Instability in a classical periodically driven string

J. Dittrich,[1] P. Duclos,[2] and P. Šeba[1]
[1]Nuclear Physics Institute, Academy of Sciences of the Czech Republic, 250 68 Řež near Prague, Czech Republic
[2]Centre de Physique Théorique, Centre National de la Recherche Scientifique, Marseille-Luminy, France
and Equipe de Physique Mathématique, Université de Toulon et du Var, Toulon, France
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The existence of instability (in the sense of unlimited growth of energy) in a classical periodically driven one-dimensional string is proven mathematically and demonstrated numerically.

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Recently much work has been devoted to the question of a possible instability of periodically driven bounded classical and quantum systems [1]. There are well known examples of systems that are classically unstable whereby their quantum counterpart displays stable behavior. One of the most prominent of them is the so-called kicked rotator [2].

In classical mechanics, the instability is closely related to the appearance of chaos (chaotic diffusion). The stability of the related quantum system has been interpreted therefore as a sign for suppression of chaos in the quantum world.

A large class of classical periodically driven systems is unstable due to parametric resonances [3–5]; the system we study below belongs also to this class.

The question of the stability and/or instability of a physical system is of relevance not only for classical and quantum mechanics. In this Brief Report, we investigate a system that is described by a one-dimensional wave equation (for instance, a one-dimensional string or a real massless scalar field in $1+1$ dimensions),

$$[\partial^2\varphi(t,x)/\partial t^2]-[\partial^2\varphi(t,x)/\partial x^2]=0\ . \tag{1}$$

The string will be driven by moving one of its end points periodically. The second end point remains at the same time fixed at the origin. Hence the system is a "string" analogue of the famous Fermi accelerator [6].

The length of the string is varied with the end-point motion without variation of the mass density, i.e., we periodically tune the string by varying its length but not its stress (a part of the string is released and absorbed at the moving end point). A possible mechanical model is shown in Fig. 1.

We investigate Eq. (1) on an interval $[0,X(t)]$ with function $X(t)$ having a continuous second derivative, being positive, periodic, and representing the motion with velocity smaller than the wave velocity (chosen as 1) inside the string:

$$X\in C^2(\mathbb{R})\ , \tag{2}$$

$$X(t)>0\ ,\quad X(t)=X(t+T)\ ,\quad |X'(t)|<1 \tag{3}$$

for all t and with zero boundary conditions at the ends of the interval:

$$\varphi(t,0)=0\ , \tag{4}$$

$$\varphi(t,X(t))=0 \tag{5}$$

for all t. The solution $\varphi(t,x)$ fulfills the following initial conditions at the time $t=0$:

$$\varphi(0,x)=f_0(x)\ ,\quad \partial\varphi(0,x)/\partial t=f_1(x) \tag{6}$$

with $f_0(0)=f_0(X(0))=0$, and a few further relations on f_0 and f_1 which guarantee the continuity of φ and its derivatives up to the second order.

The energy $E(t)$ of the string at time t is given by the expression

$$E(t)=\tfrac{1}{2}\int_0^{X(t)}[(\partial\varphi/\partial t)^2+(\partial\varphi/\partial x)^2]dx\ . \tag{7}$$

Our aim is to investigate under which conditions the energy $E(t)$ grows without limit, i.e., under which conditions

$$\limsup_{t\to\infty} E(t)=\infty \tag{8}$$

or (more strongly)

$$\lim_{t\to\infty} E(t)=\infty\ . \tag{9}$$

To answer these questions let us first briefly discuss the general solution of Eq. (1) with the boundary conditions (4) and (5). Taking into account the condition (4) we can express the solution as

$$\varphi(t,x)=f(t+x)-f(t-x) \tag{10}$$

with f being an arbitrary smooth function. The second boundary condition (5) leads to the relation

$$f(F(t))=f(t) \tag{11}$$

with

$$F=k\circ h^{-1} \tag{12}$$

and with functions h and k defined by

$$h(t)=t-X(t)\ , \tag{13}$$

$$k(t)=t+X(t)\ . \tag{14}$$

In order to construct the solution of Eq. (1), we start at time $t=0$. The function f is chosen in such a way that the initial conditions (6) are satisfied. These conditions determine the function f on the interval $[-X(0),X(0)]$:

