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Oscillation mode and "nonlinear" radiation of the double sine-Gordon 2π kink

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The collective coordinate method has been used to investigate the oscillation mode of the static and moving π-π kink system of the double sine-Gordon equation. A condition for the linear stability of the system has been found, the interaction parameter $\lambda < 0.774$. We have shown a pronounced effect of a constant external perturbation generating nonlinear periodic nonsinusoidal oscillations. This radiation appears due to interplay of the nonlinearity and of the perturbation.

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I. INTRODUCTION

The double sine-Gordon equation (DSGE) appears in many physically relevant systems [1–4]. Usually, the DSGE is related to the sine-Gordon equation when adding certain specific interactions. This is the case, for example, of the charge- and spin-density waves in $(1+1)D$ phase models with SU(2) symmetry and with a half-filled band when also including Coulomb interactions [3].

It is known that the DSGE

$$\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}+\lambda\sin\phi+\sin 2\phi=0 \tag{1}$$

is not integrable. However, it is possible to find a reasonable form of an ansatz for a solution in a specific range of the interaction parameter λ, $\lambda > 0$.

The potential related to Eq. (1),

$$V[\phi]=\lambda(1-\cos\phi)+\tfrac{1}{2}(1-\cos 2\phi)\ , \tag{2}$$

has its minima at $\phi=0$ (mod2π) and a local minimum at $\phi=\pi$ for $\lambda<2$. Two free (uncoupled) π kinks which are solutions of (1) at $\lambda=0$ become weakly coupled for $0<\lambda<2$. In other words, the 2π kink of the SGE splits into two separated π kinks for λ sufficiently small. For $\lambda>2$ the local minimum at $\phi=\pi$ disappears. The two π kinks are no longer distinguishable objects so that with increasing λ they shrink to a 2π kink. [In this case the last term in (1) can be neglected]. Therefore, for λ sufficiently small, it is reasonable to choose the solution to (1) in the form

$$\phi_K=2\arctan[\exp(\theta+\Delta)]+2\arctan[\exp(\theta-\Delta)]\ , \tag{3}$$

where

$$\theta=(2+\lambda)^{1/2}\xi/\xi_0\ ,\quad \xi=x-ut-x_0\ ,$$

$$\xi_0=(1-u^2)^{1/2}\quad\text{and}\quad \cosh^2\Delta=1+2/\lambda\ .$$

The form (3) can be rewritten, alternatively, as

$$\begin{aligned}\phi_K&=2\arccos\{-\tanh\theta/[\cosh^2\Delta-\sinh^2\Delta\tanh^2\theta]^{1/2}\}\\&=-2\arctan\{-\sinh\Delta/\sinh\theta\}\ .\end{aligned} \tag{4}$$

The centers of the π solitons $x_\pm$ are shifted due to the repulsion parameter λ:

$$x_\pm=x_0\pm\Delta(2+\lambda)^{-1/2}\xi_0\ . \tag{5}$$

The excitation spectrum of the static solution ($u=0$) has been investigated in [1], [2], and [5]. Except for the zero (Goldstone) mode and the phononlike spectrum $\omega^2(k)=2+\lambda+k^2$, there was found an internal oscillation mode which can be understood as related to the relative oscillations of two π kinks. The perturbation calculation of the frequency and the related eigenfunctions of this mode for arbitrary values of λ has been given by Hudák [5]. However, there are still open questions regarding the stability of this mode and of the related approximate solution (3).

In this paper we shall use the method of collective coordinates [6] for investigation of the DSG kink dynamics in the case of the moving π-π system (3). The translational invariance in the presence of the soliton is manifested by the arbitrary value of the soliton center x_0. As mentioned above, in the presence of the coupling λ, there appears a new degree of freedom which represents internal motion of the π-π kink system. The related quantity Δ (3) determines then the shift of x_0 due to the effective repulsion of the π kinks (5) and it is obviously translational invariant as well. Consequently, besides the collective coordinate $x_0(t)$ [as usual, we generalize $ut-x_0\equiv x_0(t)$], we choose $\Delta(t)$ as a collective coordinate related to the internal degree of freedom.

In Sec. III we shall investigate the oscillation mode of the π-π system and find conditions for its stability in both static and dynamic cases. In real systems, perturbations are likely to be present in Eq. (1). The simplest case of a small and constant field is shown to have a greater effect on the kink excitation spectra in a nonlinear range than in a linear one. In Sec. IV we shall show that a small constant force is able to generate traveling nonlinear oscillations, with the amplitude proportional to the field. We show that this radiation is of the same nature as that we have found by nonlinear perturbation analysis of the SGE in a small external constant field [7]. Generation of this radiation is a result of an interplay of the nonlinearity and of a small constant force.

