discrete (and, in particular, deformable) NLS equation describes interactions of partial TE modes in an array of (focusing or defocusing) wave guides (see, e.g., Ref. [19]). Thus, in spite of the fact that the AL model itself seems not physical (see discussions below), it may appear indeed as a particular case of a more general and physically better justified discrete model (1).

To analyze modulational instability for Eq. (1), first we note that it has the exact plane-wave solution $\psi_n(t) = \psi_0 \exp(i\theta_n)$ with $\theta_n = qna - \omega t$, where q denotes the wave number of the carrier wave and the frequency ω obeys the nonlinear dispersion relation

$$\omega = 4D\sin^2(qa/2) - \epsilon\psi_0^2 - 2\lambda\psi_0^2\cos(q\,a), \tag{9}$$

a being the lattice spacing. The linear stability of the nonlinear plane wave can be investigated by looking for solutions in the form $\psi_n(t) = (\psi_0 + b_n)\exp(i\,\theta_n)$, where the complex function $b_n(t)$ is assumed to be small in comparison with the amplitude of the carrier wave. In the linear approximation an equation for $b_n(t)$ yields the dispersion relation for the evolution of small perturbations,

$$\begin{aligned}[\Omega - 2(D+\lambda\psi_0^2)\sin(qa)\sin(Qa)]^2 \\ = 4(D+\lambda\psi_0^2)\sin^2(Qa/2)\cos(qa) \\ \times[4(D+\lambda\psi_0^2)\sin^2(Qa/2)\cos(qa) \\ -4\lambda\psi_0^2\cos(qa) - 2\epsilon\psi_0^2],\end{aligned} \tag{10}$$

where the wave number Q and frequency Ω characterize linear properties of the modulation wave. The dispersion relation given above determines the condition for the stability of a plane wave with the wave number q in the lattice. This stability condition explicitly depends on the nonlinearity parameters ϵ and λ. For $\epsilon > 2\lambda$ the instability conditions are very close to those described for the standard NLS model [6], i.e., for positive ϵ and λ a plane wave may be unstable to small modulations provided $\cos(qa) > 0$, and in the case $\epsilon\psi_0^2 > 2D$ all the waves with the wave numbers $q < \pi/2a$ become unstable. The most interesting properties of modulational instability may be observed for $\epsilon < 2\lambda$. In this case the instability region appears for high-frequency oscillations as well, so that we can expect to find *two types* of localized modes with the frequency lying respectively *below or above* the linear spectrum band. All possible types of instabilities are shown in Fig. 1 as the shadow regions on the (q,Q) plane. It is important to note that the AL model itself [its instability region is shown by the dashed line in Fig. 1(c)] does not display the dependence of the threshold of the modulational instability on q. This result seems not physical and, in particular, it does not correspond to any kind of instabilities observed in realistic lattice models of solids (see, e.g., Ref. [6]).

To investigate modulational instability numerically, we have integrated Eq. (1) with a fourth-order Runge-Kutta scheme and selected several sets of initial data. Figure 2 shows the time evolution of a modulated plane wave with the initial parameters marked as a, b, c, and d in Fig. 1(c) and for the initial condition $\psi_n(t) = [\psi_0 + \psi_1\cos(Q\,n)]\cos(q\,n)$, with ψ_0 fixed to 0.5 and $\psi_1 = 0.01$. It is clear from these figures that for the parameters se-

