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# Logarithmic decay of $\phi^4$ breathers of energy $E \lesssim 1$

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In a numerical experiment until $t = 2 \times 10^6$ it is shown that $\phi^4$ breathers of energy $E \lesssim 1$ and after an initial stage of their formation decay logarithmically in time. This result is consistent with and extends the one of Kruskal and Segur that small-amplitude $\phi^4$ breathers of energy $E \ll 1$ decay asymptotically as $(\ln t)^{-1}$. Analytic arguments are presented in support of our numerical result. It is estimated that a $\phi^4$ breather of energy $E \approx 0.97$ at $t = 10^4$ in approximately $10^{11}$ time units will radiate away half its energy and at $t = 10^{22}$ will still have an energy $E \approx 0.25$.

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Since the $\phi^4$ equation

$$\phi_{xx} - \phi_{tt} = \phi^3 - \phi \tag{1}$$

does not possess the Painlevé property [1] the $\phi^4$ field theory in 1+1 space-time dimensions is considered not to be completely integrable. Also, and in consequence, it is believed that exact nonradiating breather solutions [localized in (finite) space, oscillating in time and interpreted as kink-antikink bound states] do not exist [2,3]. Once because of this more mathematical aspect, and on the other hand, because $\phi^4$ theory is an interesting and important model in various fields of physical research as condensed-matter, nuclear, molecular and elementary-particle physics [4], the time evolution of supposed approximate $\phi^4$ breathers (commonly also named $\phi^4$ pulsons) has been studied in numerical experiments by various authors [3,5,6], however only for relatively small times not greater than $10^3$ to $10^4$. [Time and length are dimensionless in (1).]

In 1987, Kruskal and Segur, using asymptotic expansions, showed that exact small-amplitude $\phi^4$ breathers indeed do not exist and that the energy of approximate small-amplitude $\phi^4$ breathers with energies $E \ll 1$ and frequencies $\omega$ tending to $\sqrt{2}$ (from below) decays logarithmically in time: $E(t) \sim (\ln t)^{-1}$ as $t \to \infty$ [7]. This result and our own numerical observation [6] that in a certain region of non-small amplitudes $\phi^4$ breathers appear to decay "much slower than exponentially in time," suggest the examination, more quantitatively, of how $\phi^4$ breathers of considerably large amplitudes or energies, say $E \lesssim 1$, decay in time. Such investigation is the purpose of the present report. The value $E \lesssim 1$ takes into account that the maximum energy of a $\phi^4$ kink-antikink bound state at rest cannot exceed the energy $\frac{4}{3}\sqrt{2} \approx 1.8856$ which is twice the energy of a $\phi^4$ kink at rest and that during the formation of a breather from a kink and an antikink a considerable amount of energy is lost by radiation.

As a starting point we take our previous result [6] that a $\phi^4$ kink and a $\phi^4$ antikink, for $t = 0$ at rest and at a mutual distance equal to 1.6 from each other, develop into a quasistable breather. This defines the initial curves

$$\phi(x, t = 0) = \tanh \frac{x - 0.8}{\sqrt{2}} - \tanh \frac{x + 0.8}{\sqrt{2}} + 1, \tag{2a}$$

$$\phi_t(x, t = 0) = 0, \tag{2b}$$

for the numerical solution of (1) which will be carried out on the finite interval $-a \leq x \leq a$ until $t = 2 \times 10^6$ by the method of characteristics [8]. The boundary conditions in $x = \pm a$ are

$$\phi_x(x = \pm a, t) \ \pm \ \phi_t(x = \pm a, t) = 0 \tag{3a}$$

and correspond to the assumption that the breather emits its radiation in the form of traveling waves of velocity $v = \pm 1$, equal to the characteristic velocity of the wave equation (1). A test that this assumption is reasonable and that the boundary parameter $a$ is chosen large enough is that considerable reflection in $x = \pm a$ or "boundary effects" are not observed in the numerical solution.

