Negative-energy modes in a magnetically confined plasma in the framework of Maxwell-drift kinetic theory

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The general expression for the second-order perturbation energy of a Maxwell-drift kinetic system derived by Pfirsch and Morrison [Phys. Fluids B **3**, 271 (1991)] is evaluated for the case of a magnetically confined plasma for which the equilibrium quantities depend on one Cartesian coordinate y. The conditions for the existence of negative-energy modes with vanishing initial field perturbations are also obtained. If the equilibrium guiding center distribution function $f_{g\nu}^{(0)}$ of any particle species ν has locally the property $v_\parallel(\partial f_{g\nu}^{(0)}/\partial v_\parallel)>0$, where $v_\parallel$ is the guiding center velocity parallel to the magnetic field, and if this holds in the minimum-energy reference frame, parallel and oblique negative-energy modes exist with no essential restriction on either the orientation or magnitude of the wave vector. This condition also holds for the equilibria of a homogeneous magnetized plasma and an inhomogeneous force-free plasma with sheared magnetic field. If $v_\parallel(\partial f_{g\nu}^{(0)}/\partial v_\parallel)<0$, the oblique negative-energy modes possible in a magnetically confined plasma are nearly perpendicular. The condition for purely perpendicular negative-energy modes reads as $(dP^{(0)}/dy)(\partial f_{g\nu}^{(0)}/\partial y)<0$, where $P^{(0)}$ is the plasma pressure. For the cases of tokamaklike and shearless stellaratorlike equilibria, which are described on the basis of, respectively, a slightly modified Maxwellian and a Maxwellian distribution function, the existence of perpendicular negative-energy modes is related to the threshold value $\frac{2}{3}$ of the quantity $\eta_\nu=\partial \ln T_\nu/\partial \ln N_\nu$, where T_ν is the temperature and N_ν the density of some particle species. This is lower than the critical η_ν value for the onset of *linear* temperature-gradient-driven modes. For various tokamaklike and stellaratorlike, analytic cold-ion equilibria with non-negative and negative values of η_e, for which the criterion above is not necessary, a substantial fraction of thermal electrons is associated with negative-energy modes (active particles). In particular, for linearly (marginally) stable equilibria with $\eta_e=1$, nearly one-third of the electrons are active. For all equilibria considered, the phase space occupied by active electrons increases as one proceeds from the center to the plasma edge region. Consequently, negative-energy modes, relating to nonlinear instabilities that could cause anomalous transport, exist equally well in both confinement systems.

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I. INTRODUCTION

The existence of negative-energy perturbations in a linearly stable plasma may be related to nonlinear instabilities and cause anomalous transport. An instability of this kind was exemplified in a transparent way for the first time in 1925 by Cherry [1]. He examined a simple, linearly stable system of nonlinearly coupled oscillators, one possessing positive energy, the other negative energy, and the frequency of one oscillator was twice that of the other, which means third-order resonance. The exact two-parameter solution set he found exhibited explosive instability for arbitrarily small initial perturbations. Pfirsch [2] considered the corresponding three-oscillator case and found the complete solution of this problem. It shows that in the resonant case almost all initial conditions lead to explosive behavior, whereas in the nonresonant case the initial perturbations must exceed threshold amplitudes which are related to the frequency mismatch. Self-sustained drift wave turbulence in a linearly stable plasma regime resembling the tokamak edge regions was demonstrated numerically by Scott [3,4] in the framework of a nonlinear, collisional two-fluid model. His study also showed that parallel particle dynamics plays an essential role in turbulence. A related result was recently obtained by Pfirsch and Correa-Restrepo [5]. From the general energy expression for linear, quasineutral, electrostatic drift modes, obtained within the framework of dissipationless multifluid theory and applied to plain configurations, they found that negative-energy modes localized at a mode-resonant surface exist only if electron parallel dynamics is included. It is therefore very likely that the results obtained by Scott are understandable in terms of nonlinearly coupled positive- and negative-energy modes. In addition, the same physical mechanism was invoked by Nordman, Pavlenko, and Weiland [6] to explain the existence of self-sustained toroidal η_i-mode turbulence below the linear instability threshold. This result was obtained numerically within the framework of nonlinear, dissipationless two-fluid theory.

The present paper discusses such problems within the framework of collisionless Maxwell-drift kinetic theory. For collisionless Maxwell-Vlasov and Maxwell-drift

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kinetic theories, general expressions for the second-order perturbation energy were derived by Pfirsch and Morrison [7,8]. Assuming strongly localized electrostatic initial perturbations ($k_\perp r_L \gg 1$, with $k_\perp$ the perpendicular component of the wave vector and r_L the Larmor radius), Morrison and Pfirsch [7] also showed, in the context of the Maxwell-Vlasov theory, that all inhomogeneous equilibria of interest allow negative-energy modes. The degree of localization actually required along with the conditions for the existence of negative-energy modes were investigated by Correa-Restrepo and Pfirsch in the context of the same theory for the cases of a magnetized homogeneous plasma [9] and an inhomogeneous force-free plasma with sheared magnetic field [10]. It turned out that negative-energy modes exist even without any strong localization of the associated wavelengths, a feature which enhances the relevance of these modes. Negative-energy modes with not strongly localized wavelengths ($k_\perp r_L < 1$) can be investigated more conveniently with the use of drift kinetic theories which have automatically eliminated from the outset all perturbations with wavelengths smaller than the gyroradii. In this context, Pfirsch and Morrison [8] examined a magnetized homogeneous plasma and found that parallel and oblique negative-energy modes ($k_\parallel \neq 0$) exist for arbitrary wave vector **k** whenever

$$v_\parallel \frac{f_{g\nu}^{(0)}}{\partial v_\parallel} > 0 \tag{1}$$

holds for some particle species ν and parallel guiding center velocity $v_\parallel$. The investigation is extended in the present paper to the more interesting equilibria of a magnetically confined plasma with sheared magnetic field for which the equilibrium quantities depend spatially on just one Cartesian coordinate. The equilibria of a homogeneous magnetized plasma and an inhomogeneous force-free plasma with sheared magnetic field are also examined as specific examples. The most important conclusions are the following.

(1) Condition (1) for the existence of parallel and oblique negative-energy modes remains valid for all the equilibria considered, without any essential restriction on **k**.

(2) In the case of a magnetically confined plasma, the existence of perpendicular negative-energy modes, which are found to be the most important modes, is related to the threshold value $\frac{2}{3}$ of $\eta_\nu = \partial \ln T_\nu / \partial \ln N_\nu$, where T_ν is the temperature and N_ν is the density of some particle species ν. This is lower than the critical η_ν value for the onset of linear, temperature-gradient-driven modes.

The derivation of the general expression for the second-order perturbation energy within the framework of Maxwell-drift kinetic theory by Pfirsch and Morrison [8], slightly adapted to the needs of the present paper, is first reviewed in Sec. II. This consists of two subsections. The first concerns the energy-momentum tensor for general nonlinear and linearized kinetic theories. In the second the linearized energy-momentum tensor is derived in the case of Maxwell-drift kinetic theory based on the Lagrangian formulation of the guiding center theory given by Littlejohn [11] and later regularized by Correa-Restrepo and Wimmel [12]. We preferred to include this introductory section because, otherwise, repeated reference to the original analysis [8] would make for tedious reading. The equilibrium properties of the magnetically confined plasma under consideration are discussed in Sec. III. The second-order perturbation energy with vanishing initial field perturbations is obtained in Sec. IV. Part of the relevant lengthy calculation is presented in Appendix A. The conditions for the existence of negative-energy modes are derived in Sec. V. First the cases of a magnetized homogeneous plasma and an inhomogeneous force-free plasma with sheared magnetic field are examined; then the conditions for parallel, oblique, and perpendicular propagation of negative-energy modes in a magnetically confined plasma are obtained separately. The consequences of the condition for the existence of perpendicular negative-energy modes in tokamaklike equilibria, described by using a slightly shifted Maxwellian distribution function, are examined in Sec. VI A. For various analytic cold-ion equilibria of the drift kinetic equilibrium equation with non-negative η_e values, as well as with negative η_e values, for which the criterion concerning the η_e threshold value does not hold, the fraction of the electrons connected with negative-energy modes is also obtained. The same issues are addressed for stellaratorlike equilibria, derived on the basis of a Maxwellian distribution function, in Sec. VI B. The main results are summarized in Sec. VII.

II. REVIEW OF THE MAXWELL-DRIFT KINETIC THEORY

A. The energy-momentum tensor

The second-order energy of perturbations around an equilibrium state is given by

$$F^{(2)} = \int d^3x \, T_0^{(2)0} , \tag{2}$$

where $T_0^{(2)0}$ is the energy component of the second-order energy-momentum tensor $T_\rho^{(2)\mu}$. To derive the tensor $T_\rho^{(2)\mu}$ in the context of kinetic theories, Pfirsch and Morrison [8] used a modified Hamilton-Jacobi approach. The main steps of the derivation are as follows.

(1) Let $H_\nu(p_i, q_i, t)$ be the Hamiltonian for particles of species ν for the perturbed state in a phase space $q_1, \ldots, q_4, p_1, \ldots, p_4$, where $(q_1, q_2, q_3) = (x_1, x_2, x_3) = \mathbf{x}$ is the position in normal space and, correspondingly, $(p_1, p_2, p_3) = \mathbf{p}$; q_4, p_4 is an additional pair of canonical variables which is needed to describe guiding center motion. Let $H_\nu^{(0)}(P_i, Q_i)$ be the equilibrium Hamiltonian in the phase space $P_1, \ldots, P_4$, $Q_1, \ldots, Q_4$, and let $S_\nu(P_i, q_i, t)$ be a mixed-variable generating function for a canonical transformation between p_i, q_i and P_i, Q_i. The theory is more generally valid for a reference Hamiltonian $H_\nu^{(0)}(P_i, Q_i, t)$ which possesses an explicit time dependence [8]. The $\mathbf{x}, t$ dependence of H_ν is given via the dependence of H_ν on the electromagnetic potentials $\phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$ and, for the drift kinetic theory, also on the electric and magnetic fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ and

derivatives of them. The derivatives occur only when Dirac's constraint theory formalism is used for constructing an appropriate Hamiltonian because the starting Lagrangian, Eq. (15) in Sec. II B, is of the nonstandard type. But even with Dirac's formalism the variation of these quantities makes vanishing contributions to the Euler-Lagrange equations and the energy-momentum tensor [see remark after Eq. (33) in Sec. II B]. The general formalism is therefore equivalent to that for Hamiltonians not depending on the derivatives of **E** and **B**. The quantities p_i and Q_i are obtained from S_ν as

$$p_i=\frac{\partial S_\nu}{\partial q_i}\ ,\quad Q_i=\frac{\partial S_\nu}{\partial P_i}\ , \tag{3}$$

and S_ν must be the solution of the modified Hamilton-Jacobi equation

$$\frac{\partial S_\nu}{\partial t}+H_\nu\left[\frac{\partial S_\nu}{\partial q_i},q_i,t\right]=H_\nu^{(0)}\left[P_i,\frac{\partial S_\nu}{\partial P_i}\right]\ . \tag{4}$$

The time-independent, zeroth-order solution $S_\nu^{(0)}$ of Eq. (4), needed to obtain $T_\rho^{(2)\mu}$, is then simply given by the identity transformation $S_\nu^{(0)}=\sum_\nu P_i q_i$.

(2) With the notation defined on page 273 of Ref. [8], the Lagrangian for the whole theory (Maxwell-Vlasov and drift kinetic), irrespective of the special choice of $H_\nu^{(0)}$, is

$$L=-\sum_\nu\int d\tilde q\, d\tilde P\,\varphi_\nu\left[\mathcal{H}_\nu\left[\frac{\partial S_\nu}{\partial\tilde q_i},\tilde q_i\right]-\mathcal{H}_\nu^{(0)}\left[\tilde P_i,\frac{\partial S_\nu}{\partial\tilde P_i}\right]\right]-\frac{1}{16\pi}\int d^3x\,F_{\mu\lambda}F^{\mu\lambda}\ . \tag{5}$$

Using the Euler-Lagrange equations resulting from the variational principle

$$\delta\int_{t_1}^{t_2}L\,dt=0\ , \tag{6}$$

with φ_ν, S_ν, and A_μ the quantities to be varied, and Noether's theorem, one obtains the following expression for the energy-momentum tensor of nonlinear theory:

$$T_\rho^\lambda=\sum_\nu\int d\hat q\, d\tilde P\left[\frac{\partial S_\nu}{\partial x^\rho}-\frac{e_\nu}{c}A_\rho\right]\frac{\delta\mathcal{L}}{\delta(\partial S_\nu/\partial x^\lambda)}+2F_{\mu\rho}\frac{\partial\mathcal{L}}{\partial F_{\mu\lambda}}-\delta_\rho^\lambda\mathcal{L}\ , \tag{7}$$

where $\mathcal{L}$ is the Lagrangian density in **x** space corresponding to L.