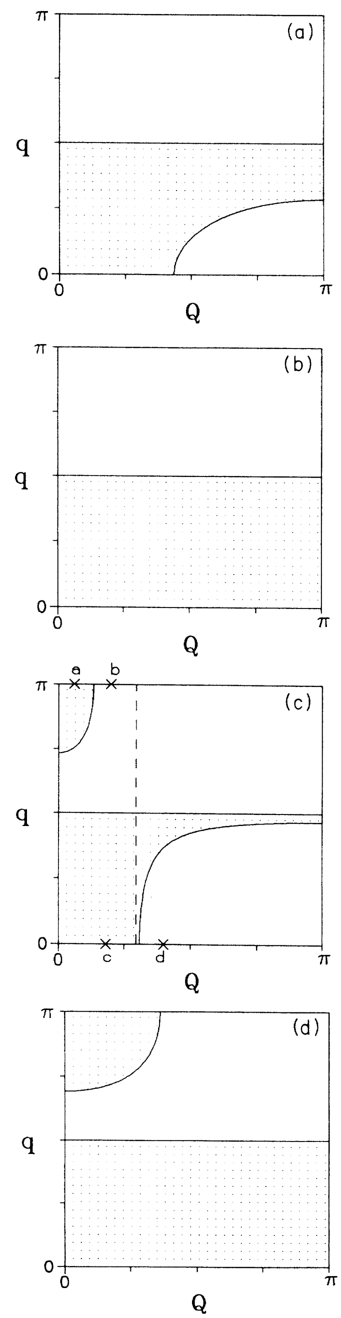

FIG. 1. Regions of modulational instability on the (q,Q) plane for $a=1$ and (a) $\epsilon > 2\lambda$ and $\epsilon\psi_0^2 < 2D$, (b) $\epsilon > 2\lambda$ and $\epsilon\psi_0^2 > 2D$, (c) $\epsilon < 2\lambda$ and $\epsilon\psi_0^2 < 2D$, and (d) $\epsilon < 2\lambda$ and $\epsilon\psi_0^2 > 2D$. The dashed line in (c) shows the threshold for modulational instability in the case of the AL model (instability region lies on the left of the dashed curve).

lected modulational instability may be displayed for low-frequency as well as for high-frequency oscillations. This simply means that in the model under consideration there exist *two kinds* of nonlinear localized modes with in-phase and out-of-phase oscillations of the neighboring particles in the lattice. Figures 2(a) and 2(c) confirm these predictions showing creation of two types of localized modes from a flat initial condition due to modulational instability.

As has been mentioned above, one of the main effects of modulational instability is to create spatially localized modes. In the model (1) localized modes may exist with the frequencies lying below (for $\epsilon > 0$) as well as above (for $0 < \epsilon < 2\lambda$) the linear frequency gap. For the former modes the particles oscillate in phase with their neighbors, whereas for the latter modes the particles in the localized state oscillate with opposite phases (in the recent work [10] these modes have been called "staggered localized states"). To understand the origin of the high-frequency (out-of-phase) localized modes in the model (1), we make the substitution $\psi_n(t) = (-1)^n \Psi_n(t) e^{i\omega_m t}$, where $\omega_m = 4D$ is the cutoff frequency of the linear spectrum band. If we assume the envelope function $\Psi_n(t)$ slowly varying, then we obtain the continuous NLS equation

$$i(\partial\Psi/\partial t) - Da^2(\partial^2\Psi/\partial x^2) + (\epsilon - 2\lambda)|\Psi|^2\Psi = 0, \qquad (11)$$

where the function $\Psi(x,t)$ is considered as slowly varying function of its arguments t and $x = an$. As is well known, the NLS equation (11) supports spatially localized (soliton) solutions provided the effect of dispersion has the same sign as that of nonlinearity. For our case this means $\epsilon < 2\lambda$, i.e., the same condition under which the modulated plane wave becomes unstable. When such a mode is strongly localized, its motion in a nonintegrable lattice is affected by an effective periodic potential which is similar to the well-known Peierls-Nabarro (PN) potential for dislocations. From the physical point of view, the amplitude of the PN potential may be viewed as the minimum barrier which must be overcome to translate the dislocation by one lattice period. Recently, the appearance of the PN potential for localized modes has been discussed in Ref. [20]. The analysis and the main conclusions of that study may be easily extended for the model (1) as