We profit by the symmetry of the problem with respect to $x = 0$ [$\phi(-x, t) = \phi(x, t)$, $\phi_t(-x, t) = \phi_t(x, t)$ and $\phi_x(-x, t) = -\phi_x(x, t)$] and execute the numerical solution only on the interval $0 \leq x \leq a$, replacing the boundary condition (3a) in $x = -a$ through

$$\phi_x(x = 0, t) = 0. \tag{3b}$$

From the numerical solution $\phi(x, t)$, $\phi_t(x, t)$, and $\phi_x(x, t)$ we calculate the energy density of the system

$$H(x, t) = \frac{1}{2} \left\{ \phi_x^2 + \phi_t^2 + \frac{1}{2}(\phi^2 - 1)^2 \right\} \tag{4a}$$

and then through numerical integration the total energy

$$E(t; b) = \int_{-b}^{b} H(x, t) dx = 2 \int_{0}^{b} H(x, t) dx. \tag{4b}$$

Like the parameter $a$, also $b$ must be chosen large enough such that the breather is located in $-b \leq x \leq b$. The normal choice will be $b = a$. However, for testing the numerical results, e.g., that $a$ and $b$ have been chosen large enough and that the boundary condition (3a) is appropriate, a choice $b < a$ may be of interest (see below).

 

The integral

$$R(t_1,t_2;b) = -2\int_{t_1}^{t_2} \phi_x(b,t)\phi_t(b,t)dt \tag{4c}$$

measures directly the amount of radiation emitted by the breather to $x < -b$ and to $x > b$ (hence the factor 2) during the time $t_1 \le t \le t_2$ [7] and is calculated through numerical integration. Conservation of energy requires that the relation

$$E(t_1;b) - E(t_2;b) = R(t_1,t_2;b) \tag{4d}$$

be fulfilled for (sufficiently large $b$ and) all non-negative times $t_1$ and $t_2$, in particular for $t_1 = 0$ and $t_2 = t$.

Both, since the numerical integration of Eqs. (1), (2), and (3) is performed until large times, and on the other hand, since the expected very slow decay of the breathers requires a high accuracy of the numerical solution, special care for a good convergence must be taken. In the beginning, until $t = 5000$, the convergence has been checked explicitly, by use of two different steplengths $\Delta = \Delta t = \Delta x = 0.05$ and 0.1 and two boundary parameters $a = 100$ and 200. The results are shown in Table I. For each time $t$ and parameter set $(\Delta, a, b)$ two values of the energy $E(t;b)$ are given, the first (smaller one) at the time when $\phi(0,t)$ takes on its first maximum and the second (larger) value when $\phi(0,t)$ takes on its first minimum, both immediately after the time $t$ indicated at the top of the column, i.e., approximately in the time interval $[t, t+5]$. The difference between these two values of $E(t;b)$ is almost time independent and about 0.004 for $\Delta = 0.1$ (it is still about 0.0035 for $t = 2\times10^6$) and about 0.001 for $\Delta = 0.05$, indicating that this difference vanishes as $\Delta^2$ for $\Delta \to 0$. For $a = b = 100$ and $\Delta = 0.1$, the greater one of the two values of $E(t;b)$ at a given time is closer to the values obtained with $\Delta = 0.05$ and therefore closer to the exact result than the smaller value for the energy $E(t;b)$. Another argument to conclude so is that the sum of the larger value for $E(t;b)$ and of $R(0,t;b)$ is approximately 1.156 for all three parameter sets with $\Delta = 0.1$ and approximately 1.157 for $\Delta = 0.05$, i.e., about 0.0015 and about 0.0005 below $E(t=0;b) = 1.157\,55...$ . For this reason, we always shall consider the energy at the first minimum of $\phi(0,t)$ after the indicated time $t$. The differences $E(t;b=100)_{a=100} - E(t;b=100)_{a=200}$ as well as $R(0,t;b=100)_{a=100} - R(0,t;b=100)_{a=200}$ are equal to 0.0032 at time $t = 203.6$ corresponding to 6% of the energy loss $\Delta E(203.6) = 0.0547$. They take on maximum values at time $t \approx 600$ which correspond to about 20% of the breather's energy loss at that time and then diminish (cf. first and third line in Table I). Thus we believe that the boundary condition (3a) is reliable and will not affect the breather's energy loss significantly. Furthermore, corresponding values of any two lines in Table I approximate each other when time increases. Altogether we think the parameters $\Delta = 0.1$ and $a = b = 100$ make up a reasonable choice for a long-time numerical solution of (1), (2), and (3). Throughout the following we shall suppress the argument $b$ in the energy (4b) and radiation (4c).