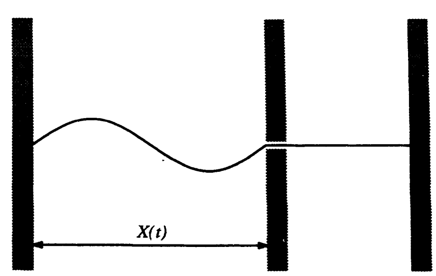

FIG. 1. A mechanical model of vibrating string with variable length.

$$f(x)=\tfrac{1}{2}f_0(x)+\tfrac{1}{2}\int_0^x f_1(y)dy \quad \text{for } x\in[0,X(0)] \tag{15}$$

and

$$f(x)=f(-x)-f_0(-x) \quad \text{for } x\in[-X(0),0) \; . \tag{16}$$

To construct the solution of (1) for time $t>0$ we use the relation (11). Applying the function F we define a sequence of points $\cdots < x_{-n} < \cdots < x_{-1} < x_0 < x_1 < \cdots < x_n < \cdots$ such that

$$x_0 = X(0) \; , \quad x_{k+1}=F(x_k) \; , \quad k=0,1,\ldots \; , \tag{17}$$

$$x_{-1}=-X(0) \; , \quad x_{-(k+1)}=F^{-1}(x_{-k}) \; , \quad k=0,1,\ldots \; . \tag{18}$$

It can be easily shown that $\lim_{n\to\infty}x_n=\infty$ and $\lim_{n\to-\infty}x_n=-\infty$. The function f is defined on the intervals (x_n,x_{n+1}) as follows. For $y\in(x_0,x_1)$ define $f(y)=f(x)$ with $y=F(x)$ and $x\in(x_{-1},x_0)$, where $f(x)$ is defined by (15) and (16). Using this rule step by step we get for $y\in(x_n,x_{n+1})$: $f(y)=f(x)$ with $y=F(x)$ and $x\in(x_{n-1},x_n)$. The same method is used to define the function f for negative arguments.

Knowing f, the solution of the wave equation is obtained by (10) and the corresponding energy can be evaluated. The results are surprising. Let us first announce a mathematical theorem that characterizes the instability of the system.

Theorem: Assume that for some $0<c<T$, X is nondecreasing in $(0,c)$, decreasing with $X'(t)<0$ in (c,T), and

$$X(0)<\tfrac{1}{2}T<X(c) \; . \tag{19}$$

Assume furthermore that the functions $f_0'+f_1$ and $f_0'-f_1$ do not vanish identically in any open subinterval of $(0,X(0))$. Then there exists $\gamma>1$ and $A>0$ such that for any natural number n and any $t\in[nT,(n+1)T]$

$$E(t)\geq A\gamma^{n-1} \; . \tag{20}$$

Remarks. (i) The assumption on $f_0'\pm f_1$ excludes the trivial case of constant field.

(ii) For simplicity, we have chosen the time origin at a moment when X passes by its absolute minimum; this is of course not necessary.

(iii) If X is not nondecreasing in $(0,c)$, we are still able to prove that $\limsup_{t\to\infty}E(t)=\infty$; however, we do not know whether the limit is indeed infinite.

Example: Let $X(t)$ be of the form

$$X(t)=x_0+\alpha\sin(\omega t) \tag{21}$$

with $x_0>\alpha>0$ and $|\alpha\omega|<1$. Further, do not let $f_0'+f_1$ and $f_0'-f_1$ vanish identically in any open subinterval of $(0,x_0)$. Then

$$\lim_{t\to\infty}E(t)=\infty \tag{22}$$

for $\pi/(x_0+\alpha)<\omega<\pi/(x_0-\alpha)$.

The detailed proof of these mathematical statements will be published elsewhere. Here we will focus ourselves on the physical interpretation of this result.

First of all the wide range in the frequencies ω of the boundary point movement clearly signifies that the instability described in the above theorem is not of usual resonant origin. Typically a resonance sourced instability takes place for discrete external frequencies only. This is the case of the so-called quantum resonance in the kicked rotator [7] or in the Fermi accelerator [8], which becomes unstable when the frequency of the external driving is rationally related to the internal frequency of the quantum system. The situation described here is different. The energy of the string is increasing for a range of driving frequencies with the width dependent on the driving amplitude α and the edge at $\alpha=0$ (no driving) which is typical for the parametric resonance [4].