(3) To obtain the linearized theory, one first considers perturbations of an equilibrium represented by

$$H_\nu^{(0)}(P_i,Q_i)\ ,\quad \varphi_\nu^{(0)}(P_i,q_i)\ ,$$
$$S_\nu^{(0)}(P_i,q_i)\ ,\quad A_\mu^{(0)}(\mathbf{x})\ ,$$

which still include all orders of the perturbations:

$$\delta\varphi_\nu(P_i,q_i,t)\ ,\quad \delta S_\nu(P_i,q_i,t)\ ,\quad \delta A_\mu(\mathbf{x},t)\ .$$

Expansion in these quantities leads to first-, second-, and higher-order expressions for the perturbed Hamiltonian $H_\nu(\partial S_\nu/\partial q_i,q_i,t)$, the equilibrium Hamiltonian $H_\nu^{(0)}(P_i,\partial S_\nu/\partial P_i)$, and the Lagrangian. The variations of the variational principle (6) can then be done in terms of the quantities $\delta\varphi_\nu$, δS_ν, and δA_μ. Variation of the first-order Lagrangian yields zero because the unperturbed quantities are solutions to the variational principle and thus variations around them vanish. The lowest-order expression of the Lagrangian that is relevant is therefore of second order. Replacement of the quantities $\delta\varphi_\nu(P_i,q_i,t)$, $\delta S_\nu(P_i,q_i,t)$, and $\delta A_\mu(\mathbf{x},t)$ in this expression by their first-order approximations $\varphi_\nu^{(1)}(P_i,q_i,t)$, $S_\nu^{(1)}(P_i,q_i,t)$, and $A_\mu^{(1)}(\mathbf{x},t)$ therefore yields the Lagrangian of linearized theory:

$$\mathcal{L}^{(2)}=-(1/16\pi)F_{\mu\lambda}^{(1)}F^{(1)\mu\lambda}-\sum_\nu d\hat q\, d\tilde P[\varphi_\nu^{(0)}(\mathcal{H}_\nu^{(2)}-\mathcal{H}_\nu^{(0)(2)})+\varphi_\nu^{(1)}(\mathcal{H}_\nu^{(1)}-\mathcal{H}_\nu^{(0)(1)})]\ , \tag{8}$$

where

$$\mathcal{H}_\nu^{(1)}=\left[\frac{\partial S_\nu^{(1)}}{\partial\tilde q_i}-\frac{e_\nu}{c}A_i^{(1)}\right]\frac{\partial\mathcal{H}_\nu^{(0)}}{\partial\tilde P_i}+F_{\mu\lambda}^{(1)}\frac{\partial\mathcal{H}_\nu^{(0)}}{\partial F_{\mu\lambda}^{(0)}}+F_{\mu\lambda,\gamma}^{(1)}\frac{\partial\mathcal{H}_\nu^{(0)}}{\partial F_{\mu\lambda,\gamma}^{(0)}}\ , \tag{9}$$

$$\begin{aligned}\mathcal{H}_\nu^{(2)}=\frac{1}{2}&\left[\frac{\partial S_\nu^{(1)}}{\partial\tilde q_i}-\frac{e_\nu}{c}A_i^{(1)}\right]\left[\frac{\partial S_\nu^{(1)}}{\partial\tilde q_\kappa}-\frac{e_\nu}{c}A_\kappa^{(1)}\right]\frac{\partial^2\mathcal{H}_\nu^{(0)}}{\partial\tilde P_i\partial\tilde P_\kappa}+\left[\frac{\partial S_\nu^{(1)}}{\partial\tilde q_i}-\frac{e_\nu}{c}A_i^{(1)}\right]F_{\mu\lambda}^{(1)}\frac{\partial^2\mathcal{H}_\nu^{(0)}}{\partial\tilde P_i\partial F_{\mu\lambda}^{(0)}}+\tfrac{1}{2}F_{\mu\lambda}^{(1)}F_{\sigma\rho}^{(1)}\frac{\partial^2\mathcal{H}_\nu^{(0)}}{\partial F_{\mu\lambda}^{(0)}\partial F_{\sigma\rho}^{(0)}}\\&+\left[\frac{\partial S_\nu^{(1)}}{\partial\tilde q_i}-\frac{e_\nu}{c}A_i^{(1)}\right]F_{\mu\lambda,\gamma}^{(1)}\frac{\partial^2\mathcal{H}_\nu^{(0)}}{\partial\tilde P_i\partial F_{\mu\lambda,\gamma}^{(0)}}+\tfrac{1}{2}F_{\mu\lambda,\gamma}^{(1)}F_{\sigma\rho,\tau}^{(1)}\frac{\partial^2\mathcal{H}_\nu^{(0)}}{\partial F_{\mu\lambda,\gamma}^{(0)}\partial F_{\sigma\rho,\tau}^{(0)}}+F_{\mu\lambda}^{(1)}F_{\sigma\rho,\gamma}^{(1)}\frac{\partial^2\mathcal{H}_\nu^{(0)}}{\partial F_{\mu\lambda}^{(0)}\partial F_{\sigma\rho,\gamma}^{(0)}}\ ,\end{aligned} \tag{10}$$

$$\mathcal{H}_\nu^{(0)(1)}=\frac{\partial S_\nu^{(1)}}{\partial\tilde P_i}\frac{\partial\mathcal{H}_\nu^{(0)}}{\partial\tilde q_i}\ , \tag{11}$$

and

$$\mathcal{H}_\nu^{(0)(2)}=\frac{1}{2}\frac{\partial S_\nu^{(1)}}{\partial\tilde P_i}\frac{\partial S_\nu^{(1)}}{\partial\tilde P_\kappa}\frac{\partial^2\mathcal{H}_\nu^{(0)}}{\partial\tilde q_i\partial\tilde q_\kappa}\ . \tag{12}$$

The tensor $T_\rho^{(2)\lambda}$ for the linearized theory is derived by replacing in Eq. (7) the quantities $\mathcal{L}$, S_ν, A_ρ, and $F_{\mu\rho}$ by the quantities $\mathcal{L}^{(2)}$, $S_\nu^{(1)}$, $A_\rho^{(1)}$, and $F_{\mu\rho}^{(1)}$:

$$T_\rho^{(2)\lambda} = -\sum_\nu \int d\hat{q}\, d\tilde{P} \left[\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_\rho} - \frac{e_\nu}{c} A_\rho^{(1)} \right] \left[f_\nu^{(0)} \left[\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_\kappa} - \frac{e_\nu}{c} A_\kappa^{(1)} \right] \frac{\partial^2 \mathcal{H}_\nu^{(0)}}{\partial \tilde{P}_\lambda \partial \tilde{P}_\kappa} + f_\nu^{(0)} F_{\tau\sigma}^{(1)} \frac{\partial^2 \mathcal{H}_\nu^{(0)}}{\partial \tilde{P}_\lambda \partial F_{\tau\sigma}^{(0)}} + \frac{\partial}{\partial \tilde{q}_i} \left[f_\nu^{(0)} \frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_i} \right] \frac{\partial \mathcal{H}_\nu^{(0)}}{\partial \tilde{P}_\lambda} \right]$$
$$-2F_{\mu\rho}^{(1)} \sum_\nu \int d\hat{q}\, d\tilde{P} \left[f_\nu^{(0)} \left[\frac{\partial S_\nu^{(1)}}{\partial \tilde{q}_\kappa} - \frac{e_\nu}{c} A_\kappa^{(1)} \right] \frac{\partial^2 \mathcal{H}_\nu^{(0)}}{\partial \tilde{P}_\kappa \partial F_{\mu\lambda}^{(0)}} + f_\nu^{(0)} F_{\sigma\tau}^{(1)} \frac{\partial^2 \mathcal{H}_\nu^{(0)}}{\partial F_{\mu\lambda}^{(0)} \partial F_{\sigma\tau}^{(0)}} \right] - \frac{1}{4\pi} F_{\mu\rho}^{(1)} F^{(1)\mu\lambda}$$
$$+\delta_\rho^\lambda \left[\sum_\nu \int d\hat{q}\, d\tilde{P} f_\nu^{(0)} (\mathcal{H}_\nu^{(2)} - \mathcal{H}_\nu^{(0)(2)}) + \frac{1}{16\pi} F_{\tau\sigma}^{(1)} F^{(1)\tau\sigma} \right], \tag{13}$$

with $f_\nu^{(0)} = \varphi_\nu^{(0)}$ the equilibrium distribution functions. In this expression the time derivatives $\partial S_\nu^{(1)}/\partial t$ are given by

$$\frac{\partial S_\nu^{(1)}}{\partial t} - e_\nu A_0^{(1)} = -[S_\nu^{(1)}, H_\nu^{(0)}] + \frac{e_\nu}{c} \mathbf{A}^{(1)} \cdot \frac{\partial H_\nu^{(0)}}{\partial \mathbf{P}} - F_{\mu\lambda}^{(1)} \frac{\partial H_\nu^{(0)}}{\partial F_{\mu\lambda}^{(0)}} , \tag{14}$$

where the mixed variable Poisson bracket has been defined as

$$[a,b] = \frac{\partial a}{\partial \tilde{q}_i} \frac{\partial b}{\partial \tilde{P}_i} - \frac{\partial a}{\partial \tilde{P}_i} \frac{\partial b}{\partial \tilde{q}_i} .$$

B. Hamiltonian for the guiding center motion

In this section the index for the particle species is suppressed. The Hamiltonian for the guiding center motion is obtained in Ref. [8] from the Lagrangian given by Littlejohn [11] in regularized form [12]. The Lagrangian is defined in terms of the variables

$$t, \mathbf{x} = (q_1, q_2, q_3) \text{ and } q_4 ,$$

where q_4 is an additional velocity variable needed to describe the motion. It is given by

$$L = (e/c)\mathbf{A}^* \cdot \dot{\mathbf{x}} - e\phi^* , \tag{15}$$

where

$$\mathbf{A}^* = \mathbf{A} + (mc/e)[v_0 g(z)\mathbf{b} + \mathbf{v}_E] , \tag{16}$$

$$e\phi^* = e\phi + \mu B + (m/2)(q_4^2 + v_E^2) , \tag{17}$$

$$\mathbf{v}_E = c(\mathbf{E} \times \mathbf{B})/B^2 , \tag{18}$$

$$\mathbf{b} = \mathbf{B}/B , \tag{19}$$

$$z = q_4/v_0 . \tag{20}$$

Here μ is the magnetic moment of the gyrating particle, and v_0 is a constant velocity. The function $g(z)$ has been introduced to regularize a singularity which occurs in the context of nonregularized theory when the guiding center velocity $v_\parallel = \mathbf{b} \cdot \dot{\mathbf{x}}$ approaches the critical value $v_c = [(eB)/(mc)]/(\mathbf{b} \cdot \nabla \times \mathbf{b})$. The nonregularized theory is obtained for $g(z) = z$, in which case $q_4 = v_\parallel$ holds [see Eq. (22) below]. Thus, $g(z)$ must have the property $g(z) \approx z, g' = dg/dz \approx 1$ for small $z(|z| \ll 1)$. For large z, however, $g(z)$ must stay finite, $g(\infty) = 1$, so that with $v_0 \gg v_{\text{thermal}}$ one has $v_0 g(\infty) \ll v_c$. A possible choice for $g(z)$ is $g(z) = \tanh z$.

Since L is linear in $\dot{\mathbf{x}}$ and does not contain $\dot{q}_4$, it is not of the standard type and therefore does not allow the standard way of obtaining a Hamiltonian from it. The corresponding equations of motion are

$$\mathbf{E}^* + \frac{1}{c} \mathbf{v} \times \mathbf{B}^* - \frac{m}{e} g' \dot{q}_4 \mathbf{b} = 0 \tag{21}$$

and

$$\mathbf{b} \cdot \dot{\mathbf{x}} = v_\parallel = q_4/g' , \tag{22}$$

where $\mathbf{v} = \dot{\mathbf{x}}$ and

$$\mathbf{E}^* = -\frac{1}{c} \frac{\partial \mathbf{A}^*}{\partial t} - \frac{\partial \phi^*}{\partial \mathbf{x}} , \quad \mathbf{B}^* = \nabla \times \mathbf{A}^* . \tag{23}$$

From these equations one obtains the guiding center velocity $\mathbf{v} = \mathbf{v}_g$ and the "velocity" $\dot{q}_4 = V_4$ as functions of $\mathbf{x}$, q_4, and t:

$$\mathbf{v} = \mathbf{v}_g = (q_4/g' B_\parallel^*)\mathbf{B}^* + (c/B_\parallel^*)\mathbf{E}^* \times \mathbf{b} , \tag{24}$$

$$\dot{q}_4 = V_4 = (e/mg')(1/B_\parallel^*)\mathbf{E}^* \cdot \mathbf{B}^* , \tag{25}$$

where $B_\parallel^* = \mathbf{B}^* \cdot \mathbf{b}$. The momenta canonically conjugated to $\mathbf{x}$ and q_4 follow from Eq. (15) as

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{e}{c} \mathbf{A}^* , \quad p_4 = \frac{\partial L}{\partial \dot{q}_4} = 0 . \tag{26}$$

Since these relations do not contain $\dot{\mathbf{x}}$ and $\dot{q}_4$, they are constraints between the momenta and the coordinates. A consequence therefore is that Hamilton's equations based on the usual Hamiltonian corresponding to the above nonstandard Lagrangian are not the equations of motion. To overcome this difficulty, Dirac's constrained theory [14] is applied. It starts with the usual or "primary" Hamiltonian

$$H_p = \dot{\mathbf{x}} \cdot \frac{\partial L}{\partial \dot{\mathbf{x}}} + \dot{q}_4 \frac{\partial L}{\partial \dot{q}_4} - L = e\phi^* . \tag{27}$$

Dirac's Hamiltonian is then given by

$$H=e\phi^*+\mathbf{v}_g\cdot[\mathbf{p}-(e/c)\mathbf{A}]+V_4p_4 , \tag{28}$$

from which the guiding center motion follows:

$$\dot{\mathbf{x}}=\frac{\partial H}{\partial \mathbf{p}}=v_g, \quad \dot{q}_4=\frac{\partial H}{\partial p_4}=V_4 . \tag{29}$$

But in general there are more solutions of the momentum equations $\dot{\mathbf{p}}=-\partial H/\partial \mathbf{x}$ and $\dot{p}_4=-\partial H/\partial q_4$ than those given by Eq. (26). These equations can be transformed to

$$\frac{d}{dt}\left[\mathbf{p}-\frac{e}{c}\mathbf{A}^*\right]=-\left[\frac{\partial}{\partial \mathbf{x}}\mathbf{v}_g\right]\cdot\left[\mathbf{p}-\frac{e}{c}\mathbf{A}^*\right]-\frac{\partial V_4}{\partial \mathbf{x}}p_4 , \tag{30}$$

$$\dot{p}_4=-\frac{\partial \mathbf{v}_g}{\partial q_4}\cdot\left[\mathbf{p}-\frac{e}{c}\mathbf{A}^*\right]-\frac{\partial V_4}{\partial q_4}p_4 . \tag{31}$$

Special solutions of Eqs. (30) and (31) are obviously constraints (26). It is, however, important to note that $\mathbf{p}-(e/c)\mathbf{A}^*=\mathbf{0}$ and $p_4=0$ do not represent special values of some constants of motion. Therefore δ functions of the constraints are not constants of motion either. The distribution function f must, however, guarantee that the constraints are satisfied. Hence it must be proportional to such δ functions, but must also be a constant of motion. Both conditions are satisfied by

$$f=\delta(p_4)\delta\left[\mathbf{p}-\frac{e}{c}\mathbf{A}^*\right]g'_v B^*_{\parallel} f_g(\mathbf{x},v_{\parallel},\mu,t) , \tag{32}$$

where the guiding center distribution function f_g is a solution of the drift kinetic differential equation

$$\frac{\partial f_g}{\partial t}+\mathbf{v}_g\cdot\frac{\partial f_g}{\partial \mathbf{x}}+V_4\frac{\partial f_g}{\partial q_4}=0 . \tag{33}$$

In f_g a dependence on the magnetic moment μ has been added which is a constant and which has therefore the character of a parameter distinguishing between different "kinds" of particles. Later [see transformation (74) in Sec. IV], one must sum over all these kinds of particles in order to obtain the total-energy-momentum tensor, i.e., one integrates over μ. Note that the form (32) of f has the consequence that in the Lagrangian (5), any variation of $\mathbf{v}_g$ and V_4 [see Eq. (28)] is multiplied by zero. Thus, although $\mathbf{v}_g$ and V_4 depend on the derivatives of $\mathbf{E}$ and $\mathbf{B}$, these dependences are unimportant for both the variational principle and the energy-momentum tensor.