The principal numerical results are shown in Table II. In the second column the energy (4b) of the breather and in the fourth column the radiation (4c), emitted during time $[0,t]$ to $|x| > b = a = 100$, are given. The third column shows the energy loss of the breather, calculated from $E(t)$ and $E(0)$ where $E(0) = 1.157\,558... \approx 1.1576$ is obtained from (4b) for all sets $(\Delta, a, b)$ used in Table I. One sees that (4d) is fulfilled very well for all times. This can be considered as another and now permanent test of the accuracy during the whole numerical integration. Columns 5 and 6 show the numerical values of two fits for the energy,

$$E_1(t) = \frac{14.395}{-2.2372 + T + 2\ln(3.0966 + T)} \tag{5a}$$

and

$$E_2(t) = \frac{12.8}{1.5048 + T}, \tag{5b}$$

where $T = \ln(t + 10^5)$. Both fits are of the form (7d) below and lead to agreement within 0.7% with $E(t)$, for $t \gtrsim 5\times10^3$. One concludes that the energy of the breather decays logarithmically in time. In view of the good agreement between the numerical results $E(t)$ and formulas (5a) and (5b), especially for larger times $t \gtrsim 10^6$ and since we did not observe any less stable behavior at smaller amplitudes [6] it is tempting to extrapolate the fits to very large times. One then finds that the energy decays to approximately 0.48 at $t = 10^{11}$ and 0.25 at $t = 10^{22}$. These decay times are by many orders of magnitudes larger than previous estimates of the breather's "half-life," based on an exponential decay [5,6]. The dimensionless time $t$ in the normalized equation (1) corresponds to real physical times $t/m$ where the square mass $m^2$ is the coefficient of $-\phi$ in the physical $\phi^4$ equation [9]. For example, with the characteristic velocity $c = 2.998\times10^{10}$ cm/sec $=1$ in (1) and $\hbar = 1.054\times10^{-27}$ erg sec$=1$, one obtains for $m = 1$ GeV$/c^2$ the physical time $t/m = 6.58\times10^{-25}t$ sec. In this case the time $t = 10^{11}$ in (1) during which the breather

TABLE I. Energy (4b) and radiation (4c) obtained from the numerical solution of (1), (2), and (3) for various sets of $a$, $b$ and $\Delta = \Delta t = \Delta x$.

| | | | $t = 600$ | | | $t = 2000$ | | | $t = 5000$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta$ | a | b | $E(t;b)$ | | $R(0,t;b)$ | $E(t;b)$ | | $R(0,t;b)$ | $E(t;b)$ | | $R(0,t;b)$ |
| 0.1 | 100 | 100 | 1.0250 | 1.0294 | 0.1260 | 0.9911 | 0.9951 | 0.1606 | 0.9754 | 0.9797 | 0.1765 |
| 0.05 | 100 | 100 | 1.0302 | 1.0313 | 0.1257 | 0.9962 | 0.9972 | 0.1599 | 0.9804 | 0.9816 | 0.1757 |
| 0.1 | 200 | 100 | 0.9996 | 1.0034 | 0.1518 | 0.9812 | 0.9853 | 0.1707 | 0.9707 | 0.9742 | 0.1813 |
| 0.1 | 200 | 200 | 1.0611 | 1.0656 | 0.0896 | 1.0079 | 1.0120 | 0.1437 | 0.9838 | 0.9878 | 0.1681 |