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II. COLLECTIVE DYNAMICS

The Hamiltonian related to Eq. (1) in terms of the collective coordinates $x_0(t)$ and $\Delta(t)$ can be found with the use of, e.g., ansatz (3) with the generalized argument

$$\theta\pm\Delta\to(2+\lambda)^{1/2}[x-x_0(t)\pm\Delta(t)]/(1-\dot{x}_0^2)^{1/2} \tag{6}$$

as

$$\begin{aligned} H&=\int_{-\infty}^{\infty}dx\left\{\frac{1}{2}\left[\frac{\partial\phi}{\partial t}\right]^2+\frac{1}{2}\left[\frac{\partial\phi}{\partial x}\right]^2+\lambda(1-\cos\phi)+\frac{1}{2}(1-\cos2\phi)\right\}\\ &=\frac{m(\Delta)}{2(1-\dot{x}_0^2)}\left[1+\dot{x}_0^2\left[1+\frac{x_0\ddot{x}_0}{1-\dot{x}_0^2}\right]^2\right]+\frac{M(\Delta)}{2(1-\dot{x}_0^2)}\left[\left[\dot{\Delta}+\frac{\Delta x_0\ddot{x}_0}{1-\dot{x}_0^2}\right]^2+2\coth^2\Delta\right]+\frac{2\lambda\Delta}{(1-\dot{x}_0^2)(2+\lambda)^{1/2}}\coth\Delta\ , \end{aligned} \tag{7}$$

where

$$\begin{aligned} m(\Delta)&=4(2+\lambda)^{1/2}\left[1+\frac{2\Delta}{\sinh2\Delta}\right],\\ M(\Delta)&=\frac{4}{(2+\lambda)^{1/2}}\left[1-\frac{2\Delta}{\sinh2\Delta}\right]. \end{aligned} \tag{8}$$

The ground state E_0 of the Hamiltonian (7) occurs for

$$\dot{x}_0=\dot{\Delta}=\ddot{x}_0=0\ ,\quad \left.\frac{\partial V}{\partial\Delta}\right|_{\Delta=\Delta_0}=0\ ,\quad \left.\frac{\partial^2 V}{\partial\Delta^2}\right|_{\Delta=\Delta_0}>0\ . \tag{9}$$

The conditions (9) imply

$$E_0=H(\Delta_0)=4(2+\lambda)^{1/2}+2^{3/2}\Delta_0\lambda\equiv2(2+\lambda)^{1/2}g^{(0)}\ , \tag{10}$$

where Δ_0 is given by its static value (3), $\cosh^2\Delta_0=1+2/\lambda$. The ground state (10) can be perturbed linearly:

$$\Delta(t)=\Delta_0+l(t)\ ,\quad |l|\ll\Delta_0\ . \tag{11}$$

Then, let us expand the Hamiltonian (7) into a series of l as

$$\begin{aligned} H=E_0&+\frac{m(\Delta_0)\dot{x}_0^2}{2(1-\dot{x}_0^2)}\left[1+\left[1+\frac{x_0\ddot{x}_0}{1-\dot{x}_0^2}\right]^2\right]+\frac{M(\Delta_0)}{2(1-\dot{x}_0^2)}\left[1+\frac{x_0\ddot{x}_0}{1-\dot{x}_0^2}\right]^2\dot{l}^2\\ &+\frac{2(2+\lambda)^{1/2}}{(1-\dot{x}_0^2)}\sum_{n=1}^{\infty}\frac{1}{n!}\left[f_{1n}\dot{x}_0^2\left[1+\frac{x_0\ddot{x}_0}{1-\dot{x}_0^2}\right]^2+g_n\right]l^n\ , \end{aligned} \tag{12}$$

where

$$\begin{aligned} &f_1=\frac{2\Delta}{\sinh2\Delta}\ ,\quad f_2=\coth^2\Delta\left[1-\frac{2\Delta}{\sinh2\Delta}\right],\quad f_3=\Delta\coth\Delta\ ,\\ &f_{in}=\left.\frac{d^n}{d\Delta^n}f_i\right|_{\Delta=\Delta_0},\quad i=1,2,3\\ &g_n=f_{1n}+\frac{2}{2+\lambda}(f_{2n}+\lambda f_{3n})\ ,\\ &m(\Delta_0)=4(2+\lambda)^{1/2}\left[1+\frac{\lambda\,\mathrm{arcsinh}(2/\lambda)^{1/2}}{[2(2+\lambda)]^{1/2}}\right],\\ &M(\Delta_0)=\frac{4}{(2+\lambda)^{1/2}}\left[1-\frac{\lambda\,\mathrm{arcsinh}(2/\lambda)^{1/2}}{[2(2+\lambda)]^{1/2}}\right], \end{aligned} \tag{13}$$