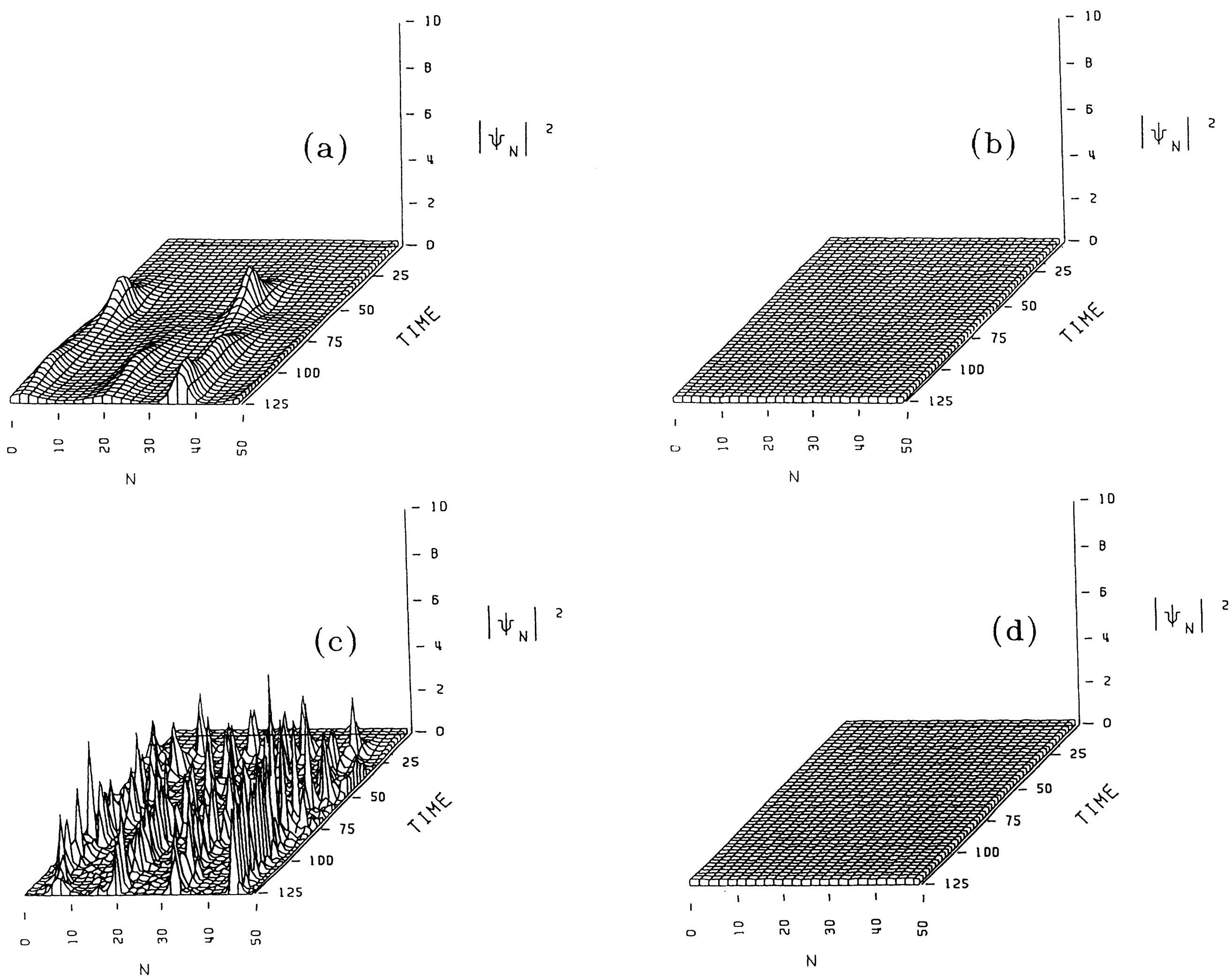

FIG. 2. Time evolution of small-amplitude modulations for $\gamma = 2.0$ and $\epsilon = 0.8$. The plots (a), (b), (c), and (d) correspond, respectively, to the set of the parameters taken at the points a, b, c, and d in Fig. 1(c).

well.