TABLE II. Energy (4b), energy loss $\Delta E(t) = E(0) - E(t) \approx 1.1576 - E(t)$ and radiation (4c), obtained from the numerical solution of (1), (2), and (3) with $a = b = 100$ and $\Delta = \Delta t = \Delta x = 0.1$. $E_1(t)$ and $E_2(t)$ present the results from the fits (5a) and (5b).

| $t$ | $E(t)$ | $\Delta E(t)$ | $R(0,t)$ | $E_1(t)$ | $E_2(t)$ |
|---|---|---|---|---|---|
| $5\times10^3$ | 0.9797 | 0.1780 | 0.1765 | 0.9796 | 0.9796 |
| $10^4$ | 0.9711 | 0.1865 | 0.1851 | 0.9761 | 0.9761 |
| $5\times10^4$ | 0.9472 | 0.2104 | 0.2094 | 0.9534 | 0.9536 |
| $10^5$ | 0.9281 | 0.2295 | 0.2288 | 0.9332 | 0.9336 |
| $3\times10^5$ | 0.8872 | 0.2704 | 0.2687 | 0.8882 | 0.8886 |
| $5\times10^5$ | 0.8645 | 0.2931 | 0.2918 | 0.8639 | 0.8643 |
| $8\times10^5$ | 0.8432 | 0.3144 | 0.3136 | 0.8410 | 0.8413 |
| $1.1\times10^6$ | 0.8258 | 0.3318 | 0.3313 | 0.8255 | 0.8257 |
| $1.4\times10^6$ | 0.8137 | 0.3439 | 0.3438 | 0.8139 | 0.8140 |
| $1.7\times10^6$ | 0.8046 | 0.3530 | 0.3530 | 0.8046 | 0.8046 |
| $2\times10^6$ | 0.7970 | 0.3606 | 0.3608 | 0.7970 | 0.7969 |
| $10^8$ | | | | 0.6449 | 0.6424 |
| $10^{10}$ | | | | 0.5270 | 0.5218 |
| $10^{11}$ | | | | 0.4833 | 0.4770 |
| $10^{20}$ | | | | 0.2790 | 0.2692 |
| $10^{22}$ | | | | 0.2553 | 0.2454 |

loses half its energy corresponds to $6.58\times10^{-14}$ sec. This time is large when compared with a typical decay time (about $10^{-23}$ sec) of strongly decaying particles, e.g., of the $\rho$ meson or of pion-nucleon resonances.

Figure 1 shows the numerical solution $\partial\phi(x,t)/\partial x$ at the early time $t \approx 2\times10^3$ (full curve) and at the large time $t \approx 2\times10^6$ (dashed curve). Both curves correspond to a phase of the breather's oscillation when $\phi(0,t)$ takes on minimum values of 0.300 and 0.454, respectively. The wavy form of $\phi_x(x,t)$ for $|x| \gtrsim 10$ at $t \approx 2\times10^3$ shows that relatively much energy is still being radiated away and the breather is still in the stage of formation. At $t \approx 2\times10^6$ almost no radiation is perceptible, and the breather is located in $|x| \lesssim 10$. Its amplitude $\max|\phi_x(x,t)|$ at that time is somewhat smaller and its width only slightly larger than at $t \approx 2\times10^3$.