$$
\begin{aligned}
g_0(z)&=2+\frac{\Delta_0}{[z(z+1)]^{1/2}}\ ,\\
g_1(z)&=0\ ,\\
g_2(z)&=-\frac{2}{z}+\frac{6}{(1+z)}+\frac{8}{z(1+z)}+\Delta_0\left[-\frac{12}{z^{3/2}(1+z)^{3/2}}-\frac{16}{z^{1/2}(1+z)^{3/2}}\right.\\
&\qquad\left.-\frac{2z}{(1+z)^{3/2}}+\frac{4}{z^{3/2}(1+z)^{1/2}}+\frac{2(1+z)^{1/2}}{z^{3/2}}\right]\ ,\\
g_3(z)&=\frac{6z^{1/2}}{(1+z)^{3/2}}+\frac{6(1+z)^{1/2}}{z^{3/2}}+\Delta_0\left[\frac{-2}{z^2}+\frac{2}{(1+z)^2}-\frac{4z}{(1+z)^2}-\frac{4(1+z)}{z^2}\right]\\
&\quad+4\frac{[3z^{1/2}(1+z)^{1/2}-3\Delta_0-2z^{1/2}\Delta_0]}{z^2(1+z)}+4\frac{(-15z^{1/2}-20z^{3/2}-5z^{5/2})}{z^2(1+z)^{3/2}}\\
&\quad+4\frac{[15(1+z)^{1/2}+15z(1+z)^{1/2}+2z^2(1+z)^{1/2}]}{z^2(1+z)^{3/2}}\Delta_0\ , \qquad (14)\\
g_4(z)&=\frac{-8}{z^2}+\frac{8}{(1+z)^2}-\frac{16z}{(1+z)^2}-\frac{16(1+z)}{z^2}-\frac{16z^{1/2}}{(1+z)^{5/2}}\Delta_0\\
&\quad+\frac{8z^{3/2}\Delta_0}{(1+z)^{5/2}}+\frac{16(1+z)^{1/2}}{z^{5/2}}\Delta_0+\frac{8(1+z)^{3/2}}{z^{5/2}}\Delta_0\\
&\quad+8[45z^{1/2}(1+z)^{1/2}+45z^{3/2}(1+z)^{1/2}+6z^{5/2}(1+z)^{1/2}-45\Delta_0-75z\Delta_0-32z^2\Delta_0-2z^3\Delta_0]/z^{5/2}(1+z)^{3/2}\\
&\quad+16[-3z^{1/2}-2z^{3/2}+3(1+z)^{1/2}\Delta_0+z(1+z)^{1/2}\Delta_0]/z^{5/2}(1+z)\ ,
\end{aligned}
$$

$$
\begin{aligned}
f_{11}&=\frac{1}{z^{1/2}(1+z)^{1/2}}-\left[\frac{1}{z}+\frac{1}{(1+z)}\right]\Delta_0\ ,\\
f_{12}&=-\frac{2}{z}-\frac{2}{1+z}+\frac{2z^{1/2}}{(1+z)^{3/2}}\Delta_0+\frac{2(1+z)^{1/2}}{z^{3/2}}\Delta_0\ ,\\
f_{13}&=\frac{6z^{1/2}}{(1+z)^{3/2}}+\frac{6(1+z)^{1/2}}{z^{3/2}}-\frac{2}{z^2}\Delta_0+\frac{2}{(1+z)^2}\Delta_0-\frac{4z}{(1+z)^2}\Delta_0-\frac{4(1+z)}{z^2}\Delta_0\ ,\\
f_{14}&=-\frac{8}{z^2}+\frac{8}{(1+z)^2}-\frac{16z}{(1+z)^2}-\frac{16(1+z)}{z^2}-\frac{16z^{1/2}}{(1+z)^{5/2}}\Delta_0+\frac{8z^{3/2}}{(1+z)^{5/2}}\Delta_0+\frac{16(1+z)^{1/2}}{z^{5/2}}\Delta_0+\frac{8(1+z)^{3/2}}{z^{5/2}}\Delta_0\ ,\\
z&=2/\lambda\ . \qquad (15)
\end{aligned}
$$

The potential $V(l,\lambda)=2(2+\lambda)^{1/2}\sum_{n=1}^{n=4}1/n!\,g_n(\lambda)l^n$ is plotted in Fig. 1. It is strongly anharmonic for l sufficiently large. In the harmonic approximation one gets for the static case ($\dot{x}_0=0$) the frequency

$$\omega_s^2(\lambda)=2(2+\lambda)^{1/2}g_2(\lambda)/M(\lambda)=1+\frac{\lambda}{2}g_2(\lambda)/\{1-\lambda\,\mathrm{arcsinh}(2/\lambda)^{1/2}/[2(2+\lambda)]^{1/2}\}\ . \qquad (16)$$

The frequency $\omega_s^2(\lambda)$ shows almost linear dependence in the whole extent of λ. This result can be compared with the results of [5]—namely, for $\lambda+2$ we get

$$\omega_s^2(\lambda=2)=2(2+\lambda)^{1/2}g_2(2)/M(2)=3.383\,58\ . \qquad (17)$$

Other values are

$$\omega^2(\lambda=10^3)=1600.27\ ,\quad \omega^2(\lambda=10^5)=163\,630\ .$$

The frequency (17) is identical with that $\omega_B^2(2)$ of [5], corresponding to the oscillations of the linearly perturbed ansatz (4). It can be shown that the frequencies of the internal oscillation mode of both linearly perturbed ansatzes (3) and (4) are identical, in contrast to the statement in [5] that they are different. Namely, when going