Whereas Eq. (32) for f is sufficient in the nonlinear theory to pick out the correct solutions, this is not the case with the linearized theory. In this case, since constraints are imposed along the perturbed orbits, a displacement vector (ξ,ξ_4) in $\mathbf{x},q_4$ space, similar to the displacement vector in macroscopic theory, is introduced [8]. That is, since the zeroth-order distribution function always selects $\mathbf{V}=\mathbf{0}$ and $P_4=0$ with

$$\mathbf{V}\equiv(1/m)[\mathbf{P}-(e/c)\mathbf{A}^{*(0)}(\mathbf{x},q_4)] , \tag{34}$$

reasonable to expand $S^{(1)}$ in powers of $\mathbf{V}$ and P_4:

$$S^{(1)}=\hat{S}^{(1)}(\mathbf{x},q_4)-\xi\cdot m\mathbf{V}-\xi_4P_4 + \text{(higher-order terms)} , \tag{35}$$

so that

$$\left.\frac{\partial S^{(1)}}{\partial \mathbf{P}}\right|_{\mathbf{V}=P_4=0}=-\xi , \quad \left.\frac{\partial S^{(1)}}{\partial P_4}\right|_{\mathbf{V}=P_4=0}=-\xi_4 . \tag{36}$$

As is shown in the Appendix, for the equilibria considered in the present work the higher-order terms in expansion (35) after imposing the constraints do not contribute to $T_0^{(2)0}$. In general, since the highest-order q_i derivatives of $S_v^{(1)}$ appearing in $T_0^{(2)0}$ are eventually of second order, e.g., in Eqs. (71) and (72) of Sec. IV, terms up to second order in expansion (35) have nonvanishing contributions. This fact was overlooked in Ref. [8].

The constraints yield the following expressions for the displacement vector:

$$\xi_4=\frac{1}{mg'B^{*(0)}_{\parallel}}\mathbf{B}^{*(0)}\cdot\left[\frac{\partial\hat{S}^{(1)}}{\partial \mathbf{x}}-\frac{e}{c}\mathbf{A}^{*(1)}\right] , \tag{37}$$

$$\xi=\xi_{\perp}^*+\lambda(\mathbf{x},q_4)\mathbf{B}^{*(0)} , \tag{38}$$

with

$$\xi_{\perp}^*=\frac{c}{eB^{*(0)2}}\left[\mathbf{b}^{*(0)}\cdot\left[\frac{\partial\hat{S}^{(1)}}{\partial \mathbf{x}}-\frac{e}{c}\mathbf{A}^{*(1)}\right]\mathbf{B}^{*(0)}\times\mathbf{b}^{(0)} -\mathbf{B}^{*(0)}\times\left[\frac{\partial\hat{S}^{(1)}}{\partial \mathbf{x}}-\frac{e}{c}\mathbf{A}^{*(1)}\right]\right] , \tag{39}$$

$$\lambda=-\frac{1}{mg'B^{*(0)}_{\parallel}}\left[\frac{\partial\hat{S}^{(1)}}{\partial q_4}+mg'\mathbf{b}^{(0)}\cdot\xi_{\perp}^*\right] . \tag{40}$$

With these relations $T_\rho^{(2)\mu}$ is a functional of

$$\mathbf{A}^{(1)} , \quad \dot{\mathbf{A}}^{(1)} , \quad \phi^{(1)} , \quad \hat{S}^1(\mathbf{x},q_4,\mu) . \tag{41}$$

Except for $\phi^{(1)}$, which is constrained to

$$\nabla\cdot\mathbf{E}^{(1)}=4\pi\rho^{(1)} , \tag{42}$$

these quantities can be freely chosen in the sense of initial conditions. The μ dependence of $\hat{S}^{(1)}$ has been added for the reason given after Eq. (33).

The tensor $T_\rho^{(2)\mu}$ for the Maxwell-drift kinetic theory based on the Hamiltonian (28) can now be evaluated for each specific equilibrium, to which only terms up to first order in the expansion (35) contribute, and for any initial conditions.

III. EQUILIBRIUM

In this paper we investigate plasmas whose equilibrium quantities depend spatially on just y in a Cartesian coordinate system x,y,z, with unit basis vectors $\mathbf{e}_x,\mathbf{e}_y$, $\mathbf{e}_z$. It is assumed that there is no equilibrium electric field $\mathbf{E}^{(0)}$, and the equilibrium vector potential and magnetic field are given by

$$\mathbf{A}^{(0)}= A_x^{(0)}(y)\mathbf{e}_x+ A_z^{(0)}(y)\mathbf{e}_z , \tag{43}$$

$$\mathbf{B}^{(0)}=B_x^{(0)}(y)\mathbf{e}_x+B_z^{(0)}(y)\mathbf{e}_z , \tag{44}$$

with

$$(A_z^{(0)})' = B_x^{(0)}, (A_x^{(0)})' = -B_z^{(0)} . \quad (45)$$

Here the prime (′) denotes differentiation with respect to y. Macroscopically, the mean Lorentz force $\mathbf{j}^{(0)}\times\mathbf{B}^{(0)}$, which is in the y direction, balances the pressure gradient $\nabla P^{(0)}$. Equation (44) implies that the drift velocity has no y component, and therefore y is a constant of motion. Since there is also no force parallel to $\mathbf{B}^{(0)}$, another constant of motion is q_4. The guiding center distribution function is therefore a function of y, q_4, and the adiabatic invariant magnetic moment μ. To calculate the current density $\mathbf{j}^{(0)}$ from $f_{g\nu}^{(0)}$, we need the guiding center velocity $\mathbf{v}_{g\nu}^{(0)}$, Eq. (24). The following quantities are prerequisites:

$$\mathbf{b}^{(0)} = \frac{B_x^{(0)}}{B^{(0)}}\mathbf{e}_x + \frac{B_z^{(0)}}{B^{(0)}}\mathbf{e}_y = b_x^{(0)}\mathbf{e}_x + b_z^{(0)}\mathbf{e}_z , \quad (46)$$

$$\mathbf{A}_\nu^{*(0)} = \mathbf{A}^{(0)} + \frac{m_\nu c}{e_\nu} v_0 g \mathbf{b}^{(0)} , \quad (47)$$

$$e_\nu \phi_\nu^{*(0)} = \mu B^{(0)} + (m_\nu/2) q_4^2 , \quad (48)$$

$$v_E = 0 , \quad (49)$$

$$\mathbf{E}_\nu^{*(0)} = -\frac{\partial \phi_\nu^{*(0)}}{\partial \mathbf{x}} = -\frac{\mu}{e_\nu}(\mathbf{B}^{(0)})'\mathbf{e}_y , \quad (50)$$

$$\mathbf{B}_\nu^{*(0)} = \mathbf{B}^{(0)} + \frac{m_\nu c}{e_\nu} v_0 g \nabla\times\mathbf{b}^{(0)} , \quad (51)$$

and

$$B_{\nu\parallel}^{*} = \mathbf{B}_\nu^{*(0)}\cdot\mathbf{b}^{(0)} = B^{(0)} + \frac{m_\nu c}{e_\nu} v_0 g Y_{xz}(y) , \quad (52)$$

with

$$Y_{xz}(y) \equiv \mathbf{b}^{(0)}\cdot(\nabla\times\mathbf{b}^{(0)}) = b_x^{(0)}(b_z^{(0)})' - (b_x^{(0)})' b_z^{(0)} . \quad (53)$$

Moreover, it can be readily shown that

$$\mathbf{b}^{(0)}\cdot(\mathbf{b}^{(0)})' = 0 \quad (54)$$

and

$$\mathbf{B}_\nu^{*(0)} = B_{\nu\parallel}^{*(0)}\mathbf{b}^{(0)} . \quad (55)$$

The guiding center velocity then takes the form

$$\mathbf{v}_{g\nu}^{(0)} = \frac{q_4}{g_\nu'}\mathbf{b}^{(0)} - \frac{c\mu}{e_\nu B_\nu^{*(0)}}(B^{(0)})'(\mathbf{e}_y\times\mathbf{b}^{(0)}) , \quad (56)$$

and therefore it consists of a component parallel to $\mathbf{B}^{(0)}$ and a component perpendicular to $\mathbf{B}^{(0)}$ due to the grad-B drift. To calculate the current density $\mathbf{j}^{(0)}$, we apply the general formula (8.15) of Ref. [13], which was derived in the context of Maxwell-drift kinetic theory. The result is

$$\mathbf{j}^{(0)} = (c/4\pi)\nabla\times\mathbf{B}^{(0)} = \sum_\nu e_\nu \int dq_4 d\mu\, g_\nu' B_\nu^{*(0)} f_{g\nu}^{(0)} \mathbf{v}_{g\nu}^{(0)} - c\sum_\nu \nabla\times\int dq_4 d\mu \left\{ g_\nu' B_\nu^{*(0)} f_{g\nu}^{(0)} \left[\mu\mathbf{b}^{(0)} + \frac{m_\nu v_0 c\mu g_\nu}{e_\nu B^{(0)} B_\nu^{*(0)}}(B^{(0)})'(\mathbf{e}_y\times\mathbf{b}^{(0)}) \right] \right\} . \quad (57)$$

The components $j_x^{(0)}$ and $j_z^{(0)}$ read

$$j_x^{(0)} = (c/4\pi)(B_z^{(0)})' = \sum_\nu \int dq_4 d\mu \left\{ B_\nu^{*(0)} q_4 b_x^{(0)} f_{g\nu}^{(0)} - g_\nu' c\mu \left[(B^{(0)})' b_z^{(0)} f_{g\nu}^{(0)} + (B_\nu^{*(0)} b_z^{(0)} f_{g\nu}^{(0)})' - \frac{m_\nu c}{e_\nu} v_0 g_\nu \left[\frac{(B^{(0)})'}{B^{(0)}} b_x^{(0)} f_{g\nu}^{(0)} \right]' \right] \right\} , \quad (58)$$

and

$$j_z^{(0)} = -(c/4\pi)(B_x^{(0)})' = \sum_\nu \int dq_4 d\mu \left\{ B_\nu^{*(0)} q_4 b_z^{(0)} f_{g\nu}^{(0)} + g_\nu' c\mu \left[(B^{(0)})' b_x^{(0)} f_{g\nu}^{(0)} + (B_\nu^{*(0)} b_x^{(0)} f_{g\nu}^{(0)})' + \frac{m_\nu c}{e_\nu} v_0 g_\nu \left[\frac{(B^{(0)})'}{B^{(0)}} b_z^{(0)} f_{g\nu}^{(0)} \right]' \right] \right\} . \quad (59)$$

Multiplying Eqs. (58) and (59) by the integrating factors $B_z^{(0)}$ and $B_x^{(0)}$, respectively, subtracting the first from the second of the resulting equations, and doing some straightforward algebraic manipulations leads to the pressure balance relation

$$\frac{d}{dy}[P^{(0)} + (B^{(0)})^2/8\pi] = 0 , \quad (60)$$

with

$$P^{(0)} = \sum_\nu \int dq_4 d\mu\, g_\nu' \mu B^{(0)} B_\nu^{*(0)} f_{g\nu}^{(0)} . \quad (61)$$

Relation (60) can also be derived by the momentum-conservation equation $(\partial/\partial x^\mu)T_\rho^\mu = 0$ $[\mu,\rho = 1,2,3, x^\mu \to x,y,z]$ for $\rho = 2$, with the tensor T_ρ^μ given in explicit form by Eq. (76) of Ref. [15]. Evidently, only two of Eqs. (58), (59), and (60) are independent in the sense that by treating any two of them one can derive the third one.

For distribution functions symmetric with respect to

q_4, Eqs. (58) and (59) take the simpler forms

$$-j_x^{(0)}=-\frac{c}{4\pi}(B_z^{(0)})'$$
$$=\sum_\nu \int dq_4 d\mu\{[m_\nu c q_4 v_0 g_\nu Y_{zx} b_x^{(0)} + g'_\nu c\mu (B^{(0)})' b_z^{(0)}] f_{g\nu}^{(0)} + c\mu g'_\nu [B_z^{(0)} f_{g\nu}^{(0)}]'\} \qquad (62)$$

and

$$j_z^{(0)}=-\frac{c}{4\pi}(B_x^{(0)})'$$
$$=\sum_\nu \int dq_4 d\mu\{[m_\nu c q_4 v_0 g_\nu Y_{xz} b_z^{(0)} + g'_\nu c\mu (B^{(0)})' b_x^{(0)}] f_{g\nu}^{(0)} + c\mu g'_\nu [B_x^{(0)} f_{g\nu}^{(0)}]'\} . \qquad (63)$$

Equations (62) and (63) impose a constraint on the y dependence on $f_{g\nu}^{(0)}$, namely $f_{g\nu}^{(0)}$ and $\partial f_{g\nu}^{(0)}/\partial y$ must be invariant under the transformation $B_x^{(0)} \Longleftrightarrow B_z^{(0)}$, $(B_x^{(0)})' \Longleftrightarrow (B_z^{(0)})'$. This condition is fulfilled if $f_{g\nu}^{(0)}$ belongs to a specific class of functions, Maxwellians included, such that its potential dependence on the magnetic field involves the magnetic-field modulus $B^{(0)}$ only. The functions $f_{g\nu}^{(0)}$, however, remain free to depend on y either explicitly or implicitly through any other quantity not related to $\mathbf{B}^{(0)}$.

IV. SECOND-ORDER PERTURBATION ENERGY

The second-order perturbation energy [see Eqs. (2) and (13)] will be calculated in the case of equilibria defined in Sec. III for initial perturbations $\mathbf{A}^{(1)}=\dot{\mathbf{A}}^{(1)}=\mathbf{0}$. It is also shown *a posteriori* that one can choose initial perturbations without changing the particle contribution to the energy, so that the corresponding charge density $\rho^{(1)}$ vanishes. Therefore, choosing initial perturbations of this kind, we set from the outset

$$F_{\mu\lambda}^{(1)}\equiv 0 , \quad A_\rho^{(1)}\equiv 0 . \qquad (64)$$

Equation (13) then reduces to

$$T_0^{(2)0}=-\sum_\nu \int d\hat{q}\, d\tilde{P} \frac{\partial S_\nu^{(1)}}{\partial t}\frac{\partial}{\partial \tilde{q}_i}\left[f_\nu^{(0)} \frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_i}\right] + \sum_\nu \int d\hat{q}\, d\tilde{P} f_\nu^{(0)} (\mathcal{H}_\nu^{(2)} - \mathcal{H}_\nu^{(0)(2)}) , \qquad (65)$$

and Eq. (14) to

$$\frac{\partial S_\nu^{(1)}}{\partial t} = -[S_\nu^{(1)}, H_\nu^{(0)}] . \qquad (66)$$

The Dirac Hamiltonian Eq. (28), with the help of relation $V_4=0$ [following from Eq. (25)], takes the form

$$H_\nu^{(0)}=e_\nu \phi_\nu^{*(0)} + \mathbf{v}_{g\nu}^{(0)} \cdot \left[\mathbf{P} - \frac{e_\nu}{c}\mathbf{A}_\nu^{*(0)} \right] . \qquad (67)$$

Because of

$$\frac{\partial^2 \mathcal{H}_\nu^{(0)}}{\partial \tilde{P}_i \partial \tilde{P}_\kappa} = 0 , \qquad (68)$$