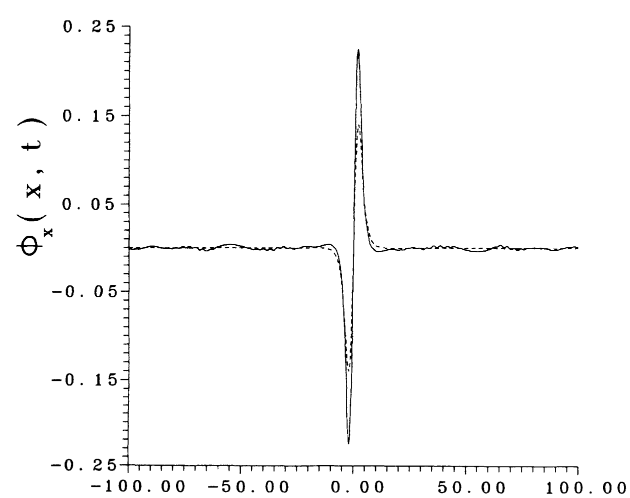


FIG. 1. The derivative $\phi_x(x,t)$ of the breather's field function at an early time $t \approx 2\times10^3$ (full curve) and at $t \approx 2\times10^6$ (dashed curve).

The amplitudes of the oscillations about the selected vacuum $\Phi \equiv +1$ decay quite similarly as the energy. In $x = 0$ the field function $\phi(0,t)$ oscillates between 0.303 and 1.440 at $t \approx 5\times10^3$, between 0.426 and 1.382 at $t \approx 10^6$, and between 0.454 and 1.369 at $t \approx 2\times10^6$.

Kruskal and Segur had already found [7] that the energy $E$ of small-amplitude $\phi^4$ breathers ($\varepsilon = \sqrt{2-\omega^2} \to 0$ or $E \to 0$) decays like

$$\frac{dE}{dt} = -A\exp\left(\frac{-\sqrt{6}\pi}{2\varepsilon}\right)[1+O(\varepsilon^2)], \tag{6}$$

where $A$ is a positive constant and $E = \frac{4}{3}\sqrt{2}\varepsilon + O(\varepsilon^3)$ [9]. A similar result was given for the leading order of energy emission from a small-amplitude sine-Gordon breather $\varphi$ in a small perturbation $\alpha\varphi^5$, $\alpha \ll 1$ [10].

To support our numerical result we now shall discuss decay laws of the form

$$\frac{dE}{dt} = -AE^{\lambda}\exp\left(\frac{-B}{E^{\mu}}\right) \tag{7a}$$

with arbitrary, but real positive constants $A, B, \lambda$, and $\mu$. Setting $z = E^{-\mu}$ and $\gamma = \frac{\lambda-\mu-1}{\mu}$ one finds through one partial integration

$$z(t)^{\gamma}e^{Bz(t)}\Big|_{t_0}^{t} - \gamma\int_{z(t_0)}^{z(t)} z^{\gamma-1}e^{Bz}dz = \mu ABt|_{t_0}^{t}. \tag{7b}$$

The result (7b) is still exact. On the left-hand side (lhs) the integral can be neglected against the first term for large $z(t)$ [although both terms diverge for $z(t) \to \infty$ or $E(t) \to 0$]. This approximation is the better the larger $B$ and the smaller $|\gamma|$ are. An important additional observation now is that for sufficiently large $B$ the interval where the integral is negligible may extend to the whole interval $1 \approx z(t_0) \lesssim z(t) \leq +\infty$. Namely, for $t = t_0$ all three terms in (7b) vanish trivially. For $B \gg 1$ and monotonically increasing $z(t) \gtrsim 1$ the first term on the lhs of (7b) increases much faster in time or with $z(t)$ than the integral and if $B \gg |\gamma|$ soon becomes the leading term on the lhs of (7b). Thus, neglecting the integral in (7b), one gets

$$\gamma\ln z(t) + Bz(t) \approx \ln(\mu ABt + C), \tag{7c}$$

where $C = C(t_0, z(t_0))$ is the constant of integration. Since on the lhs the logarithmic term is much smaller than the term linear in $z(t)$, an iterative solution gives

$$E(t) \approx \left\{\frac{B}{\ln(\mu ABt+C) + \gamma\ln\frac{B}{\ln(\mu ABt+C)}}\right\}^{1/\mu}. \tag{7d}$$

(7c) and (7d) are exact for $\lambda = \mu + 1$ or $\gamma = 0$. (7d) remains still valid in the leading order when in (7a) instead of $AE^{\lambda}$ a power series with leading term $AE^{\lambda}$ for $E \to 0$ is considered.