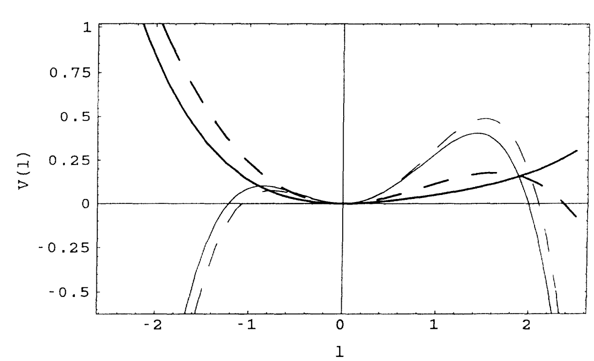

FIG. 1. Static potential $V(\lambda,l)$ for $\lambda=0.1$ (thick solid), $\lambda=0.3$ (thick dashed), $\lambda=1.6$ (thin solid), and $\lambda=2$ (thin

from ansatz (3) to ansatz (4), we change the perturbation $\Delta \rightarrow \Delta_1$, or

$$\cosh^2\Delta_1 = \cosh^2\Delta - 1 = 2/\lambda \ , \quad \lambda \leq 2$$

as has been done in [5]. Then, the functions $g_2(\Delta)$ and $M(\Delta)$ change as follows:

$$\begin{aligned} g_2(\Delta_1) &= g_2(\Delta)(d\Delta/d\Delta_1)^2 + g_1(\Delta) d^2\Delta/d\Delta^2 \\ &= g_2(\Delta)(d\Delta/d\Delta_1)^2 \ , \end{aligned} \tag{18}$$

as $g_1(\Delta)=0$ according to (14) and

$$M(\Delta_1) = M(\Delta)(d\Delta/d\Delta_1)^2 \ . \tag{19}$$

From (18) and (19) it follows that $\omega_s^2(\lambda)$ given by (16) is invariant against the choice of the perturbation as the functions $g_2(\Delta)$ and $M(\Delta)$ scale by the same scale factor.

III. DYNAMICS OF THE MOVING π-π KINK SYSTEM

The dynamic equations for the coordinates x_0 and l defined by (11) related to the Hamiltonian (7) in a simplified case, when $\ddot{x}_0 = 0$ and $\dot{x}_0^2 \ll 1$, are

$$\dot{x}_0(t) = \frac{P_0}{m[\Delta(t)]} \tag{20}$$

and

$$\ddot{l} + [2(1-f_1)]^{-1} \sum_{n=1}^{N} \frac{1}{(n-1)!}(f_{1n}\dot{x}_0^2 + g_n) l^{n-1} = 0 \ . \tag{21}$$

Here, f_1, f_{1n}, and g_n are given by (13), (14), and (15). Due to the definition (11), the dynamic equations (20) and (21) are coupled. However, in zero-order approximation when we put $\Delta = \Delta_0$ in (20), these equations decouple and we get

$$\begin{aligned} &\dot{x}_0 \approx \frac{P_0}{m_0} \ , \\ &\ddot{l} + \left[\omega_s^2 + \lambda_2 \frac{P_0^2}{m_0^2}\right] l + \lambda_1 \frac{P_0^2}{m_0^2} = 0 \ , \end{aligned} \tag{22}$$

where $m_0 = m[\Delta_0(\lambda)]$.

Here, we consider a harmonic approximation [$N=2$ in (21)]. In Eq. (22), ω_s^2 is given by (16) and

$$\lambda_i = \frac{(1+\lambda/2)}{(1-f_1)} f_{1i} \ , \quad i=1,2 \ . \tag{23}$$

f_{1i} are given by (15).

The solution to Eq. (22) is simple:

$$l = -\lambda_1 \frac{P_0^2}{m_0^2 \Omega^2}(1 \pm \sin\Omega t) \ , \tag{24}$$

where

$$\Omega^2 = \omega_s^2 + \lambda_2 P_0^2/m_0^2 \ . \tag{25}$$

From Eqs. (24) and (25) it is evident that for a moving soliton the plateau Δ between two coupled π kinks increases by $[-\lambda_1(\lambda)P_0^2]/m_0^2\Omega^2$ [$\lambda_1(\lambda)<0$ as $f_{11}<0$] and the frequency Ω^2 (25) increases as well [$\lambda_2(\lambda)>0$ for $\lambda < \lambda_c = 2.5125$]. Equation (25) implies also the condition for the instability of the oscillation mode, $\Omega^2 < 0$. For a given λ, this possibility occurs for $0 \leq P_0^2 \leq P_{0,\mathrm{crit}}^2$, $P_{0,\mathrm{crit}}^2(\lambda)$ given by (Fig. 2):

$$P_{0,\mathrm{crit}}^2(\lambda) = -m_0^2(\lambda)\omega_s^2(\lambda)/\lambda_2(\lambda) \ , \quad \lambda_2 \neq 0 \ . \tag{26}$$

As $\lambda_2(\lambda)$ changes its sign at $\lambda_c = 2.5125$, the instability of the oscillation mode and of the ansatz (3) appears at this point. For the static kink, $P_0 \rightarrow 0$, the value λ_c persists as the critical value for its stability. For $\lambda < \lambda_c$ the mode Ω^2 is stable for any value of $P_0^2 \geq 0$. However, we have to have in mind that the linear approximation of the dynamic equation (21) leads apparently to the overestimation of λ_c. In view of what has been said in the Introduction, λ_c is expected in the range $0 < \lambda_c < 2$. We shall return to this point in the Sec. VI where we shall determine λ_c from other conditions.