In the other case, $\epsilon > 2\lambda$, the NLS equation (11) has no localized solutions, instead solitary waves may propagate as dark solitons on a modulationally stable background $\Psi(t) = \Psi_0 \exp(i\Omega t)$, where $\Omega = (\epsilon - 2\lambda)\Psi_0^2$. It is interesting to analyze how the "nonintegrable contribution" $\epsilon|\psi_n|^2\psi_n$ to the integrable AL model may change properties of dark solitons. This may be already seen taking into account the next-order terms in Eq. (11) which describe, as a matter of fact, a contribution of the higher-order dispersion into the continuous model due to the effect of discreteness [cf. Eq. (11)],

$$i(\partial\Psi/\partial t) - Da^2(\partial^2\Psi/\partial x^2) - (Da^4/12)(\partial^4\Psi/\partial x^4) + (\epsilon - 2\lambda)|\Psi|^2\Psi - \lambda a^2|\Psi|^2(\partial^2\Psi/\partial x^2) = 0. \quad (12)$$

Properties of dark solitons in the model (12) may be analyzed in the so-called small-amplitude approximation (see details in Ref. [21]) looking for a solution of Eq. (12) in the form, $\Psi(x,t) = [\Psi_0 + A(x,t)]\exp[i\Omega t + i\phi(x,t)]$, where the amplitude A is assumed to be small in comparison with the background amplitude $A \ll \Psi_0$. For the amplitude A and phase ϕ we obtain a system of two coupled equations which may be analyzed by applying an asymptotic expansions $A = \epsilon^2 A_0 + \epsilon^4 A_1 + \dots$, $\phi = \epsilon\phi_0 + \epsilon^3\phi_1 + \dots$, and introducing new ("slow") variables $\tau = \epsilon^3 t$ and $z = \epsilon(x - Ct)$, where ϵ is a small parameter and C is the velocity of linear waves propagating on the plane wave background $C^2 = 2a^2(\epsilon - 2\lambda)(D + \lambda\Psi_0^2)$. As a result, in the lowest order in the small parameter ϵ we obtain the Korteweg–de Vries equation for the amplitude A_0,

$$2C(\partial A_0/\partial\tau) + G_1 A_0(\partial A_0/\partial z) - G_2(\partial^3 A_0/\partial z^3) = 0, \quad (13)$$

where $G_1 = 4a^2\Psi_0(\epsilon - 2\lambda)(3D + 4\lambda\Psi_0^2)$ and $G_2 = a^4\left[(D + \lambda\Psi_0^2)^2 - (D/6)(\epsilon - 2\lambda)\Psi_0^2\right]$. Equation (13) has a soliton solution

$$A_0(z,\tau) = -(12\nu^2 G_2/G_1)\mathrm{sech}^2[\nu(z - V\tau)], \quad (14)$$

where $V = -2\nu^2 G_2/C$ is the soliton velocity in the reference frame moving with the sound speed C and ν is an arbitrary parameter. In the case when G_2/G_1 is positive, the solution (14) corresponds to a dark soliton of the standard type. However, it is important to note that for the case $(\epsilon/\lambda - 2) > 6(1 + \Delta)^2/\Delta$, where $\Delta \equiv \lambda\Psi_0^2/D$, the function G_2 becomes negative and the soliton (14) changes the sign, and it transforms into a bright soliton on a pedestal (i.e., a dark soliton of the reverse-sign amplitude [22]). Therefore, such a transformation of dark solitons is one of the effects which may be expected in the nonintegrable deformable NLS equation (1).

It is known that integrable models display a recurrence for periodic boundary conditions and, for example, this effect was studied numerically [23] and analytically [24] for the continuous NLS equation. As a matter of fact, the influence of the discreteness effect on the recurrence phenomenon in integrable and nearly integrable lattice models is not understood yet and, as a matter of fact, the model (1) seems to be the most suitable one for this purpose: It has two integrable simplifications, for the continuous limit and at $\epsilon = 0$, when the recurrence should be recovered.

In conclusion, we have analyzed modulational instability in the framework of the discrete model (1), and we have shown how the interplay between the nonlinear onsite and intersite interactions may change modulational instability properties and, consequently, the conditions for the localized (bright and dark) nonlinear modes to exist in the lattice.

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