Eq. (10) yields

$$\mathcal{H}_\nu^{(2)}=0 . \qquad (69)$$

Integration by parts of the term which contains derivatives of $f_\nu^{(0)}$ in Eq. (65),

$$-\sum_\nu \int d\hat{q}\, d\tilde{P} \frac{\partial S_\nu^{(1)}}{\partial t}\frac{\partial}{\partial \tilde{q}_i}\left[f_\nu^{(0)} \frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_i}\right] = \sum_\nu \int d\hat{q}\, d\tilde{P} f_\nu^{(0)} \frac{\partial S_\nu^{(1)}}{\partial \tilde{P}_i}\frac{\partial}{\partial \tilde{q}_i}\frac{\partial S_\nu^{(1)}}{\partial t} , \qquad (70)$$

and use of Eqs. (66), (69), and (12) for $\mathcal{H}_\nu^{(0)(2)}$ leads to

$$T_0^{(2)(0)} = \sum_\nu \int d\hat{q}\, d\tilde{P} f_\nu^{(0)} \mathcal{A} , \qquad (71)$$

with

$$\mathcal{A} \equiv -\frac{\partial^2 S_\nu^{(1)}}{\partial q_i \partial P_\kappa}\frac{\partial H_\nu^{(0)}}{\partial q_\kappa}\frac{\partial S_\nu^{(1)}}{\partial P_i} - \frac{\partial^2 S_\nu^{(1)}}{\partial q_i \partial q_\kappa}\frac{\partial H_\nu^{(0)}}{\partial P_\kappa}\frac{\partial S_\nu^{(1)}}{\partial P_i} - \frac{\partial^2 H_\nu^{(0)}}{\partial q_i \partial P_\kappa}\frac{\partial S_\nu^{(1)}}{\partial q_\kappa}\frac{\partial S_\nu^{(1)}}{\partial P_i} + \frac{1}{2}\frac{\partial^2 H_\nu^{(0)}}{\partial q_i \partial q_\kappa}\frac{\partial S_\nu^{(1)}}{\partial P_\kappa}\frac{\partial S_\nu^{(1)}}{\partial P_i} . \qquad (72)$$

After a lengthy calculation, which is presented in the Appendix, one obtains

$$\mathcal{A}=\tfrac{1}{2}m_\nu g'_\nu \frac{d}{dq_4}\left[\frac{q_4}{g'_\nu}\right]\xi_4^2-\frac{1}{2}\frac{c}{e_\nu}\frac{q_4}{g'_\nu}B_\nu^{*(0)}Y_{xz}\xi_y^2-\tfrac{1}{2}\mu B_\nu^{*(0)}\left[\frac{(B^{(0)})'}{B_\nu^{*(0)}}\right]'\xi_y^2$$

$$+\frac{q_4}{g'_\nu}\left[-\frac{1}{m_\nu g'_\nu}\frac{\partial \hat{S}_\nu^{(1)}}{\partial q_4}\left[\mathbf{b}^{(0)}\cdot\frac{\partial}{\partial \mathbf{x}}\right]\left[\mathbf{b}^{(0)}\cdot\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]-\frac{c}{e_\nu B_\nu^{*(0)}}\left[\mathbf{b}^{(0)}\times\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]\cdot\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{b}^{(0)}\cdot\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]\right.$$

$$\left.+\frac{c}{eB_\nu^{*(0)}}\left[b_x^{(0)}\frac{\partial^2 \hat{S}_\nu^{(1)}}{\partial x\,\partial y}+b_z^{(0)}\frac{\partial^2 \hat{S}_\nu^{(1)}}{\partial y\,\partial z}\right]\left[\mathbf{b}^{(0)}\times\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]\cdot\mathbf{e}_y\right]$$

$$-\frac{c\mu}{e_\nu}\left\{\left[\frac{(B^{(0)})'}{B_\nu^{*(0)}}b_z^{(0)}\right]'\frac{\partial \hat{S}_\nu^{(1)}}{\partial x}-\left[\frac{(B^{(0)})'}{B_\nu^{*(0)}}b_x^{(0)}\right]'\frac{\partial \hat{S}_\nu^{(1)}}{\partial z}\right\}\xi_y$$

$$+\frac{c\mu}{e_\nu}\frac{(B^{(0)})'}{B_\nu^{*(0)}}\left\{(\boldsymbol{\xi}\cdot\mathbf{b}^{(0)})\left[\mathbf{b}^{(0)}\cdot\frac{\partial}{\partial \mathbf{x}}\right]\left[b_x^{(0)}\frac{\partial \hat{S}_\nu^{(1)}}{\partial z}-b_z^{(0)}\frac{\partial \hat{S}_\nu^{(1)}}{\partial x}\right]\right.$$

$$+\frac{c}{e_\nu B_\nu^{*(0)}}\left[\mathbf{b}^{(0)}\times\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]\cdot\left[b_z^{(0)}\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{e}_x\cdot\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]-b_x^{(0)}\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{e}_z\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\right]\right]$$

$$\left.+\left[b_z^{(0)}\frac{\partial^2 \hat{S}_\nu^{(1)}}{\partial x\,\partial y}-b_x^{(0)}\frac{\partial^2 \hat{S}_\nu^{(1)}}{\partial y\,\partial z}\right]\xi_y\right\}$$

$$+\frac{q_4}{g'_\nu}\left[\frac{\partial}{\partial q_4}\left[\mathbf{b}^{(0)}\cdot\frac{\partial \hat{S}_\nu^{(1)}}{\partial x}\right]-\left[\mathbf{b}^{(0)}\cdot\frac{\partial}{\partial \mathbf{x}}\right]\frac{\partial \hat{S}_\nu^{(1)}}{\partial q_4}\right]\xi_4$$

$$-\frac{c\mu}{e_\nu}\frac{(B^{(0)})'}{B_\nu^{*(0)}}\left\{\left[\frac{\partial}{\partial q_4}\left[\mathbf{e}_x\cdot\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]-\left[\mathbf{e}_x\cdot\frac{\partial}{\partial \mathbf{x}}\right]\frac{\partial \hat{S}_\nu^{(1)}}{\partial q_4}\right]b_z^{(0)}\xi_4-\left[\frac{\partial}{\partial q_4}\left[\mathbf{e}_z\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\right]-\left[\mathbf{e}_z\cdot\frac{\partial}{\partial \mathbf{x}}\right]\frac{\partial \hat{S}_\nu^{(1)}}{\partial q_4}\right]b_x^{(0)}\xi_4\right\}. \tag{73}$$

We note here that the last two terms in this expression, $(q_4/g'_\nu)[\ ]$ and $-(c\mu/e_\nu)[(B^{(0)})'/B_\nu^{*(0)}]\{\ \}$, vanish after the ansatz (76) is used for $\hat{S}_\nu^{(1)}$ below. By substituting the integral over the momentum space according to the rule (a proof is given in Ref. [13])

$$\int d\tilde{P}\, f_\nu^{(0)}(\cdots)\to\int d\mu\, g'_\nu B_{\nu\parallel}^{*(0)}f_{g\nu}^{(0)}(\cdots)\,, \tag{74}$$

and by means of Eq. (55) the second-order perturbation energy can be written in the form

$$F^{(2)}=\int d^3x\, T_0^{(2)0}=\int d^3x\, dq_4 d\mu\, g'_\nu B_\nu^{*(0)}f_{g\nu}^{(0)}\mathcal{A}\ . \tag{75}$$

Since the equilibrium is independent of x and z, an appropriate ansatz for the functions $\hat{S}_\nu^{(1)}$ is

$$\hat{S}_\nu^{(1)}=G_\nu^{(1)}(y,q_4,\mu)e^{i(\mathbf{k}_{xz}\cdot\mathbf{x})}\ . \tag{76}$$

The wave vector $\mathbf{k}_{xz}$ introduced here is defined by

$$\mathbf{k}_{xz}=\kappa_x\mathbf{e}_x+\kappa_z\mathbf{e}_z\ , \tag{77}$$

and therefore it lies in magnetic surfaces. By introducing real quantities by the rule

$$AB\to\tfrac{1}{2}\mathrm{Re}\,A^*B\ , \tag{78}$$

the second-order energy, after some algebra, can be put in the form

$$F^{(2)}=S\sum_{\nu}\int dq_4 d\mu\, dy\, f_{g\nu}^{(0)}\left\{\frac{B_\nu^{*(0)}}{m_\nu}k_\parallel^2\frac{\partial}{\partial q_4}\left[\frac{q_4}{g'_\nu}|G_\nu^{(1)}|^2\right]-q_4\frac{c}{e_\nu}\frac{\partial}{\partial y}(k_\parallel|G_\nu^{(1)}|^2)\kappa_\perp\right.$$
$$+\left[\frac{c}{e_\nu}\right]^2\mu g'_\nu k_\perp^2\frac{\partial}{\partial y}\left[\frac{(B^{(0)})'}{B_\nu^{*(0)}}|G_\nu^{(1)}|^2\right]+2\frac{(B^{(0)})'}{B_\nu^{*(0)}}\left[\frac{c}{e_\nu}\right]^2\mu g'_\nu k_\perp|G_\nu^{(1)}|^2\frac{dk_\perp}{dy}$$
$$\left.-\frac{c\mu}{e_\nu m_\nu}(B^{(0)})'k_\parallel k_\perp\frac{\partial}{\partial q_4}|G_\nu^{(1)}|^2\right\}$$
$$=S\sum_{\nu}\int dq_4 d\mu\, dy\left\{-\frac{B_\nu^{*(0)}}{m_\nu}k_\parallel^2\frac{q_4}{g'_\nu}|G_\nu^{(1)}|^2\frac{\partial f_{g\nu}^{(0)}}{\partial q_4}-\frac{k_\parallel^2}{m_\nu}\frac{q_4}{g'_\nu}|G_\nu^{(1)}|^2 f_{g\nu}^{(0)}\frac{\partial B_\nu^{*(0)}}{\partial q_4}\right.$$
$$+q_4\frac{c}{e_\nu}k_\parallel k_\perp|G_\nu^{(1)}|^2\frac{\partial f_{g\nu}^{(0)}}{\partial y}+q_4\frac{c}{e_\nu}k_\parallel\frac{dk_\perp}{dy}|G_\nu^{(1)}|^2 f_{g\nu}^{(0)}$$
$$-\left[\frac{c}{e_\nu}\right]^2\mu g'_\nu\frac{(B^{(0)})'}{B_\nu^{*(0)}}k_\perp^2|G_\nu^{(1)}|^2\frac{\partial f_{g\nu}^{(0)}}{\partial y}-2\left[\frac{c}{e_\nu}\right]^2\mu g'_\nu\frac{(B^{(0)})'}{B_\nu^{*(0)}}k_\perp\frac{dk_\perp}{dy}|G_\nu^{(1)}|^2 f_{g\nu}^{(0)}$$
$$\left.+\frac{c\mu}{e_\nu m_\nu}(B^{(0)})'k_\parallel k_\perp|G_\nu^{(1)}|^2\frac{\partial f_{g\nu}^{(0)}}{\partial q_4}+2\left[\frac{c}{e_\nu}\right]\mu g'_\nu\frac{(B^{(0)})'}{B_\nu^{*(0)}}k_\perp\frac{dk_\perp}{dy}|G_\nu^{(1)}|^2 f_{g\nu}^{(0)}\right\}, \tag{79}$$

where

$$k_\parallel=(\mathbf{k}_{xz}\cdot\mathbf{b}^{(0)})\ ,\quad k_\perp=(\mathbf{b}^{(0)}\times\mathbf{k}_{xz})\cdot\mathbf{e}_y\ , \tag{80}$$

and S is a normalization surface. It can readily be shown that

$$\frac{d}{dy}k_\perp=Y_{xz}k_\parallel\ , \tag{81}$$

on the basis of which the second term cancels the fourth term on the right-hand side of Eq. (79). $F^{(2)}$ can then be cast in the neat form

$$F^{(2)}=-S\sum_{\nu}\int dq_4 d\mu\, dy\left\{\frac{B_\nu^{*(0)}}{m_\nu}|G_\nu^{(1)}|^2(\mathbf{k}_{xz}\cdot\mathbf{v}_{g\nu}^{(0)})\left[k_\parallel\frac{\partial f_{g\nu}^{(0)}}{\partial q_4}-k_\perp\frac{g'_\nu}{\omega_\nu^{*(0)}}\frac{\partial f_{g\nu}^{(0)}}{\partial y}\right]\right\}, \tag{82}$$

with

$$\omega_\nu^{*(0)}\equiv\frac{e_\nu B_\nu^{*(0)}}{m_\nu c} \tag{83}$$

and $\mathbf{v}_{g\nu}^{(0)}$ as given by Eq. (56). We note that $F^{(2)}$ depends on $G_\nu^{(1)}$ only via $|G_\nu^{(1)}|^2$.

Since the first-order charge density $\rho^{(1)}$ is a q_4,μ integral over an expression that is linear in $S_\nu^{(1)}$ and therefore also in $G_\nu^{(1)}$, one can satisfy the relation $\rho^{(1)}=0$ (invoked at the beginning of this section) by a proper distribution of positive and negative values of $G_\nu^{(1)}$, on which $F^{(2)}$ does not depend.

V. CONDITIONS FOR THE EXISTENCE OF NEGATIVE-ENERGY MODES

The conditions for the existence of negative-energy modes are obtained if the chosen frame of reference is that of minimum energy. For the equilibria of a homogeneous magnetized plasma, this is the frame in which the center-of-mass velocity parallel to $\mathbf{B}^{(0)}$ vanishes. This simple case is first examined; then some more complicated equilibria are considered.

A. Homogeneous magnetized plasma

For $\mathbf{B}^{(0)}=$const the guiding center velocity $\mathbf{v}_{g\nu}^{(0)}$ is parallel to $\mathbf{B}^{(0)}$, and $f_{g\nu}^{(0)}$ is independent of y. Since the plasma is homogeneous, the perturbations can be chosen as

$$S_\nu^{(1)}=G_\nu^{(1)}(q_4,\mu)e^{\mathbf{k}\cdot\mathbf{x}}\ , \tag{84}$$

where $\mathbf{k}=k_x\mathbf{e}_x+k_y\mathbf{e}_y+k_z\mathbf{e}_z$. $F^{(2)}$ then takes the simpler form

$$F^{(2)}=-V\sum_{\nu}\int dq_4 d\mu\left[\frac{B^{(0)}}{m_\nu}|G_\nu^{(1)}|^2(\mathbf{b}^{(0)}\cdot\mathbf{k})^2\times\frac{q_4}{g'_\nu}\frac{\partial f_{g\nu}^{(0)}}{\partial q_4}\right], \tag{85}$$

V being a normalization volume. Thus $F^{(2)}<0$ if

$$\frac{q_4}{g'_\nu}\frac{\partial f_{g\nu}^{(0)}}{\partial q_4}>0 \tag{86}$$

holds for some q_4 (we recall that q_4/g'_ν is the velocity parallel to $\mathbf{B}^{(0)}$) and μ for any particle species ν. Condition (86), which was first derived by Pfirsch and Morrison [8], guarantees the existence of negative-energy modes without any restrictions on the magnitude or orientation of the wave vector other than $k_\parallel \neq 0$: it suffices to localize $G_\nu^{(1)}$ to the region in q_4, μ where $(q_4/g'_\nu)(\partial f_{g\nu}^{(0)}/\partial q_4)>0$. Outside this region $G_\nu^{(1)}$ vanishes. All the other $G_\lambda^{(1)}$, i.e., with $\lambda \neq \nu$, are set equal to zero. The sign of $F^{(2)}$ is then determined only by the sign of the integrand in the region of localization. For $f_{g\nu}^{(0)}$ symmetric with respect to q_4, condition (86) is satisfied if a minimum with respect to q_4 exists in $f_{g\nu}^{(0)}$.