IV. EXACT TRAVELING "NONLINEAR" OSCILLATIONS GENERATED BY A CONSTANT EXTERNAL PERTURBATION

In this section we shall show that in the presence of a constant force f on the right-hand side (rhs) of Eq. (1), there are generated qualitatively new "nonlinear" oscillations. Indeed, in [7] we have shown that in sine-Gordon systems a small constant external field generates a periodic nonsinusoidal radiation with the amplitude proportional to the field. On the other hand, the respective linear perturbation spectrum is suppressed by the field.

The ground state ϕ_0 of the Hamiltonian related to Eq. (1) with additional force term $f>0$ given by

$$\lambda \sin\phi_0 + \sin 2\phi_0 = -f \ , \tag{27}$$

where $f + \lambda \leq 2$. In the linear approximation, $f^2 \ll 1$, the solution of (27) reads

$$\begin{aligned} &\sin\phi_{01} \approx (-f)/(2+\lambda) \ , \quad \cos\phi_{01} > 0 \\ &\sin\phi_{02} \approx f/(2-\lambda) \ , \quad \cos\phi_{02} < 0 \ . \end{aligned} \tag{28}$$

Hence, the zero mode splits into two states; the lower energy evidently yields the state $\phi_{01} = \phi_0$. Linear excitations

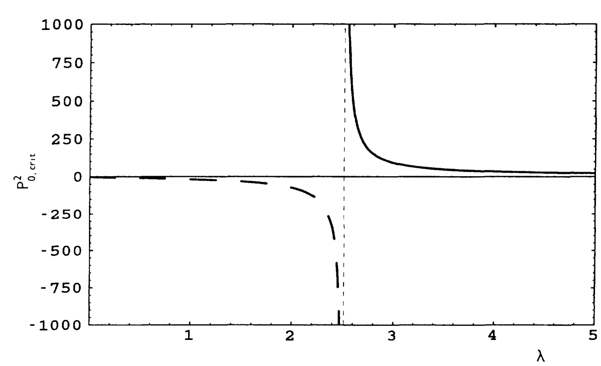

FIG. 2. P_0^2 [Eq. (26)] is discontinuous at $\lambda_c = 2.5125$, where $\lambda_2 = 0$.

of the ground state $\psi=\phi-\phi_0$ obey the equation

$$\ddot{\psi}-\psi''+\omega_0^2\psi=0\ , \tag{29}$$

where

$$\omega_0^2(f)=\lambda\cos(\phi_0)+2\cos(2\phi_0)$$
$$=\lambda[1-f^2/(2+\lambda)^2]^{1/2}+2[1-2f^2/(2+\lambda)^2]\ .$$

Let

$$\psi\equiv\psi_0\sum_k \exp[ikx+i\omega(k)t]\ ;$$

then the phonon energy becomes $\omega^2(k)=\omega_0^2(f)+k^2$. The gap for phonon excitations yields

$$W=2\omega_0^2(f)=2(\omega_0^2-\lambda\{1-[1-f^2/(2+\lambda)^2]^{1/2}-4f^2/(2+\lambda)^2\})$$
$$=W_0-2\lambda(\{1-[1-f^2/(2+\lambda)^2]\}^{1/2})-f^2/(1+\lambda/2)^2\ , \tag{30}$$
$$W_0=2(2+\lambda)\ ,$$

W_0 being the unperturbed gap.

On the other hand, when going over the linear approximation in Eq. (1) for the fluctuations ψ in the filed f, we get

$$\ddot{\psi}-\psi''+F+A\psi-B\psi^3=0\ , \tag{31}$$

where

$$F=\frac{1}{6^3}\frac{(f-3\sin 2\phi_0)^3}{\cos^2 2\phi_0}\ ,$$
$$A=\frac{1}{3.2^2}\frac{(f-3\sin 2\phi_0)^2}{\cos 2\phi_0}\ , \tag{32}$$
$$B=\cos 2\phi_0\ .$$

For $f^2\ll 1$, we have $A>0$, $B>0$. Then Eq. (31) admits an exact solution [7,8],

$$\psi=a\frac{n_1+\cos[\omega(x-vt)]}{n_2+\cos[\omega(x-vt)]}\ , \tag{33}$$

where

$$a^2=\frac{A}{B}\frac{n_2}{(n_2^2+1)}\ ,\quad \omega^2=\frac{2A(n_2^2-1)}{(1-v^2)(n_2^2+2)}>0\ ,$$
$$n_1=-\left[n_2-\frac{2}{n_2}\right]\ ,\quad n_2^2>1\ ,\quad v^2<1\ .$$

The relation $-Aa+Ba^3=F$ has to be fulfilled as well. Equation (33) represents traveling periodic nonsinusoidal oscillations with the amplitude $a\sim f$, $f^2\ll 1$. Let us note that the coefficients A, B, and F (32) imply the same values for oscillations of the ground state ϕ_0 as for the oscillations of the plateau at $\phi=\phi_0+\pi$; $f>0$.