One sees that the parameters $B$ and $\mu$ are much more important than $A$ and $\lambda$ except that $|\lambda - \mu|$ should not be large. For example, the fits (5a) and (5b) are of the form (7d) (the denominator can be written as $\ln(\mu AB) + \gamma\ln B + T' - \gamma\ln[\ln(\mu AB) + T']$, $T' = \ln(t + \frac{C}{\mu AB})$) with

$\mu = 1$, $\lambda = 0$, $A = 1.5368$, $B = 14.395$ and $\mu = 1$, $\lambda = 2$, $A = 0.3518$, $B = 12.8$, respectively. (5a) and (5b) have quite different values of $A$ and $\lambda$, but equal or similar values of $\mu$ and $B$. Since $B$ is large, $\gamma = -2$ or 0 and $E \lesssim 1$, the conditions for validity of the approximations (7c) and (7d) are satisfied; a logarithmic decay is observed already for $E \lesssim 1$, and not only asymptotically for $E \ll 1$. For $E > 1$ these approximations begin to lose their validity because the two terms on the lhs of (7b) begin to compete more and both terms on the lhs of (7c) will do so. Furthermore, in this energy region (and during the initial stage of formation in general) the system corresponds rather to a kink-antikink system, colliding with low velocity, than to a formed breather, and the decay of the energy is no longer determined by (6). [Therefore the fits (5a) and (5b) are not good for relatively small times.]

We should like to know whether the values for $B$ in our fits (5a) and (5b) correspond with the asymptotic one of Ref. [7]. From (6) where $\lambda = 0$ one obtains asymptotically $E(t) \sim \frac{4\pi}{\sqrt{3}\ln t} \approx \frac{7.252}{\ln t}$ as $t \to \infty$ where we have used $2\varepsilon \sim \frac{3}{2\sqrt{2}}E$ as $\varepsilon \to 0$ or $E \to 0$. The numerator in $E(t)$ here is approximately only half the numerator in (5a) or (5b). The reason that our result is still not in the asymptotic region would be unlikely in view of the discussion just given. Indeed, formula (15) and the following one [without number and with its right-hand side (rhs) bilinear in $\phi$] in Ref. [7] show that there should stand a factor $\left[\exp\left(-\frac{\pi\sqrt{6}}{2\varepsilon}\right)\right]^2$ instead of $\exp\left(\frac{-\pi\sqrt{6}}{2\varepsilon}\right)$ in formula (16) of Ref. [7], i.e., in (6) above. Then the general result (7d) implies $E(t) \sim \frac{8\pi}{\sqrt{3}\ln t} \approx \frac{14.503}{\ln t}$ as $t \to \infty$, and our numerical results for $0.8 \lesssim E \lesssim 0.97$ [yielding $B = 14.395$ for $\lambda = 0$ in (5a)] are in good correspondence with the asymptotic approximation found for $E \to 0$ in Ref. [7].

This quantitative agreement together with the extension of the asymptotic approximation to higher energies as shown above is another—now *a posteriori*—indication that, first, the boundary condition (3a) does not affect the breather's energy loss significantly and, secondly, our numerical results, though obtained with a relatively large steplength $\Delta = 0.1$, describe the energy loss with considerable accuracy. The latter is also suggested by the good agreement between $\Delta E(t)$ and $R(0,t)$ for all times as shown in Table II if one observes that the integrals (4b) and (4c) are obtained from different integrations over space and over time, respectively. In conclusion, $\phi^4$ breathers even of nonasymptotic energies are classically quasistable objects. This is relevant for their semiclassical quantization [9, 11] and for the structure of bound states in field or particle theory [4]. Besides, the results provide an interesting physical example that in certain circumstances an asymptotic approximation may be valid in a quite larger interval [$z(t) \gtrsim 1$] than *a priori* expected [$z(t) \to \infty$].

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