B. Inhomogeneous force-free plasma with sheared magnetic field

The equilibrium magnetic field now has a constant twist as one proceeds along the y axis. It is given by

$$\mathbf{B}^{(0)}=B^{(0)}(\sin\alpha y\,\mathbf{e}_x+\cos\alpha y\,\mathbf{e}_z)\ , \tag{87}$$

with $B^{(0)}=$ const and α^{-1} the twist length. The electric current associated with this sheared magnetic field is

$$\mathbf{j}^{(0)}=-(c/4\pi)\alpha\mathbf{B}^{(0)}\ , \tag{88}$$

and therefore the mean Lorentz force vanishes. In order to guarantee a uniform plasma pressure, $f_{g\nu}^{(0)}$ (as in the case of a homogeneous magnetized plasma) must not depend on y. Since $B^{(0)}=$const, the perpendicular component of $\mathbf{v}_{g\nu}^{(0)}$ due to grad-B drift vanishes, and the second-order wave energy [Eq. (82)] reduces to

$$F^{(2)}=-S\sum_\nu \int dy\, dq_4 d\mu \left[\frac{B_\nu^{*(0)}}{m_\nu}|G_\nu^{(1)}|^2 k_\parallel^2 \times \frac{q_4}{g'_\nu}\frac{\partial f_{g\nu}^{(0)}}{\partial q_4}\right]. \tag{89}$$

This form again implies that if condition (86) is satisfied locally in q_4 and μ for any particle species ν, with the localization of $G_\nu^{(1)}(y,q_4,\mu)$ in q_4 and μ as in Sec. V A, negative-energy modes exist without restriction on the magnitude or orientation of $\mathbf{k}_{xz}$ (other than $\kappa_\parallel \neq 0$). This result agrees with that obtained by Correa-Restrepo and Pfirsch [10], condition (67), in the context of Maxwell-Vlasov theory.

C. Magnetically confined plasma

1. Parallel modes ($k_\perp=0$)

In this case Eq. (82) again reduces to Eq. (89), and therefore negative-energy modes exist if condition (86) holds for some y, q_4, and μ. Since $f_{g\nu}^{(0)}$ is now y dependent, the perturbations $G_\nu^{(1)}(y,q_4,\mu)$ are localized around the values of y, q_4, and μ at which $(q_4/g'_\nu)(\partial f_{g\nu}^{(0)}/\partial q_4)>0$. Outside this region $g_\nu^{(1)}$ vanishes. The functions $G_\lambda^{(1)}$ for the other particle species are set equal to zero.

2. Oblique modes ($k_\parallel \neq 0$ and $k_\perp \neq 0$)

With the definitions

$$\mathcal{C}=\mathbf{k}_{xz}\cdot\mathbf{v}_{g\nu}^{(0)} \quad\text{and}\quad \mathcal{D}=k_\parallel\frac{\partial f_{g\nu}^{(0)}}{\partial q_4}-k_\perp\frac{g'_\nu}{\omega_\nu^{*(0)}}\frac{\partial f_{g\nu}^{(0)}}{\partial y}\ , \tag{90}$$

Eq. (82) yields $F^{(2)}<0$ if

$$\mathcal{C}>0 \quad\text{and}\quad \mathcal{D}>0$$

or

$$\mathcal{C}<0 \quad\text{and}\quad \mathcal{D}<0\ . \tag{91}$$

The following cases are now considered separately.

(a) Let us first assume that

$$\frac{q_4}{g'_\nu}\frac{\partial f_{g\nu}^{(0)}}{\partial q_4}>0$$

again holds locally in y, q_4, and μ for any particles species ν. It then follows from inequalities (91), with the help of the equilibrium condition (60), that negative-energy modes exist, provided that

$$\frac{k_\parallel}{k_\perp}<\min(\Lambda_\nu,M_\nu) \quad\text{or}\quad \frac{k_\parallel}{k_\perp}>\max(\Lambda_\nu,M_\nu)\ , \tag{92}$$

with

$$\Lambda_\nu\equiv-\frac{4\pi g'_\nu\mu(P^{(0)})'}{m_\nu q_4 B^{(0)}\omega_\nu^{*(0)}}\ ,\quad M_\nu\equiv\frac{g'_\nu(\partial f_{g\nu}^{(0)}/\partial y)}{\omega_\nu^{*(0)}(\partial f_{g\nu}/\partial q_4)}\ . \tag{93}$$

The perturbations $G_\nu^{(1)}$ are localized as in the previous case of parallel propagation. The orders of magnitude of Λ_ν and M_ν depend on the particle energy. For particles with velocities of the order of the thermal velocities $(v_\nu)_{\text{th}}$ (thermal particles), these being the most representative particles, one can use the unregularized theory [$g(z)=z$, $g'_\nu=1$, $q_4=v_\parallel$] because $(v_\nu)_{\text{th}}$ is far lower than the critical velocity at which the singularity discussed in Sec. II B appears in this theory. With

$$R_\nu(y,v_\parallel)\equiv\frac{m_\nu c}{e_\nu B^{(0)}}v_\parallel Y_{xz}(y)=\frac{v_\parallel}{\omega_\nu^{(0)}}Y_{xz}(y)\ , \tag{94}$$

and with the help of Eq. (53) for Y_{xz}, one has

$$R_\nu(y,(v_\nu)_{\text{th}})\approx\frac{(v_\nu)_{\text{th}}}{\omega_\nu^{(0)}}\frac{1}{L}=\frac{(r_{L\nu})_{\text{th}}}{L}\ll 1\ . \tag{95}$$

From Eqs. (51), (52), and (55) it then follows that $B_\nu^{*(0)}=B^{(0)}[1+R_\nu(y,(v_\nu)_{\text{th}})]\approx B^{(0)}$, and from Eq. (83) that $\omega_\nu^{*(0)}\approx\omega_\nu^{(0)}$. Therefore,

$$\Lambda_\nu \approx -\frac{\mu B^{(0)}}{m_\nu (v_\nu)_{\rm th}} \frac{1}{\omega_\nu^{(0)}} \frac{\max\{(P^{(0)})'\}}{(B^{(0)})^2/8\pi}$$
$$\approx \frac{(v_\nu)_{\rm th}}{\omega_\nu^{(0)}} \frac{P^{(0)}(0)}{L} \frac{1}{(B^{(0)})^2/8\pi} \approx \frac{(r_{L\nu})_{\rm th}}{L}\beta_l(0) \ll 1 \tag{96}$$

and

$$M_\nu \approx \frac{1}{\omega_\nu^{(0)}} \frac{(v_\nu)_{\rm th}}{L} = \frac{(r_{L\nu})_{\rm th}}{L} \ll 1 \ . \tag{97}$$

Here $(r_{L\nu})_{\rm th}$ is the Larmor radius at a thermal velocity, L is the macroscopic scale length, and $P^{(0)}(0)$ and $\beta_l(0)$ are the pressure and local beta, respectively, at $y=0$. Consequently, condition (92) imposes no essential restriction on the magnitude or orientation of the $\mathbf{k}_{xz}$ connected with negative-energy modes.

(b) If

$$\frac{q_4}{g'_\nu} \frac{\partial f_{g\nu}^{(0)}}{\partial q_4} < 0 \ , \tag{98}$$

at some y, q_4, and μ for any ν, a condition which is more frequently satisfied, e.g., in the case of a Maxwellian distribution function, it follows from inequalities (91) that negative-energy modes exist if, in addition to (98),

$$\min(\Lambda_\nu, M_\nu) < \frac{k_\parallel}{k_\perp} < \max(\Lambda_\nu, M_\nu) \tag{99}$$

holds. For particles with thermal velocities, the latter condition, in conjunction with (96) and (97), implies that

$$\frac{k_\parallel}{k_\perp} \approx \frac{(r_{L\nu})_{\rm th}}{L} \ll 1 \ . \tag{100}$$

Therefore, the most important negative-energy perturbations, in the sense that the less restrictive condition (98) is involved, concern *nearly perpendicular modes.*

3. Perpendicular modes ($k_\parallel = 0$)

Using the equilibrium condition (60), Eq. (82) reduces to

$$F^{(2)} = 4\pi S \sum_\nu \int dy\, dq_4 d\mu \left[\frac{\mu g'_\nu}{B^{(0)} B_\nu^{*(0)}} (c/e_\nu)^2 |G_\nu^{(1)}|^2 k_\perp^2 \times \frac{dP^{(0)}}{dy} \frac{\partial f_{g\nu}^{(0)}}{\partial y} \right] . \tag{101}$$

The condition for the existence of negative-energy modes without any restriction on $k_\perp$, and irrespective of the sign of the quantity $(q_4/g'_\nu)(\partial f_{g\nu}^{(0)}/\partial q_4)$, is therefore that

$$\frac{dP^{(0)}}{dy} \frac{\partial f_{g\nu}^{(0)}}{\partial y} < 0 \tag{102}$$

is satisfied locally in y, q_4, and μ for any ν. We note that in the cases of a homogeneous magnetized plasma and an inhomogeneous force-free plasma with sheared magnetic field, in which gradients are not present, propagation of perpendicular negative-energy modes is not possible [for $k_\parallel = 0$, Eqs. (85) and (89) yield $F^{(2)} = 0$].

The consequences of condition (102) for tokamaklike and stellaratorlike equilibria are examined in Sec. VI.

VI. PERPENDICULAR NEGATIVE-ENERGY MODES IN EQUILIBRIA RELATED TO MAGNETIC CONFINEMENT SYSTEMS

A. Tokamaklike equilibria

To describe equilibria of this kind, we use a shifted Maxwellian distribution function. Since it is thermal particles that we are interested in, the unregularized theory is again employed, in the context of which the shifted Maxwellian distribution function reads [to simplify the notation, the superscript (0) is suppressed in the rest of this section on the understanding that all the quantities pertain to equilibrium]

$$f_{g\nu} = \left[\frac{m_\nu}{2\pi}\right]^{1/2} \frac{1}{1+R_\nu(y, V_\nu(y))} \frac{N_\nu(y)}{T_\nu^{3/2}(y)} \times \exp\left\{ -\frac{\mu B(y) + 1/2 m_\nu [q_4 - V_\nu(y)]^2}{T_\nu(y)} \right\} . \tag{103}$$

Here, $V_\nu(y)$ is a parallel shift velocity so small that

$$\frac{V_\nu}{(v_\nu)_{\rm th}} \approx \frac{(r_{L_\nu})_{\rm th}}{L} \ll 1 \ . \tag{104}$$

N_ν and T_ν, respectively, are the number density and temperature (in energy units) for particles of species ν, and $R_\nu(y, V_\nu(y)) = (1/\omega_\nu) V_\nu(y) Y_{xz}(y)$[$Y_{xz}$ as defined by Eq. (53)]. It will be shown later that V_ν produces a net "toroidal" current (the coordinates x and z correspond to the poloidal and toroidal directions, respectively). The distribution function has been normalized so that

$$\int_{-\infty}^{\infty} dq_4 \int_0^\infty d\mu\, B^*_{\nu\parallel} f_{g\nu} = N_\nu \ . \tag{105}$$

In addition, performing the integrations in Eq. (61) one obtains, as expected,

$$P = \sum_\nu \int dq_4 d\mu\, \mu B B^*_\nu f_{g\nu} = \sum_\nu N_\nu T_\nu \ . \tag{106}$$

Insertion of the distribution function (103) into condition (102) yields

$$\frac{dP}{dy}\frac{\partial f_{g\nu}}{\partial y} = P' \left[\frac{N'_\nu}{N_\nu} - \frac{3}{2}\frac{T'_\nu}{T_\nu} + \frac{\mu B}{T_\nu}\left[\frac{T'_\nu}{T_\nu} - \frac{B'}{B}\right] + \frac{m_\nu}{2}\frac{(q_4 - V_\nu)^2}{T_\nu}\frac{T'_\nu}{T_\nu} - \frac{R'_\nu}{1+R_\nu} + m_\nu \frac{q_4 - V_\nu}{T_\nu} V'_\nu \right] f_{g\nu} < 0 \ . \tag{107}$$

The terms $R'_\nu/(1+R_\nu)$ and $m_\nu[(q_4 - V_\nu)/T_\nu]V'_\nu$ in Eq. (107) can be neglected because

$$|R_\nu(y,V_\nu(y))| \ll |R_\nu(y,(v_\nu)_{\rm th})| \ll 1 \quad (108)$$

and

$$m_\nu \frac{(q_4-V_\nu)}{T_\nu} V'_\nu \approx \frac{1}{(v_\nu)_{\rm th}} V'_\nu \approx \frac{1}{(v_\nu)_{\rm th}} \frac{V_\nu}{L} = \frac{1}{L}\frac{(r_{L\nu})_{\rm th}}{L} \ll 1 \ . \quad (109)$$

Condition (107) can then be written in the form

$$\frac{dP}{dy}\frac{\partial f_{g\nu}}{\partial y} = P' \left[\frac{N'_\nu}{N_\nu}\right] Q_\nu f_{g\nu} < 0 \ . \quad (110)$$

Here, it holds that

$$P' = \sum_\mu N_\mu T_\mu \left[\frac{N'_\mu}{N_\mu}\right](1+\eta_\mu) \ , \quad (111)$$

with

$$\eta_\mu = \partial \ln T_\mu / \partial \ln N_\mu$$

and

$$Q_\nu \equiv 1 - \tfrac{3}{2}\eta_\nu + \frac{\mu B}{T_\nu}\left[\eta_\nu + \frac{4\pi}{B^2}\left[\frac{N'_\nu}{N_\nu}\right]^{-1} P'\right] + \tfrac{1}{2} m_\nu \frac{q_4^2}{T_\nu}\eta_\nu \ . \quad (112)$$

1. Singly peaked density and temperature profiles

It is now assumed that both the density and temperature profiles have only one maximum for all particle species ν, which is the most common case in tokamak equilibria, and therefore $\eta_\nu \geq 0$ for all ν. [Equilibria which exhibit singly peaked density and hollow temperature profiles, or vice versa ($\eta_\nu < 0$) will be examined later.] This implies that $P'(N'_\nu/N_\nu) > 0$ and consequently condition (110) is satisfied if $Q_\nu < 0$. Since the last two terms of Q_ν, which involve the perpendicular and parallel particle energies, are non-negative, by taking the limit $\mu \Longrightarrow 0$ and $q_4 \Longrightarrow 0$ the inequality $Q_\nu < 0$ is satisfied if

$$\eta_\nu > 2/3 \equiv \eta_\nu^{\rm sc} \quad (113)$$

holds for *some* particle species ν. $\eta_\nu^{\rm sc}$ means "subcritical" η_ν. The existence of perpendicular negative-energy modes for any wave number $k_\perp$ is therefore related to the threshold value $\frac{2}{3}$ of the quantity η_ν, a quantity which usually governs the onset of temperature-gradient-driven modes. The linear stability properties of these modes have been extensively investigated. To be specific, performing a kinetic stability analysis of the ion temperature-gradient-driven mode, Hahm and Tang [16] obtained a critical value for instability, $\eta_i^c \geq 1$. Hassam *et al.* [17] examined the same instability for short and long wavelengths in a wide range of collisionality. For collisionless modes of arbitrary wavelength, a domain which corresponds to that of the present analysis, they calculated $\eta_i^c = 2$. In addition, Guo and Romanelli [18] recently studied the linear η_i^c threshold in various domains of collisionality, wavelength, and shear. For singly peaked density profiles they also calculated a threshold value $\eta_i^c \geq 1$ (see Eqs. (21) and (23) of Ref. [18]). Accordingly, the value $\eta_\nu^{\rm sc} = \frac{2}{3}$ appears to be subcritical in the sense that it is lower than the linear threshold η_ν^c value, and therefore the possible existence of negative-energy modes below the instability threshold implies that self-sustained turbulence may be present in a linearly stable tokamak regime. This result agrees with numerical results on drift-wave turbulence obtained by Nordman, Pavlenko, and Weiland [6] within the framework of a nonlinear dissipationless fluid model. Specifically, in this paper self-sustained η_i-mode turbulence substantially below the linear stability threshold was demonstrated and the driving mechanism is attributed to the interaction between negative- and positive-energy modes. The subcritical value $\eta_i^{\rm sc}$ (depending, according to the authors, on the finite Larmor radius parameter $k^2 r_L^2$), however, is not uniquely specified. We also note that self-sustained drift-wave turbulence in a linearly stable plasma slab resembling the edge region of tokamaks was demonstrated numerically by Scott [3,4] in the context of a nonlinear collisional fluid model.