V. NUMERICAL SIMULATIONS

With these preliminary results in mind, we can understand the results of the following numerical simulations: For $t<0$ we assume the π-π kink ansatz (3) with the traveling velocity u of the unperturbed double sine-Gordon equation (1). At $t=0$ we switch on (immediately) the external constant force f. Numerical simulation of the solution to the equation

$$\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}+\lambda\sin\phi+\sin 2\phi=-f$$

was performed by using the difference approximation. We introduce the following replacements:

$$\frac{\partial^2\phi}{\partial t^2}\approx[\phi(x_i,t_{j+1})-2\phi(x_i,t_j)+\phi(x_i,t_{j-1})]/(\Delta t)^2\ , \tag{34}$$

$$\frac{\partial^2\phi}{\partial x^2}\approx[\phi(x_{i+1},t_j)-2\phi(x_i,t_j)+\phi(x_{i+1},t_j)]/(\Delta x)^2\ . \tag{35}$$

Then, from Eq. (34) we express $\phi(x_i,t_{j+1})$. Further, if we suppose that we know $\phi(x_i,t_j)$ and $\phi(x_i,t_{j-1})$ for $x_i=\pm i(\Delta t)$, $i=0,\ldots,I$, we are able to determine $\phi(x_i,t_{j+1})$ at the points $x_i=\pm i(\Delta t)$ for $i=0,\ldots,I-1$, according to Eq. (35). At each time step Δt (from t_i to t_{i+1}) we lose one point (the region at the x axis where the kink is defined is getting narrower). However, this does not create a problem if the starting number of the points is much larger when compared with the number of time steps. We started with $I=500$ and $\Delta x=0.04$, i.e., at $t=0$ the region of the definition of the kink is $x\epsilon[-20,20]$. The time step was $\Delta t\approx 0.05$ and the number of the time steps was ≈ 250. (Hence, the finite time at which the time development of the kink was considered reached ≈ 10.) Finally, the kink was determined at the interval $x\epsilon[-10,10]$. As the dynamics during the transient regime was observed in the region $x\epsilon[-5,5]$, the present consideration is sufficiently reliable.

The initial conditions, i.e., $\phi(x_i,t_0)$, $\phi(x_i,t_0-\Delta t)$ at $x_i=\pm i(\Delta x)$, $i=0,\ldots,I=500$, are determined in accordance with the unperturbed kink.

$$\phi(x,t=t_0)=\phi_K(x-vt)$$

and

$$\phi_t(x,t-t_0)=\left.\frac{\partial\phi_K}{\partial t}\right|_{t=0}\approx[\phi(x,t_0)-\phi(x,t_0-\Delta t)]/\Delta t\ .$$

The force f is switched on at $t=0$. The method was tested by comparing with the behavior of the numerical simulation of the solution to (1) (i.e., for $f=0$): We checked on the time, when the numerical solution was

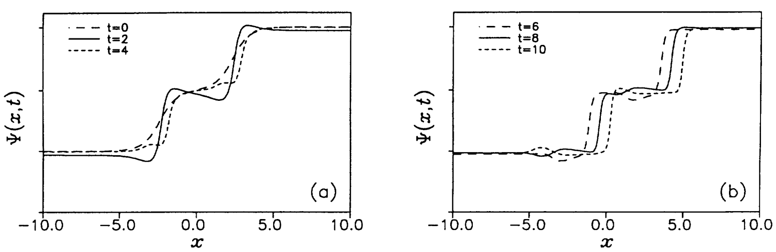

FIG. 3. Time evolution of the π-π pair (3) for $\lambda=10^{-2}$, $f=0.2$, and $u=0$, as described by Eq. (1) with additional term $(-f)$ on the rhs and with initial condition (3).

starting to differ from the analytical solution (which is known for $f=0$). As this time was much larger when compared with the duration of the transient regime, we believe that the procedure yields a convergent solution with $f\neq 0$ for a given Δx and Δt as well.

The numerical simulations of the time development of the solution to Eq. (1) are given in Figs. 3–11 for different values of the parameters λ, f and of the initial velocity $u=P_0/m_0$ (see below). The solutions exhibit the following features.

(i) The kink walls are preserved for small times and generate oscillations behind them traveling in the opposite direction and with the amplitude increasing with increasing f.

(ii) For small λ when the π kinks are well separated from each other, the oscillations of the uniform parts of both kinks are identical. In the strong coupling regime oscillations of the plateau disappear (Fig. 11). We identify these oscillations with those given above (33) as solutions of Eq. (31). Large-time behavior clearly shows the instability of the perturbed π-π kink system (Figs. 8 and 10).

VI. COLLECTIVE DYNAMICS IN THE CASE WITH EXTERNAL FIELD

If we switch to the collective coordinates again, in the case with external field f we arrive at the dynamic equation for x_0:

$$\frac{m_0(\lambda)\dot{x}_0(t)}{[1-\dot{x}_0^2(t)]^2}=P_0+2\pi f t \ . \tag{36}$$

From here we get the asymptotic value of $\dot{x}_0$ for $t\to\infty$, $\dot{x}_0\to 1$. For $ft\ll 1$ we can neglect $\dot{x}_0^2$ in the denominator as in the case $f=0$, Eq. (20).