There is even one more difference between the quantum resonance and the instability of the periodically driven string. In the case of a quantum resonance the increase of the mean energy $E(t)$ of the system is a quadratic function of time:

$$E(t)\approx at^2 \tag{23}$$

for $t\to\infty$ and some constant $a>0$. In the case of a driven string, however, the energy increase is given by

$$E(nT)\approx b^n \tag{24}$$

with $b>1$ and n denoting the number of oscillations of the moving boundary point.

The formula (24) is a direct consequence of the iterative use of the relation (11). The energy of the string (7) depends on the derivatives of the solution of the corresponding wave equation, i.e.,

$$E(t)=\int_{t-X(t)}^{t+X(t)}[f'(y)]^2dy \; . \tag{25}$$

Using the formula (11) we get for f'

$$f'(F(t))F'(t)=f'(t) \tag{26}$$

and hence by iteration

$$f'(F\circ F\circ\cdots\circ F(t))=f'(t)\Big/\prod_k F'(t_k) \tag{27}$$

with $t_1=t$ and $t_{k+1}=F(t_k)$. Under the assumptions of the above theorem, all the factors $F'(t_k)$ are smaller than 1 in a part of the integration interval. This leads to the multiplicative energy increase indicated by the formula (24).

The physical content of the theorem can be easily un-

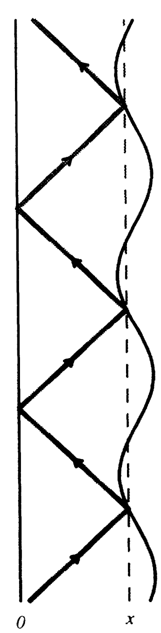

FIG. 2. A ray bouncing in phase with the wall at distance x. If $x\in(X(0),X(c))$ the condition $2x=T$ (for unit ray velocity) leads immediately to the condition (19) in the theorem. At the end points $X(0)=X(T)$ and $X(c)$ the wall has zero velocity and the ray gains no energy.

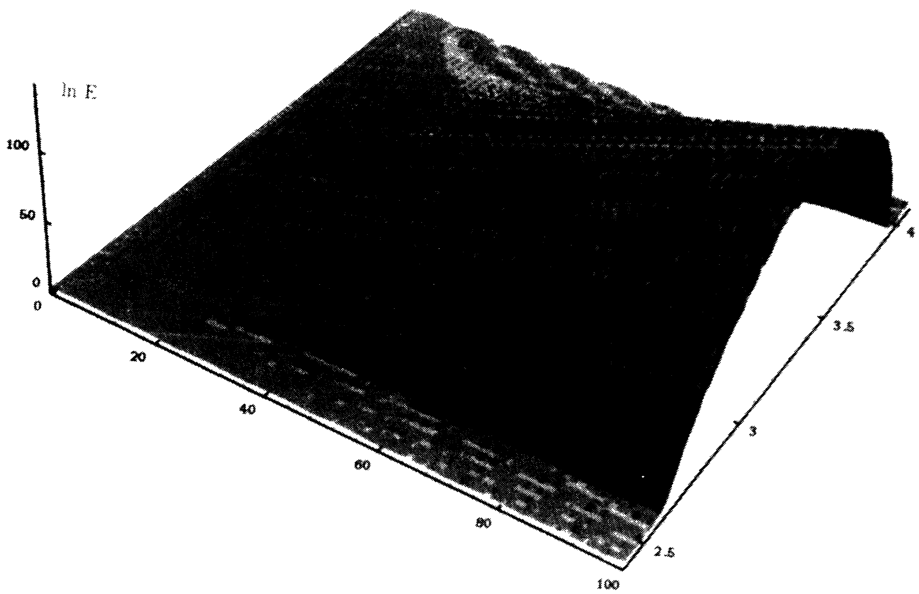

FIG. 3. The energy of the system plotted in a logarithmic scale as a function of time t and frequency ω with $\alpha=0.2$.