We now calculate the phase space occupied by the particles associated with negative-energy modes on the basis of analytic solutions. Henceforth, particles of this kind will be called active particles.

When inserting the distribution function (103) into the equilibrium equations (58) and (59) after carrying out the integrations with respect to q_4 and μ, one obtains

$$-j_x = -\frac{cB'_z}{4\pi} = c\frac{b_z}{B}P' - b_x \sum_\nu e_\nu N_\nu V_\nu - C \sum_\nu \left[N_\nu T_\nu V_\nu \frac{B'_z}{B^2}\frac{1}{\omega_\nu}\right]' \quad (114)$$

and

$$j_z = -\frac{cB'_x}{4\pi} = c\frac{b_x}{B}P' + b_z \sum_\nu e_\nu N_\nu V_\nu + c\sum_\nu \left[N_\nu T_\nu V_\nu \frac{B'_x}{B^2}\frac{1}{\omega_\nu}\right]' \ . \quad (115)$$

The last term in each of these two equations,

$$\left[cN_\nu T_\nu V_\nu \frac{B'_i}{B^2}\frac{1}{\omega_\nu}\right]' \approx \frac{cN_\nu T_\nu}{LB}\frac{V_\nu}{(v_\nu)_{\rm th}}\frac{(r_{L\nu})_{\rm th}}{L} \approx \frac{cN_\nu T_\nu}{LB}\left[\frac{(r_{L\nu})_{\rm th}}{L}\right]^2 , \quad (116)$$

with $i = x$ and z, is much smaller than the other terms, namely $c(b_z/B)P'$ and

$$b_z e_\nu N_\nu V_\nu \approx \frac{e_\nu B}{cm_\nu}\frac{cm_\nu}{B} N_\nu V_\nu \frac{T_\nu}{m_\nu (v_\nu)_{\rm th}^2} \approx \frac{cN_\nu T_\nu}{LB}\frac{V_\nu}{(v_\nu)_{\rm th}}\frac{L}{(r_{L\nu})_{\rm th}} \approx \frac{cN_\nu T_\nu}{LB} \ . \quad (117)$$

They can therefore be neglected. For simplicity, we now restrict discussion to $T_i=0$. For cold ions and a constant "toroidal" magnetic field $B_z=B_0$, Eqs. (114) and (115), respectively, yield

$$c\frac{b_z}{B}P'+eb_xN_eV_e=0 \tag{118}$$

and

$$c\frac{b_x}{B}P'-eb_zN_eV_e,=-\frac{cB_x'}{4\pi} \tag{119}$$

with $e_e=-e$ and

$$P=N_eT_e\ . \tag{120}$$

Let us briefly discuss here the meaning of V_e: For $V_e=0$ one obtains from Eq. (115) the "toroidal" current density

$$j_z=\frac{cb_x}{B}P'\ . \tag{121}$$

On the other hand, Eq. (118) for this case yields $P'=0$. Hence there is neither a pressure gradient nor a "toroidal" current.

For a y-dependent "toroidal" magnetic-field component, $B_z(y)$, and $V_e=0$, Eqs. (114) and (115) become

$$-\frac{B_z'}{4\pi}=\frac{b_z}{B}P'\ , \tag{122}$$

$$-\frac{B_x'}{4\pi}=\frac{b_x}{B}P'\ , \tag{123}$$

and their solutions satisfy the relation $B_z=cB_x$ with $c=\text{const}$. The magnetic field is therefore shearless and the only possible equilibrium which can be described by any of Eqs. (122) and (123) is a stellaratorlike configuration with vanishing "toroidal" current, a case which will be examined in Sec. VI A 2.

To obtain analytic tokamaklike equilibria with $V_e\neq0$, it is convenient to use, instead of Eqs. (118) and (119), Eq. (118) and the equilibrium condition

$$P+\frac{B^2}{8\pi}=\frac{B_\infty^2}{8\pi}\ ,\quad B_\infty=\text{const}\ . \tag{124}$$

Two of the quantities B, P, V_e, N_e, and T_e appearing in Eqs. (118), (120), and (124) can be arbitrary functions of y. Accordingly, assigning the y dependence of P and V_e, one can obtain from Eq. (124) the magnetic-field modulus B and, since the toroidal magnetic field B_z is given ($B_z=B_0=\text{const}$), the "poloidal" component B_x (strictly speaking, the absolute value of B_x). N_e can then be determined from Eq. (118) and, subsequently, T_e from Eq. (120).

Choosing the singly peaked pressure profile

$$P=\frac{B_s^2}{8\pi}\frac{1}{\cosh^2\rho}\ , \tag{125}$$

with $B_s=\text{const}$, $\rho=y/L$, L corresponding to the plasma radius, and

$$B_s^2+B_0^2=B_\infty^2\ , \tag{126}$$

one obtains a hollow B profile

$$B=(B_0^2+B_s^2\tanh^2\rho)^{1/2}\ , \tag{127}$$

an antisymmetric "poloidal" magnetic field

$$B_x=B_s\tanh\rho\ , \tag{128}$$

and a peaked "toroidal" current density

$$j_z=\frac{-cB_x'}{4\pi}=\frac{j_z(0)}{\cosh^2\rho}\ , \tag{129}$$

with

$$j_z(0)=-\frac{cB_s}{4\pi L}\ . \tag{130}$$

We note that Eq. (119) is then satisfied identically.

Phase space occupied by active particles. In the following we find the phase space occupied by the active particles for the three cases of Table I, which are characterized by three constant values of η_e. We note here that only four of the constants B_0, B_s, B_∞, $N_e(0)$, $T_e(0)$, $j_z(0)$, $V_e(0)$, and L appearing in the various expressions, e.g., B_0, $N_e(0)$, $T_e(0)$, and L, can be treated as free parameters. The others can be expressed in terms of the free parameters via relations (126) and (130);

$$N_e(0)V_e(0)=\frac{cB_s}{4\pi eL} \tag{131}$$

and

$$N_e(0)T_e(0)=\frac{1}{8\pi}B_s^2\ . \tag{132}$$

The last two relations follow, respectively, from Eqs. (119) and (125) evaluated at $\rho=0$.

(i) $\eta_e=1$

Condition (110) then yields

$$[1+\Gamma(\rho)]\frac{W_\perp}{T_e}+\frac{W_\parallel}{T_e}<\frac{1}{2}\ , \tag{133}$$

with $W_\perp=\mu B$, $W_\parallel=\frac{1}{2}m_ev_\parallel^2$ and

$$\Gamma(\rho)\equiv\frac{8\pi N_e(0)T_e(0)}{B^2}\frac{1}{\cosh^2\rho}\ . \tag{134}$$

The fraction of active particles is represented by the dotted area of Fig. 1. Invoking the spherical symmetry of the distribution function, and since the maximum value of $\Gamma(\rho)$, $\max\Gamma(\rho)=\Gamma(0)=\beta_l(0)$, is an order of magnitude

TABLE I. Equilibrium quantities for non-negative η_e. The shift velocity $V_e(y)$ is an assigned function of y. B is given by Eq. (126).

$V_e(y)$	$N_e(y)$	$T_e(y)$	η_e
$V_e(0)\dfrac{B_0}{B}\dfrac{1}{\cosh\rho}$	$\dfrac{N_e(0)}{\cosh\rho}$	$\dfrac{T_e(0)}{\cosh\rho}$	1
$V_e(0)\dfrac{B_0}{B}\dfrac{1}{\cosh^2\rho}$	const	$\dfrac{T_e(0)}{\cosh^2\rho}$	∞
$V_e(0)\dfrac{B_0}{B}$	$\dfrac{N_e(0)}{\cosh^2\rho}$	const	0

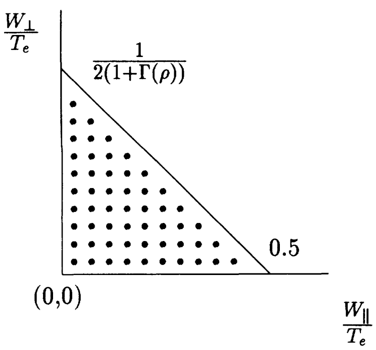

FIG. 1. The phase space occupied by the active electrons for $\eta_e = 1$.

lower than unity, relation (133) implies that nearly one-third of the thermal electrons are active. Thus, since the value $\eta_e = 1$ is close to the critical value for linear stability, negative-energy modes involving a considerable number of thermal electrons exist in a linearly marginally stable (or stable) regime.

(ii) $\eta_e \to \infty$

This case means a flat density profile. It is given by the second line of Table I. Since $N_e' = 0$, condition (107), instead of (110), is now evaluated and leads to

$$[2+\Gamma(\rho)]\frac{W_\perp}{T_e}+\frac{W_\parallel}{T_e}<3 \ . \tag{135}$$

The phase space occupied by the active electrons, following from inequality (135), is represented by the dotted area in Fig. 2. All thermal electrons are now active, as expected, because η_e approaches an extremely large value.

(iii) $\eta_e = 0$

This equilibrium exhibits a flat temperature and a peaked density profile (see the third line of Table I). In this case condition (110) yields

$$2+\frac{\mu B}{T_e}\Gamma(\rho)<0 \ , \tag{136}$$

and therefore no negative-energy modes exist, as again expected, because η_e takes its lowest non-negative value well below the subcritical one.

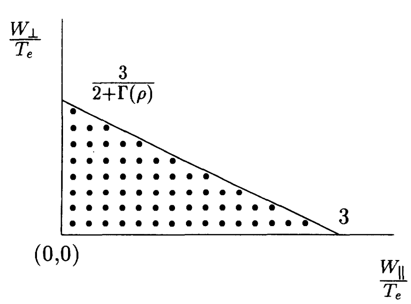

FIG. 2. The phase space occupied by the active electrons for $\eta_e \to \infty$. All thermal electrons are now active, as expected, because η_e approaches an extremely large value.

It follows from inequalities (133), (135), and (136) and from $\Gamma(\rho)$ being a decreasing function of ρ that for all the equilibria considered the dotted areas in Figs. 1 and 2, and therefore the fraction of active electrons, slightly increases as one proceeds from the center ($\rho = 0$) to the edge ($\rho = 1$). This indicates that self-sustained turbulence exists to a higher degree in the edge region.

2. *Hollow temperature or hollow density profile*

For equilibria with negative values of η_e criterion (113) does not obtain. Equilibria of this kind have been experimentally observed in *H*-mode confinement in tokamaks [19], as well as in discharges with electron cyclotron resonance heating in stellarators [20]. For this reason two equilibria with pressure, magnetic field, and current density profiles identical to those previously considered, Eqs. (125)–(130), but with negative values of η_e, are examined below.

a. Singly peaked density and hollow temperature profiles. This situation is realized by the first line of Table II.

Condition (110) yields

$$\frac{W_\parallel}{T_e}<\frac{3}{2}-\frac{1}{\eta_e}=\frac{3}{2}+\left[1+\frac{2B^2}{B_s^2}\cosh^2\rho\right] \ . \tag{137}$$

We note that the perpendicular particle energy $W_\perp$ does not appear in inequality (137), because the factor by which μB is multiplied in Eq. (112) vanishes. Inequality (137) imposes no restriction on the active thermal electrons.

b. Hollow density and singly peaked temperature profiles. This situation is realized by the second line of Table II. Condition (110) leads to

$$2[1+\Gamma(\rho)]\frac{W_\perp}{T_e}+[2+\Gamma(\rho)]\frac{W_\parallel}{T_e}<\Gamma(\rho)+\tfrac{3}{2}[2+\Gamma(\rho)] \ , \tag{138}$$

and the phase space of active electrons is depicted in Fig. 3. Nearly all thermal electrons are active.

For the considered equilibria with negative values of η_e the phase space of active electrons slightly increases as one proceeds from the center to the edge, as can be seen from inequalities (137) and (138). This is similar to the cases of equilibria with non-negative values of η_e.

B. Shearless stellaratorlike equilibria

The distinguishing feature of these equilibria in comparison with the tokamaklike type is that the net plasma current vanishes. To derive equilibria of this kind, an appropriate distribution function is a y-dependent Maxwellian

$$f_{g\nu}=\left[\frac{m_\nu}{2\pi}\right]^{1/2}\frac{N_\nu(y)}{T_\nu^{3/2}(y)}\exp\left[-\frac{\mu B(y)+\frac{1}{2}m_\nu q_4^2}{T_\nu(y)}\right] , \tag{139}$$

which is a special case of Eq. (103) for $V_\nu = R_\nu = 0$. Con-

TABLE II. Equilibrium quantities for negative η_e. B is given by Eq. (126), and $\Gamma(\rho)$ by Eq. (133).

$V_e(y)$	$N_e(y)$	$T_e(y)$	$\eta_e(y)$
const	$N_e(0)\dfrac{B_0}{B}\dfrac{1}{\cosh^2\rho}$	$T_e(0)\dfrac{B}{B_0}$	$-\left[\dfrac{\Gamma(\rho)}{2+\Gamma(\rho)}\right]$
$V_e(0)\left[\dfrac{B_0^2}{B^2}\dfrac{1}{\cosh^2\rho}\right]$	$N_e(0)\dfrac{B}{B_0}$	$T_e(0)\dfrac{B_0}{B}\dfrac{1}{\cosh^2\rho}$	$-\left[\dfrac{2+\Gamma(\rho)}{\Gamma(\rho)}\right]$

sequently, if one performs an analysis similar to that in Sec. VI A 2, first-order terms in $(r_{L\nu})_{\rm th}/L$ do not appear, and only those in Eqs. (105)–(124) which contain V_ν and R_ν must be modified by replacing $V_\nu=R_\nu=0$. Thus, the condition $(dP/dy)(\partial f_{g\nu}/\partial y)<0$ yields, through (110), the same subcritical value $\eta_\nu^{\rm sc}=\frac{2}{3}$.