For the fluctuation of the oscillation mode l, we get, instead of (21), the equation

$$\ddot{l}+\left[\Omega^2+\lambda_2\frac{4\pi f t}{m_0^2}(P_0+\pi f t)\right]l+\lambda_1\frac{4\pi f t}{m_0^2}(P_0+\pi f t)=0 \ . \tag{37}$$

Here, the transient nature of the oscillations of l is evident. For small ft, $\pi f t\ll P_0$, we get

$$\frac{d^2L}{d\xi^2}+\xi L-\frac{\gamma_1}{\gamma_2^{5/3}}\Omega^2=0 \ , \tag{38}$$

where $\xi=\gamma_2^{-2/3}(\Omega^2+\gamma_2 t)$, $\gamma_1=(4\pi f P_0)/m_0^2\lambda_1$, $\gamma_2=(4\pi f P_0)/m_0^2\lambda_2$, and $L=l+\gamma_1/\gamma_2$; Ω^2 is defined by (25). Equation (38) is an inhomogeneous Airy equation, the solution to which is given by [9]

$$L=c_1L_1+c_2L_2+\frac{\gamma_1}{\gamma_2^{5/3}}\Omega^2\left[L_2\int\frac{L_1}{W}d\xi-L_1\int\frac{L_2}{W}d\xi\right], \tag{39}$$

$$L_1=\xi^{1/2}Z_{\pm 1/3}(\tfrac{2}{3}\xi^{3/2}) \ , \quad L_2=\xi^{1/2}Z_{\pm 1/3}(\tfrac{2}{3}i\xi^{3/2}) \ . \tag{40}$$

Z_ν are cylindric functions. (Airy functions for $\nu=\pm\frac{1}{3}$ and $W=L_1(dL_2/d\xi)-(dL_1/d\xi)L_2$.

For large t, one can neglect the inertial term in (37) and l approaches, asymptotically,

$$l\to\left[-\frac{\lambda_1}{\lambda_2}\right]=\left[-\frac{f_{11}}{f_{12}}\right]>0 \ . \tag{41}$$

According to (11), the perturbation approach is approved if $l<\Delta_0=\operatorname{arccosh}2^{1/2}$. This relation with the use of (41) and (15) implies a constraint on λ, $\lambda<0.774\equiv\lambda_c$. This value of λ_c is smaller than that implied by the condition of the instability of the oscillation mode Ω (25), $\lambda_c=2.5125$ (Sec. III). For the determination of the weak coupling regime of the π kinks, we choose the smaller critical value $\lambda_c=0.774<2$ as the more reliable one (see the note at the end of Sec. III). Namely, for $\lambda>\lambda_c$, e.g., $\lambda=1$ (Fig. 11), the π kinks are no longer distinguishable and the object represents a perturbed 2π soliton [7].

VII. CONCLUSIONS

The linear perturbation analysis of both the static and moving π-π systems enabled us to ascertain precisely the conditions for the stability of the ansatz (3) of the nonin-

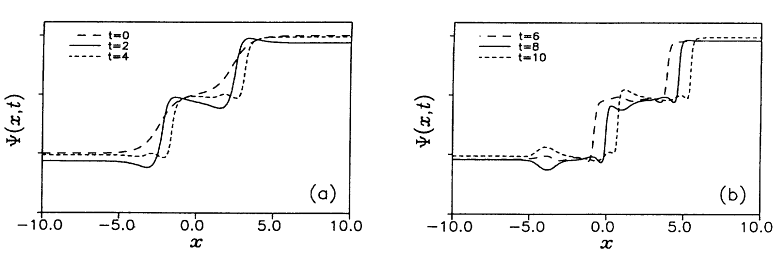

FIG. 4. The same as Fig. 3 for $\lambda=10^{-2}$, $f=0.4$, and $u=0$.

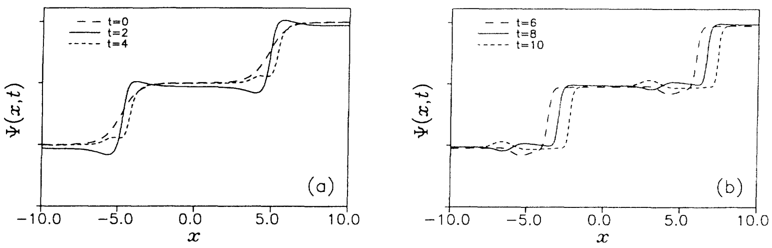

FIG. 5. The same as Fig. 3 for $\lambda=10^{-5}$, $f=0.2$, and $u=0$.

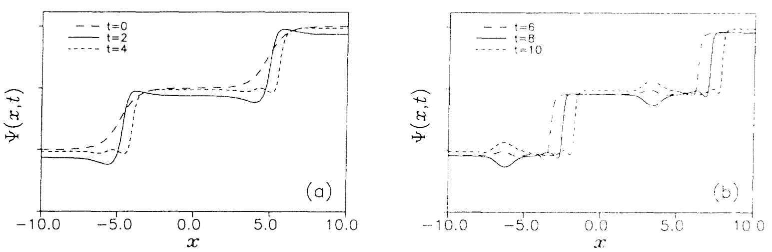

FIG. 6. The same as Fig. 3 for $\lambda=10^{-5}$, $f=0.4$, and $u=0$.