derstood as a consequence of a "cumulative" Doppler effect. In order to see this let us investigate the reflection of a plane wave with frequency ν (propagating in the direction from $-\infty$ to ∞) on a mirror moving with a constant velocity v. The frequency after the reflection changes to

$$\nu'=\nu(1-v)/(1+v) . \tag{28}$$

The energy of the plane wave is proportional to its squared frequency. Consequently denoting by E and E' the energy of the wave before and after the reflection, respectively, we get

$$E'=E[1-v)/(1+v]^2 . \tag{29}$$

So the energy increase or decrease of the wave is (due to the Doppler effect) proportional to its energy before the reflection. A multiple reflection leads then to the exponential increase of the energy under the conditions described by the theorem. Heuristically, we obtain Eq. (24) with

$$b=[1-v)/(1+v]^2 \tag{30}$$

from (29), where $v<0$ for the wall moving against the

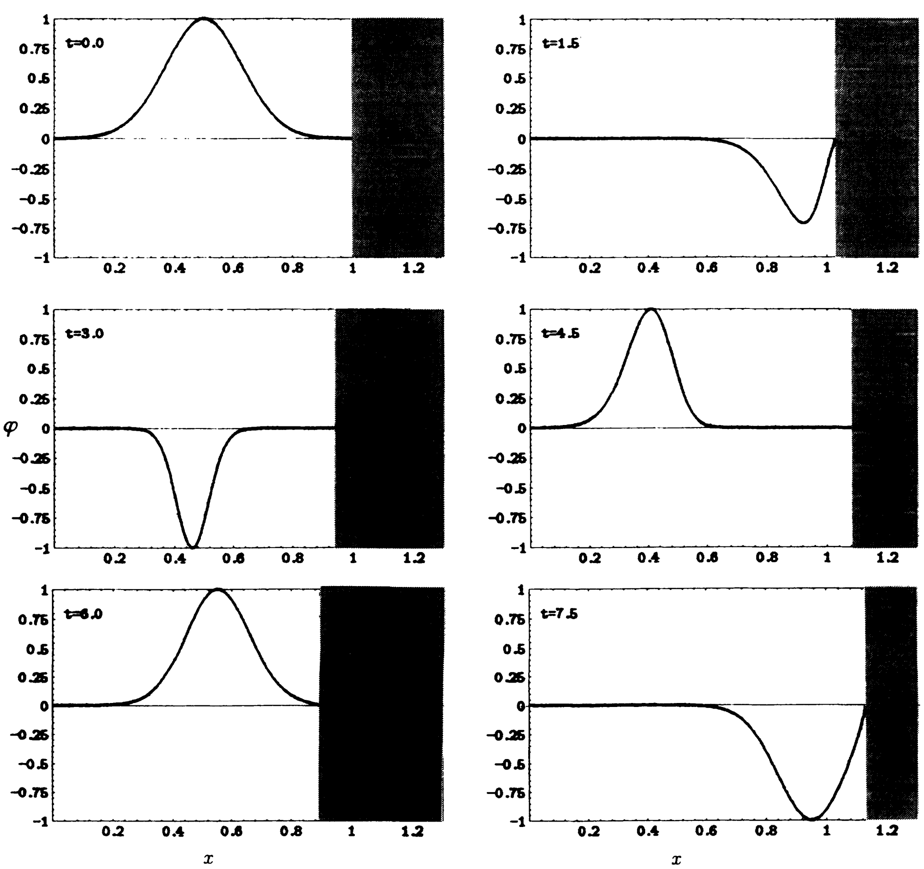

FIG. 4. The time dependence of the solution φ of the wave equation (1) is shown in a series of plots for $\omega=2$ and $\alpha=0.2$. The time is indicated in the upper left corner of each plot. The shaded rectangle indicates the position of the end of the interval $X(t)$.