To obtain the fraction of active particles in the case of cold ions, we first consider the equilibrium equation

$$-j_x=-\frac{cB_z'}{4\pi},=c\frac{b_z}{B}P' , \tag{140}$$

which contains the single "toroidal" magnetic-field component B_z. The equilibrium condition (124) is not an independent equation and therefore the pressure $P(y)$ can be an arbitrary function of y, as it is in the case of tokamaklike equilibria. If one chooses the same singly peaked pressure profile given by Eq. (125), the solution of Eq. (140) is

$$B_z=B=(B_0^2+B_s^2\tanh^2\rho)^{1/2} . \tag{141}$$

It should be noted that, to prevent B_z from vanishing at $y=0$, the constant magnetic field B_0 must not be zero; otherwise a singularity would appear because the Larmor radius would approach infinity at $y=0$ and the drift kinetic theory would become invalid. The "poloidal" current density corresponding to B_z,

$$j_x=-\frac{c}{4\pi L}\frac{1}{B}\frac{\tanh\rho}{\cosh^2\rho} , \tag{142}$$

is an odd function of y and therefore no net current flows through the plasma. The electron density profile $N_e(y)$ can be freely chosen; the temperature $T_e(y)$ can then be determined from the relation $P=N_eTe$. The same $N_e(y)$

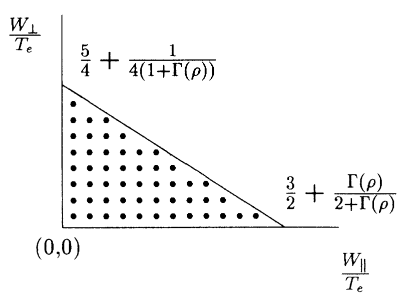

FIG. 3. The phase space of active electrons for hollow density and peaked temperature profiles (negative η_e).

and therefore the same η_e profiles examined for tokamaklike equilibria are adopted. It should be noted that the scale length L and three of the constants B_s, B_0, B_∞, $N_e(0)$, and $T_e(0)$, e.g., B_0, $N_e(0)$, and $T_e(0)$, can be used as free parameters. The other two can be expressed in terms of the free parameters via relations (126) and (132). The phase space occupied by active electrons can be obtained from relation (110) for $N_e'\neq 0$ and (107) for $N_e'=0$. In these relations the only macroscopic functions involved are P, N_e, T_e, B, and their derivatives. Thus, since these functions are identical in form to the corresponding functions considered in the case of tokamaklike equilibria, the results of Sec. VI A 2 that concern the phase space of active particles are also valid in the stellaratorlike regime. It therefore turns out that, as far as the existence of negative-energy modes is concerned, the two confinement systems are equivalent.

VII. CONCLUSIONS

The conditions for the existence of negative-energy modes with vanishing initial field perturbations were investigated for the cases of homogeneous magnetized, inhomogeneous force-free, and magnetically confined plasmas with plane equilibria. To this end, the second-order perturbation energy was obtained [Eq. (82)] by evaluating the general expression derived by Pfirsch and Morrison in the framework of collisionless Maxwell-drift kinetic theory. The conditions need only be satisfied for some particle species ν, locally in q_4 and μ for a homogeneous magnetized and an inhomogeneous force-free plasma, and locally in y, q_4, and μ for a magnetically confined plasma. They obtain if the reference frame is that of minimum energy. The conditions are as follows.

(i) Homogeneous magnetized plasma and inhomogeneous force-free plasma with sheared magnetic field.

If $(q_4/g_\nu')(\partial f_{g\nu}^{(0)}/\partial q_4)>0$, parallel and oblique modes ($k_\parallel\neq 0$) exist with no restriction on either the orientation or magnitude of the wave vector $\mathbf{k}$.

(ii) Magnetically confined plasma.

(1) For parallel and oblique modes the above condition is also valid with no essential restriction on $\mathbf{k}$.

(2) If $(q_4/g_\nu')(\partial f_{g\nu}^{(0)}/\partial q_4)<0$, a condition which is more frequently satisfied, the possible oblique negative-energy modes are nearly perpendicular.

(3) Purely perpendicular negative-energy modes ($k_\parallel=0$) also exist for *any* $k_\perp$ if $(dP/dy)(\partial f_{g\nu}^{(0)}/\partial y)<0$, irrespective of the sign of the quantity $(q_4/g_\nu')(\partial f_{g\nu}^{(0)}/\partial q_4)$ [$P(y)$ is the equilibrium plasma pressure].

The consequences of the last condition were examined for tokamaklike and stellaratorlike equilibria, described, respectively, on the basis of a slightly modified Maxwellian distribution and a Maxwellian distribution function. It turned out that the existence of perpendicular negative-energy modes is related to the threshold value $\frac{2}{3}$ for η_ν, which is lower than the critical value of η_ν for the onset of *linear* temperature-gradient-driven modes.

For various analytic tokamaklike and stellaratorlike cold-ion equilibria with non-negative as well as negative η_e for which the criterion $\eta_e > \frac{2}{3}$ is not necessary, a considerable fraction of thermal electrons is associated with negative-energy modes (active particles). In particular, for linearly (marginally) stable equilibria ($\eta_e = 1$) nearly one-third of the thermal electrons are active. For all equilibria considered the phase space occupied by active electrons increases as one proceeds from the center to the plasma edge. It is shown that the above results are exactly the same for stellaratorlike and tokamaklike equilibria if their density and temperature profiles are identical. It therefore turns out that negative-energy modes relating to nonlinear instabilities which could cause anomalous transport in a linearly stable regime exist equally well in both confinement systems.

APPENDIX: CALCULATION OF THE EXPRESSION $\mathcal{A}$ INVOLVED IN THE SECOND-ORDER PERTURBATION ENERGY

From Eq. (67), one obtains

$$\left.\frac{\partial H_\nu^{(0)}}{\partial q_4}\right|_{\mathbf{V}=0} = 0 \tag{A1}$$

and

$$\left.\frac{\partial H_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{V}=0} = \mathbf{0} \; . \tag{A2}$$

We note that the constraint $P_4 = 0$ is not involved here, because P_4 does not appear in $H_\nu^{(0)}$. If one recalls that $P_\lambda = (\mathbf{P}, P_4)$ and $q_\lambda = (\mathbf{x}, q_4)$, these relations imply that

$$\frac{\partial^2 S_\nu^{(1)}}{\partial q_i \partial P_\kappa} \frac{\partial H_\nu^{(0)}}{\partial q_\kappa} \frac{\partial S_\nu^{(1)}}{\partial P_i} = 0 \; , \tag{A3}$$

that is, the first term in $\mathcal{A}$, Eq. (72), vanishes. The other three terms are calculated separately as follows.

The term $\frac{\partial^2 S_\nu^{(1)}}{\partial q_i \partial q_\kappa} \frac{\partial H_\nu^{(0)}}{\partial P_\kappa} \frac{\partial S_\nu^{(1)}}{\partial P_i}$

It is convenient to write this term in the form

$$\frac{\partial^2 S_\nu^{(1)}}{\partial q_i \partial q_\kappa} \frac{\partial H_\nu^{(0)}}{\partial P_\kappa} \frac{\partial S_\nu^{(1)}}{\partial P_i} = \left[\left[\frac{\partial}{\partial \mathbf{x}} \frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}} \right] \cdot \frac{\partial H_\nu^{(0)}}{\partial \mathbf{P}} \right] \cdot \frac{\partial S_\nu^{(1)}}{\partial \mathbf{P}} + \left[\frac{\partial}{\partial q_4} \frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}} \right] \cdot \frac{\partial H_\nu^{(0)}}{\partial \mathbf{P}} \frac{\partial S_\nu^{(1)}}{\partial P_4}$$
$$+ \left[\frac{\partial}{\partial \mathbf{x}} \left[\frac{\partial S_\nu^{(1)}}{\partial q_4} \right] \frac{\partial H_\nu^{(0)}}{\partial P_4} \right] \cdot \frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}} + \frac{\partial^2 S_\nu^{(1)}}{\partial q_4^2} \frac{\partial H_\nu^{(0)}}{\partial P_4} \frac{\partial S_\nu^{(1)}}{\partial P_4} \; . \tag{A4}$$

By virtue of $\partial H_\nu^{(0)} / \partial P_4 = 0$, the last two terms on the right-hand side of Eq. (A4) vanish.

To calculate the first term, use of the relation $\partial H_\nu^{(0)} / \partial \mathbf{P} = \mathbf{v}_{g\nu}^{(0)}$ and Eq. (56) for $\mathbf{v}_{g\nu}^{(0)}$ yields

$$\left.\frac{\partial}{\partial \mathbf{x}} \left[\left.\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\right|_{\mathbf{P}} \right]\right|_{\mathbf{P}} \cdot \frac{\partial H_\nu^{(0)}}{\partial \mathbf{P}} = \frac{q_4}{g_\nu'} \left[\left.\frac{\partial}{\partial \mathbf{x}} \left[\mathbf{b}^{(0)} \cdot \left.\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\right|_{\mathbf{P}} \right]\right|_{\mathbf{P}} - \frac{\partial \mathbf{b}^{(0)}}{\partial \mathbf{x}} \cdot \left.\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\right|_{\mathbf{P}} \right]$$
$$- \frac{c\mu}{e} \frac{(B^{(0)})'}{B_\nu^{*(0)}} \left[b_z^{(0)} \left.\frac{\partial}{\partial \mathbf{x}} \left[\mathbf{e}_x \cdot \left.\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\right|_{\mathbf{P}} \right]\right|_{\mathbf{P}} - b_x^{(0)} \left.\frac{\partial}{\partial \mathbf{x}} \left[\mathbf{e}_z \cdot \left.\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\right|_{\mathbf{P}} \right]\right|_{\mathbf{P}} \right] \; . \tag{A5}$$

Equation (34) implies that

$$\left.\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\right|_{\mathbf{P}} = \left.\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\right|_{\mathbf{V}} - \left.\frac{\partial \mathbf{P}}{\partial \mathbf{x}}\right|_{\mathbf{V}} \cdot \left.\frac{\partial S_\nu^{(1)}}{\partial \mathbf{P}}\right|_{\mathbf{x}}$$
$$= \left.\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\right|_{\mathbf{V}} - \left.\frac{e_\nu}{c} \frac{\partial \mathbf{A}^*}{\partial \mathbf{x}} \cdot \frac{\partial S_\nu^{(1)}}{\partial \mathbf{P}}\right|_{\mathbf{x}} \; . \tag{A6}$$

Since $\mathbf{A}_\nu^{*(0)}$ depends only on y for any vector $\mathbf{r}_{xz}$ perpendicular to the y axis (such as the vectors $\mathbf{b}^{(0)}$, $\mathbf{e}_x$, and $\mathbf{e}_z$), the relation

$$\mathbf{r}_{xz} \cdot \left[\left.\frac{\partial \mathbf{A}_\nu^{*(0)}}{\partial \mathbf{x}} \cdot \frac{\partial S_\nu^{(1)}}{\partial \mathbf{P}}\right|_{\mathbf{x}} \right] = 0 \tag{A7}$$

holds and therefore

$$\mathbf{r}_{xz}\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\bigg|_{\mathbf{P}}=\mathbf{r}_{xz}\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\bigg|_{\mathbf{V}} . \tag{A8}$$

Applying the operator $\partial/\partial\mathbf{x}|_{\mathbf{P}}$ to the last equation yields

$$\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{r}_{xz}\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\bigg|_{\mathbf{P}}\right]\bigg|_{\mathbf{P}}=\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{r}_{xz}\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\bigg|_{\mathbf{V}}\right]\bigg|_{\mathbf{V}}-\frac{\partial \mathbf{P}}{\partial \mathbf{x}}\bigg|_{\mathbf{V}}\cdot\frac{\partial}{\partial \mathbf{P}}\left[\mathbf{r}_{xz}\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\bigg|_{\mathbf{V}}\right]\bigg|_{\mathbf{x}}$$

$$=\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{r}_{xz}\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\bigg|_{\mathbf{V}}\right]\bigg|_{\mathbf{V}}-\frac{e_\nu}{c}\frac{\partial \mathbf{A}_\nu^{*(0)}}{\partial \mathbf{x}}\cdot\left\{\left[\mathbf{r}_{xz}\cdot\frac{\partial}{\partial \mathbf{x}}\right]\left[\frac{\partial S_\nu^{(1)}}{\partial \mathbf{P}}\bigg|_{\mathbf{x}}\right]\bigg|_{\mathbf{V}}\right\} . \tag{A9}$$

Relation (A7) has the consequence that higher-order terms in expansion (35) for $S_\nu^{(1)}$, after imposing the constraint $\mathbf{V}=0$, do not contribute to Eq. (A9). Using this expansion, Eqs. (A6) and (A9), respectively, yield

$$\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\bigg|_{\mathbf{P}}\bigg|_{\mathbf{V}=0}=\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}+\frac{e_\nu}{c}\frac{\partial \mathbf{A}_\nu^{*(0)}}{\partial \mathbf{x}}\cdot\xi$$

$$=\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}-\frac{e_\nu}{c}\frac{\partial \hat{S}_\nu^{(1)}}{\partial y} \tag{A10}$$

and

$$\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{r}_{xz}\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\bigg|_{\mathbf{P}}\right]\bigg|_{\mathbf{P}}\bigg|_{\mathbf{V}=0}$$

$$=\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{r}_{xz}\cdot\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]+\frac{e_\nu}{c}\frac{\partial \mathbf{A}_\nu^{*(0)}}{\partial \mathbf{x}}\cdot\left[\left[\mathbf{r}_{xz}\cdot\frac{\partial}{\partial \mathbf{x}}\right]\xi\right] , \tag{A11}$$

where

$$\xi=-\frac{c}{e_\nu B_\nu^{*(0)}}\left[\mathbf{b}^{(0)}\times\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]-\frac{1}{m_\nu g_\nu'}\frac{\partial \hat{S}_\nu^{(1)}}{\partial q_4}\mathbf{b}^{(0)} . \tag{A12}$$