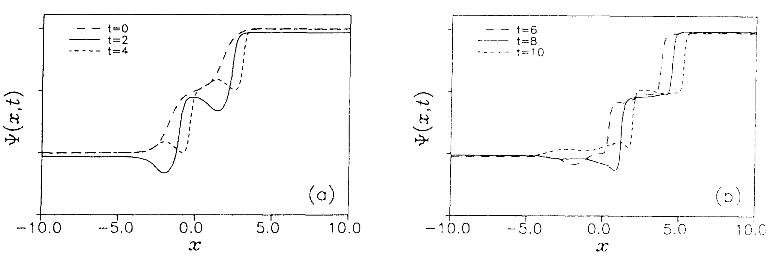

FIG. 7. Time evolution of the π-π pair (3) as described by Eq. (1) with additional term $(-f)$ on the rhs and with initial condition (3) for $\lambda=10^{-2}$, $f=0.2$, and $u=0.7$.

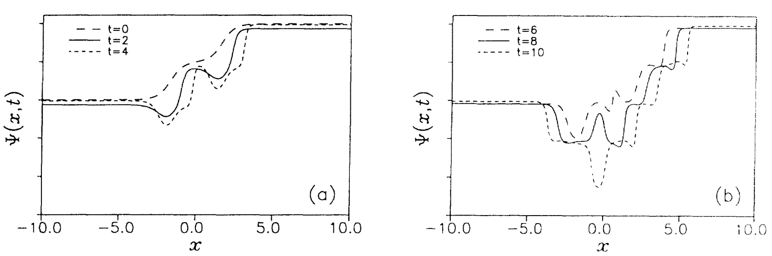

FIG. 8. The same as Fig. 7 for $\lambda=10^{-2}$, $f=0.4$, and $u=0.7$.

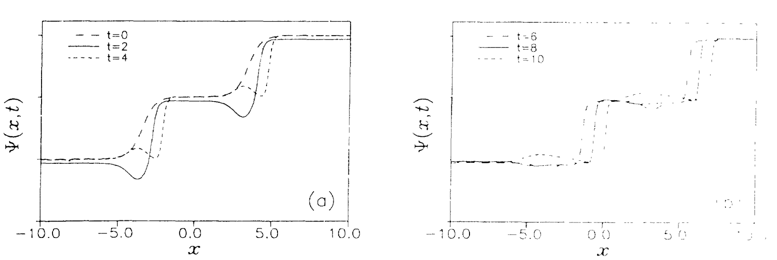

FIG. 9. Time evolution of the π-π pair (3) as described by Eq. (1) with additional term $(-f)$ on the rhs and with initial condition (3) for $\lambda=10^{-5}$, $f=0.2$, and $u=0.7$.

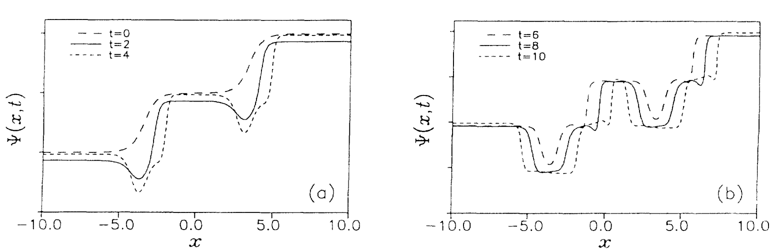

FIG. 10. The same as Fig. 9 for $\lambda=10^{-5}$, $f=0.4$, and $u=0.7$.

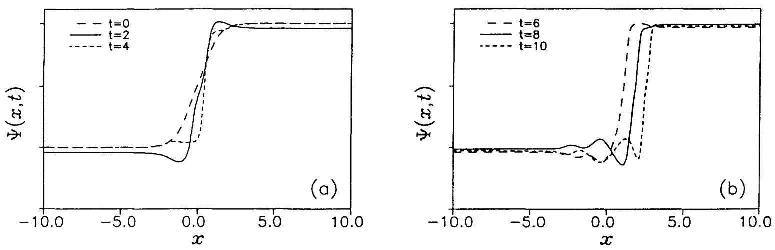

FIG. 11. Time evolution of the same object as in Figs. 4–11 for $\lambda=1.0$, $f=0.4$, and $u=0$. Here, two π kinks are no longer distinguishable.

tegrable DSGE. We have found that only for a sufficiently small λ, $\lambda<0.774$, is this ansatz approved, i.e., the object behaves as two distinguishable π kinks. Moreover, starting from small values of constant field f in the DSGE there appear transient nonlinear oscillations with the amplitude proportional to the field. Thus, the nonlinearity is able to transform the energy supplied by the field to the radiation. In the range of λ, where the π kinks are well separated, the nonlinear fluctuations for $\phi=\phi_0+\pi$ are identical with those for $\phi=\phi_0$ (at low constant branches of the kinks for $f>0$), as can be seen from expressions (32). For λ so large that the kinks are not more distinguishable, the oscillations appear only at the low branch of the 2π kink (Fig. 11) and the case goes over to the case we have investigated in [7].

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