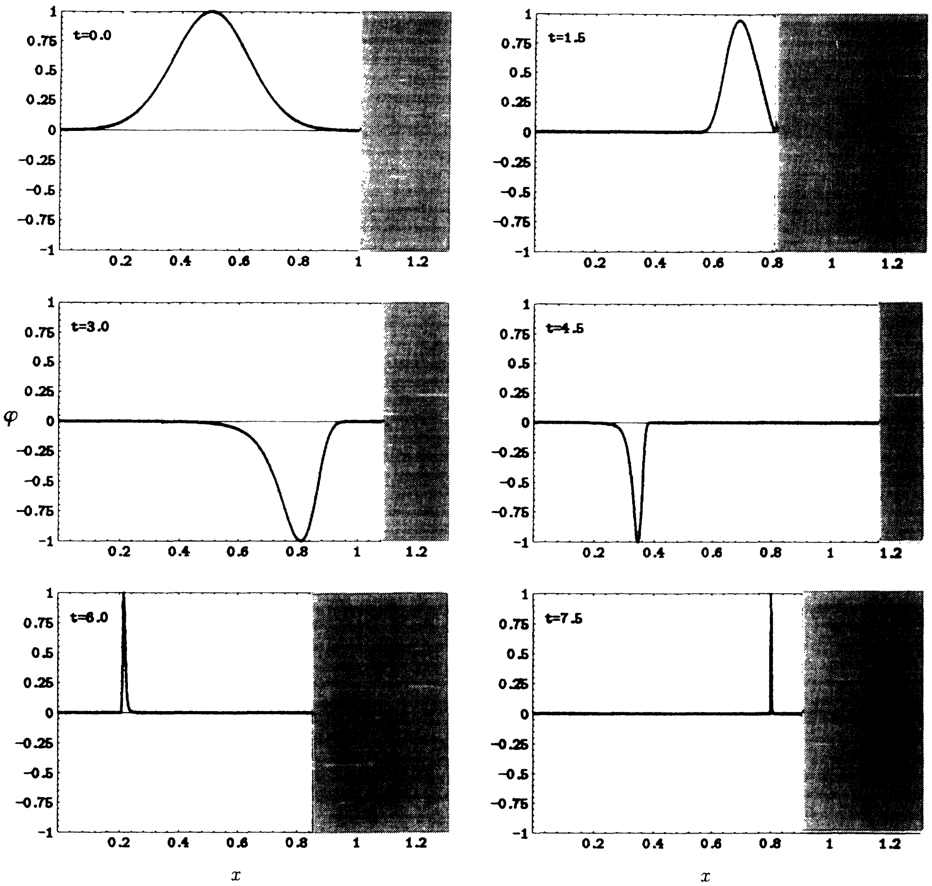

FIG. 5. The same as Fig. 4, but evaluated for $\omega=3$.

wave. The condition (19) means that there exists $t_0 \in (c,T)$ such that $2X(t_0)=T$, that is, there exists a wave which is in phase with the moving boundary. This wave meets the boundary with the same velocity v at all reflections (Fig. 2). Such a physical reasoning should also work for the cases $2X(t_0)=NT$ with $N=1,2,\ldots$, we are, in fact, also able to replace (19) by the condition $X(0)<(N/2)T<X(c)$ in the theorem. Such instability near the higher harmonic frequencies is also typical for the parametric resonance.

Let us now pass to the numerical investigation of the system. In order to demonstrate the validity of the theorem, we have solved the corresponding wave equation on a computer. We choose

$$X(t)=1+\alpha\sin(\omega t) \tag{31}$$

and start the evaluations with initial conditions

$$\varphi(0,x)=e^{-30(x-0.5)^2}, \quad \partial\varphi(0,x)/\partial t=\partial\varphi(0,x)/\partial x \tag{32}$$

at time $t=0$. The consistency conditions at $x=0$ and $x=X(0)$ are satisfied here to a good approximation; the initial shape $ax^3(1-x)^3$ exactly satisfying them gives qualitatively the same picture. Using the relation (26) iteratively we have evaluated the energy by the formula (25).

To check numerically the above example, we fixed the parameter α in (31) and evaluated the energy $E(t)$ for different values of ω ranging from $\pi/(1+\alpha)-0.2$ to $\pi/(1-\alpha)+0.2$. The result is plotted in Fig. 3. It is seen that the energy increases without bounds only for ω in $(\pi/(1+\alpha),\pi/(1-\alpha))$. Out of this frequency interval the energy behaves quasiperiodically and the system remains stable for our choice of parameter α. For the values of α which allow the cyclic frequencies $\omega=N\pi$, $N>1$, $|\alpha\omega|<1$ (higher harmonics), further instability intervals appear in their neighborhoods; with our choice of $\alpha=0.2$ in Fig. 3 this does not occur.

Figures 4 and 5 show the shape of the string in a series of plots for different times t. In Fig. 4 the driving frequency is out of the instability interval ($\omega=2$), while in Fig. 5 the frequency $\omega=3$ (inside the instability interval). $\alpha=0.2$ in both cases. The difference in the shape of the two solutions is clearly visible. In the instable case, the solution becomes nearly discontinuous in the course of time.

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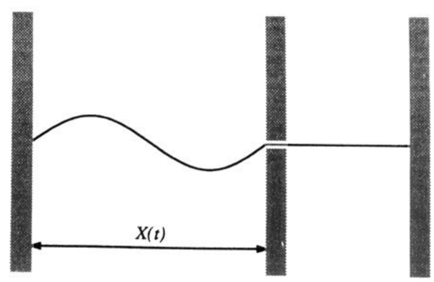

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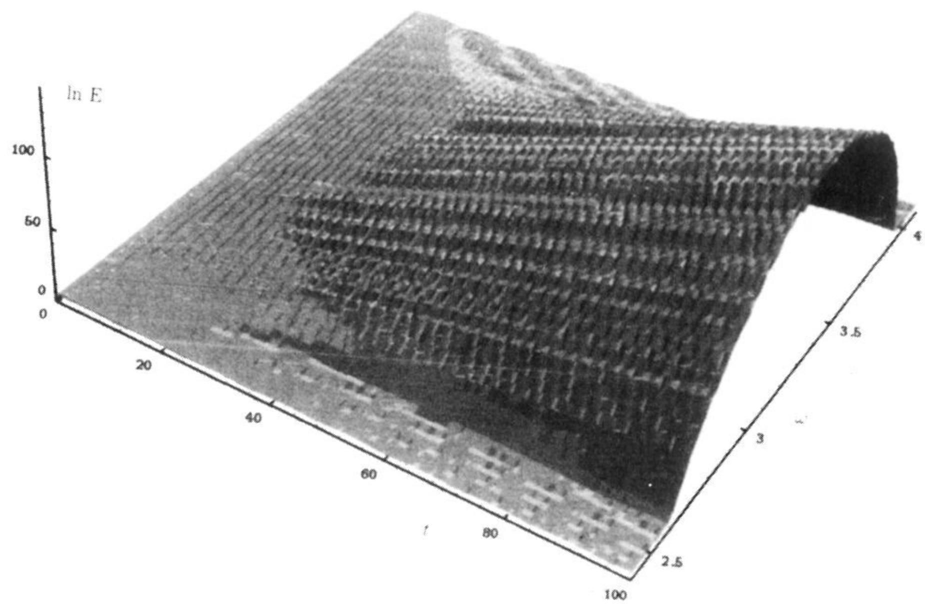

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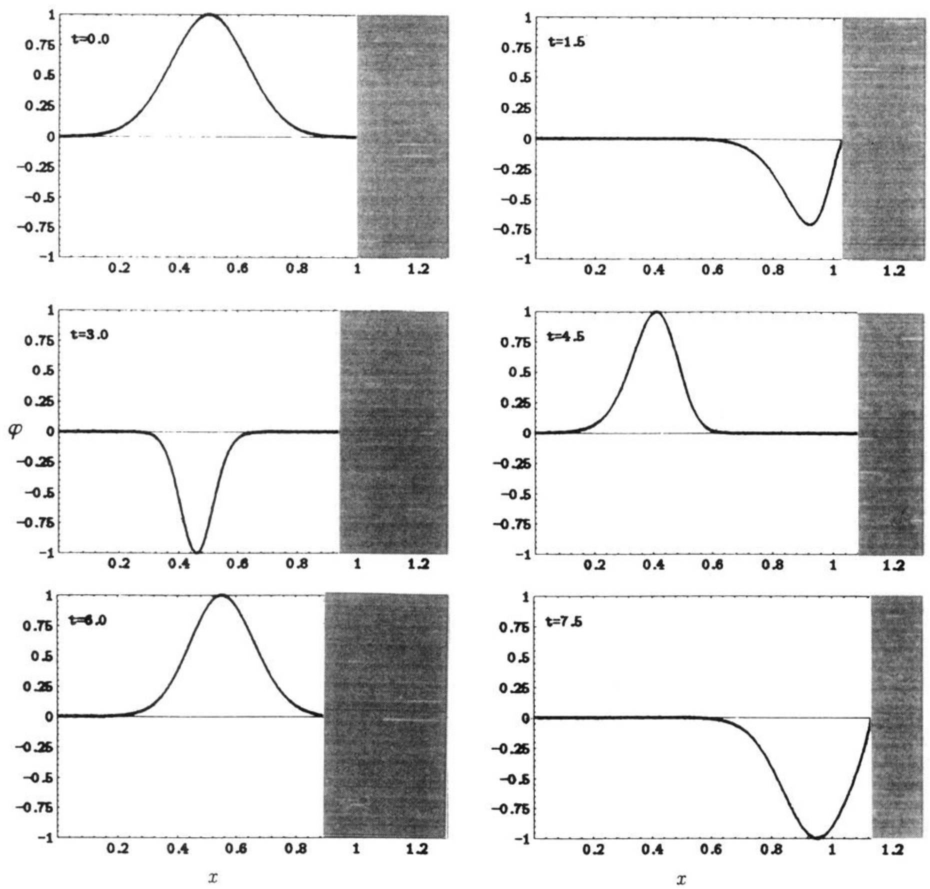

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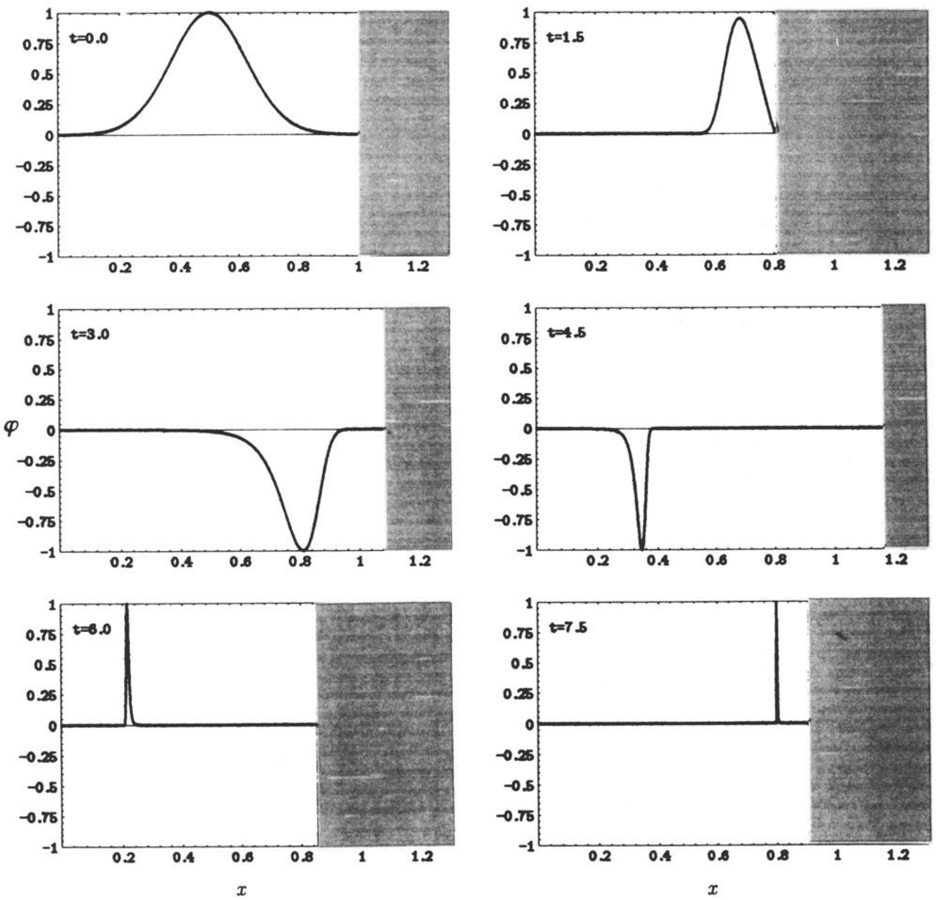

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