We note that, in order to calculate the right-hand side of Eq. (A10) and Eqs. (A16), (A17), and (A18) below, relations (55),

$$\frac{\partial \mathbf{A}}{\partial \mathbf{x}}\cdot\mathbf{b}^{(0)}=\mathbf{0} , \tag{A13}$$

and

$$\frac{\partial \mathbf{b}^{(0)}}{\partial \mathbf{x}}\cdot\mathbf{b}^{(0)}=\mathbf{0} \tag{A14}$$

are helpful. The second term on the right-hand side of Eq. (A5) can then be calculated on the basis of

$$\frac{\partial \mathbf{b}^{(0)}}{\partial \mathbf{x}}\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\bigg|_{\mathbf{P}}=\left[(b_x^{(0)})'\frac{\partial \hat{S}_\nu^{(1)}}{\partial x}+(b_z^{(0)})'\frac{\partial \hat{S}_\nu^{(1)}}{\partial z}\right]\mathbf{e}_y , \tag{A15}$$

following from (A10). The other three terms in Eq. (A5) can be calculated by applying relation (A11) to $\mathbf{r}_{xz}=\mathbf{b}^{(0)}$, $\mathbf{e}_x$ and $\mathbf{e}_y$:

$$\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{b}^{(0)}\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\bigg|_{\mathbf{P}}\right]\bigg|_{\mathbf{P}}$$

$$=\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{b}^{(0)}\cdot\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]$$

$$-\left[b_x^{(0)}\frac{\partial^2 \hat{S}_\nu^{(1)}}{\partial x\,\partial y}+b_z^{(0)}\frac{\partial^2 \hat{S}_\nu^{(1)}}{\partial y\,\partial z}\right]\mathbf{e}_y , \tag{A16}$$

$$\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{e}_x\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\bigg|_{\mathbf{P}}\right]\bigg|_{\mathbf{P}}=\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{e}_x\cdot\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]-\frac{\partial^2 \hat{S}_\nu^{(1)}}{\partial x\,\partial y}\mathbf{e}_y , \tag{A17}$$

$$\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{e}_z\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\bigg|_{\mathbf{P}}\right]\bigg|_{\mathbf{P}}=\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{e}_z\cdot\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]-\frac{\partial^2 \hat{S}_\nu^{(1)}}{\partial y\,\partial z}\mathbf{e}_y . \tag{A18}$$

Inserting Eqs. (A15) and (A16)–(A18) into Eq. (A5), and taking the inner product of the resulting equation with $\partial S_\nu^{(1)}/\partial\mathbf{P}=-\xi$, one obtains, for the first term in expression (A4),

$$\left[\left[\frac{\partial}{\partial \mathbf{x}}\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\right]\cdot\frac{\partial H_\nu^{(0)}}{\partial \mathbf{P}}\right]\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{P}}$$

$$=\frac{q_4}{g'_\nu}\left\{-(\boldsymbol{\xi}\cdot\mathbf{b}^{(0)})\left[\mathbf{b}^{(0)}\cdot\frac{\partial}{\partial \mathbf{x}}\right]\left[\mathbf{b}^{(0)}\cdot\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]\right.$$

$$+\frac{c}{e_\nu B_\nu^{*(0)}}\left[\mathbf{b}^{(0)}\times\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]\cdot\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{b}^{(0)}\cdot\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]+\left[b_x^{(0)}\frac{\partial^2 \hat{S}_\nu^{(1)}}{\partial x\,\partial y}+b_z^{(0)}\frac{\partial^2 \hat{S}_\nu^{(1)}}{\partial y\,\partial z}\right]\xi_y$$

$$\left.+\left[(b_x^{(0)})'\frac{\partial \hat{S}_\nu^{(1)}}{\partial x}+(b_z^{(0)})'\frac{\partial \hat{S}_\nu^{(1)}}{\partial z}\right]\xi_y\right\}$$

$$-\frac{c\mu}{e_\nu}\frac{(B^{(0)})'}{B_\nu^{*(0)}}\left\{(\boldsymbol{\xi}\cdot\mathbf{b}^{(0)})\left[\mathbf{b}^{(0)}\cdot\frac{\partial}{\partial \mathbf{x}}\right]\left[b_x^{(0)}\frac{\partial \hat{S}_\nu^{(1)}}{\partial z}-b_z^{(0)}\frac{\partial \hat{S}_\nu^{(1)}}{\partial x}\right]+\frac{c}{e_\nu B_\nu^{*(0)}}b_z^{(0)}\left[\mathbf{b}^{(0)}\times\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]\cdot\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{e}_x\cdot\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]\right.$$

$$\left.-\frac{c}{e_\nu B_\nu^{*(0)}}b_x^{(0)}\left[\mathbf{b}^{(0)}\times\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]\cdot\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{e}_z\cdot\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]+\left[b_z^{(0)}\frac{\partial^2 \hat{S}_\nu^{(1)}}{\partial x\,\partial y}-b_x^{(0)}\frac{\partial^2 \hat{S}_\nu^{(1)}}{\partial y\,\partial z}\right]\xi_y\right\}, \tag{A19}$$

with $\xi_y=(\mathbf{e}_y\cdot\mathbf{x})$.

A similar but simpler procedure is used, because q_4 is a scalar variable and $\mathbf{b}^{(0)}$ does not depend on q_4, to calculate the second term of Eq. (A4):

$$\left[\frac{\partial}{\partial q_4}\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\bigg|_{\mathbf{P}}\right]\cdot\frac{\partial H_\nu^{(0)}}{\partial \mathbf{P}}\frac{\partial S_\nu^{(1)}}{\partial P_4}=\frac{\partial}{\partial q_4}\left[\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\bigg|_{\mathbf{P}}\cdot\frac{\partial H_\nu^{(0)}}{\partial \mathbf{P}}\bigg|_{\mathbf{P}}\right]\frac{\partial S_\nu^{(1)}}{\partial P_4}$$

$$=-\frac{q_4}{g'_\nu}\xi_4\frac{\partial}{\partial q_4}\left[\mathbf{b}^{(0)}\cdot\frac{\partial \hat{S}_\nu^{(1)}}{\partial x}\right]+\frac{q_4}{g'_\nu}\xi_4\left[\mathbf{b}^{(0)}\cdot\frac{\partial}{\partial \mathbf{x}}\right]\frac{\partial \hat{S}_\nu^{(1)}}{\partial q_4}$$

$$-\frac{c\mu}{e_\nu}\frac{(B^{(0)})'}{B_\nu^{*(0)}}\left\{b_z^{(0)}\left[-\xi_4\frac{\partial}{\partial q_4}\left[\mathbf{e}_x\cdot\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\right]+\xi_4\left[\mathbf{e}_x\cdot\frac{\partial}{\partial \mathbf{x}}\right]\frac{\partial \hat{S}_\nu^{(1)}}{\partial q_4}\right]\right.$$

$$\left.-b_x^{(0)}\left[-\xi_4\frac{\partial}{\partial q_4}\left[\mathbf{e}_z\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\right]+\xi_4\left[\mathbf{e}_z\cdot\frac{\partial}{\partial \mathbf{x}}\right]\frac{\partial \hat{S}_\nu^{(1)}}{\partial q_4}\right]\right\}, \tag{A20}$$

with

$$\xi_4=\frac{1}{m_\nu g'_\nu}\mathbf{b}^{(0)}\cdot\frac{\partial \hat{S}_\nu^{(1)}}{\partial \mathbf{x}}\ . \tag{A21}$$

The expression (A4) is then the sum of Eqs. (A19) and (A20).

The term $\dfrac{\partial^2 H_\nu^{(0)}}{\partial q_i \partial P_\kappa}\dfrac{\partial S_\nu^{(1)}}{\partial q_\kappa}\dfrac{\partial S_\nu^{(1)}}{\partial P_i}$

Since $\partial H_\nu^{(0)}/\partial P_4=0$, this term can be written in the form

$$\frac{\partial^2 H_\nu^{(0)}}{\partial q_i\partial P_\kappa}\frac{\partial S_\nu^{(1)}}{\partial q_\kappa}\frac{\partial S_\nu^{(1)}}{\partial P_i}=\frac{\partial}{\partial q_4}\left[\frac{\partial H_\nu^{(0)}}{\partial \mathbf{P}}\right]\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\frac{\partial S_\nu^{(1)}}{\partial P_4}+\left[\left[\frac{\partial}{\partial \mathbf{x}}\frac{\partial H_\nu^{(0)}}{\partial \mathbf{P}}\right]\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{x}}\right]\cdot\frac{\partial S_\nu^{(1)}}{\partial \mathbf{P}}\ . \tag{A22}$$

This expression requires calculation of

$$\frac{\partial}{\partial q_4}\left[\frac{\partial H_\nu^{(0)}}{\partial \mathbf{P}}\right]=\frac{\partial}{\partial q_4}\mathbf{v}_{g\nu}^{(0)}=\frac{d}{dq_4}\left[\frac{q_4}{g'_\nu}\right]\mathbf{b}^{(0)}-\frac{c\mu}{e_\nu}\mathbf{b}^{(0)}(B^{(0)})'\frac{\partial}{\partial q_4}\left[\frac{1}{B_\nu^{*(0)}}\right](\mathbf{e}_y\times\mathbf{b}^{(0)})$$

$$=\frac{d}{dq_4}\left[\frac{q_4}{g'_\nu}\right]\mathbf{b}^{(0)}+m_\nu\left[\frac{c}{e_\nu}\right]^2\mu g'_\nu Y_{xz}\frac{(B^{(0)})'}{(B_\nu^{*(0)})^2}(b_z^{(0)}\mathbf{e}_x-b_x^{(0)}\mathbf{e}_z) \tag{A23}$$

and, with Eq. (A10) for $\partial S_\nu^{(1)}/\partial\mathbf{x}|_{\mathbf{P}}$, of

$$\left[\frac{\partial}{\partial\mathbf{x}}\frac{\partial H_\nu^{(0)}}{\partial\mathbf{P}}\right]\cdot\frac{\partial S_\nu^{(1)}}{\partial\mathbf{x}}\Bigg|_{\mathbf{P}}\Bigg|_{\mathbf{V}=0}=\frac{q_4}{g'_\nu}\frac{\partial\mathbf{b}^{(0)}}{\partial\mathbf{x}}\cdot\frac{\partial S_\nu^{(1)}}{\partial\mathbf{x}}\Bigg|_{\mathbf{P}}\Bigg|_{\mathbf{V}=0}-\mu\frac{c}{e_\nu}\frac{\partial}{\partial\mathbf{x}}\left[\frac{(B^{(0)})'}{B_\nu^{*(0)}}(\mathbf{e}_y\times\mathbf{b}^{(0)})\right]\cdot\frac{\partial S_\nu^{(1)}}{\partial\mathbf{x}}\Bigg|_{\mathbf{P}}\Bigg|_{\mathbf{V}=0}$$

$$=\frac{q_4}{g'_\nu}\left[(b_x^{(0)})'\frac{\partial\hat{S}_\nu^{(1)}}{\partial x}+(b_z^{(0)})'\frac{\partial\hat{S}_\nu^{(1)}}{\partial z}\right]\mathbf{e}_y$$

$$-\mu\frac{c}{e_\nu}\left\{\left[\frac{(B^{(0)})'}{B_\nu^{*(0)}}b_z^{(0)}\right]'\frac{\partial\hat{S}_\nu^{(1)}}{\partial x}-\left[\frac{(B^{(0)})'}{B_\nu^{*(0)}}b_x^{(0)}\right]'\frac{\partial\hat{S}_\nu^{(1)}}{\partial z}\right\}\mathbf{e}_y\ . \tag{A24}$$

On the basis of Eqs. (A23) and (A24), one then obtains

$$\frac{\partial^2 H_\nu^{(0)}}{\partial q_i\partial P_\kappa}\frac{\partial S_\nu^{(1)}}{\partial q_\kappa}\frac{\partial S_\nu^{(1)}}{\partial P_i}=-\frac{q_4}{g'_\nu}\left[(b_x^{(0)})'\frac{\partial\hat{S}_\nu^{(1)}}{\partial x}+b_z^{(0)}\frac{\partial\hat{S}_\nu^{(1)}}{\partial z}\right]\xi_y+\frac{c\mu}{e_\nu}\left\{\left[\frac{(B^{(0)})'}{B_\nu^{*(0)}}b_z^{(0)}\right]'\frac{\partial\hat{S}_\nu^{(1)}}{\partial x}-\left[\frac{(B^{(0)})'}{B_\nu^{*(0)}}b_x^{(0)}\right]'\frac{\partial\hat{S}_\nu^{(1)}}{\partial z}\right\}\xi_y$$

$$-\frac{d}{dy}\left[\frac{q_4}{g'_\nu}\right]\left[b_x^{(0)}\frac{\partial\hat{S}_\nu^{(1)}}{\partial x}+b_z^{(0)}\frac{\partial\hat{S}_\nu^{(1)}}{\partial z}\right]\xi_4$$

$$-m_\nu\left[\frac{c}{e_\nu}\right]^2\mu g'_\nu Y_{xz}\frac{(B^{(0)})'}{(B_\nu^{*(0)})^2}\left[b_z^{(0)}\frac{\partial\hat{S}_\nu^{(1)}}{\partial x}-b_x^{(0)}\frac{\partial\hat{S}_\nu^{(1)}}{\partial z}\right]\xi_4\ . \tag{A25}$$

The term $\dfrac{\partial^2 H_\nu^{(0)}}{\partial q_i\partial q_\kappa}\dfrac{\partial S_\nu^{(1)}}{\partial P_\kappa}\dfrac{\partial S_\nu^{(1)}}{\partial P_i}$

After some algebra one calculates

$$\frac{\partial^2 H_\nu^{(0)}}{\partial q_4^2}\Bigg|_{\mathbf{V}=0}=-m_\nu g'_\nu\frac{d}{dq_4}\left[\frac{q_4}{g'_\nu}\right], \tag{A26}$$

$$\frac{\partial^2 H_\nu^{(0)}}{\partial\mathbf{x}\partial q_4}\Bigg|_{\mathbf{V}=0}=-\frac{e_\nu}{c}\frac{\partial\mathbf{v}_{g\nu}^{(0)}}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{A}_\nu^{*(0)}}{\partial q_4}=m_\nu\frac{c}{e_\nu}g'_\nu\mu\frac{(B^{(0)})'}{B_\nu^{*(0)}}Y_{xz}\mathbf{e}_y\ , \tag{A27}$$

and

$$\left[\frac{\partial}{\partial\mathbf{x}}\left[\frac{\partial H_\nu^{(0)}}{\partial\mathbf{x}}\right]\right]\cdot\frac{\partial S_\nu^{(1)}}{\partial\mathbf{P}}\Bigg|_{\mathbf{V}=0}=-\xi_y\frac{\partial}{\partial\mathbf{x}}\left[\frac{\partial H_\nu^{(0)}}{\partial y}\right]\Bigg|_{\mathbf{V}=0}$$

$$=B_\nu^{*(0)}\left\{\frac{e_\nu}{c}\frac{q_4}{g'_\nu}Y_{xz}+\mu\left[\frac{(B^{(0)})'}{B_\nu^{*(0)}}\right]'\right\}\xi_y\mathbf{e}_y\ . \tag{A28}$$

Inserting Eqs. (A26)–(A28) into the expression

$$\frac{\partial^2 H_\nu^{(0)}}{\partial q_i\partial q_\kappa}\frac{\partial S_\nu^{(1)}}{\partial P_\kappa}\frac{\partial S_\nu^{(1)}}{\partial P_i}=\frac{\partial^2 H_\nu^{(0)}}{\partial q_4^2}\left[\frac{\partial S_\nu^{(1)}}{\partial P_4}\right]^2+2\left[\frac{\partial}{\partial q_4}\frac{\partial H_\nu^{(0)}}{\partial\mathbf{x}}\right]\cdot\frac{\partial S_\nu^{(1)}}{\partial\mathbf{P}}\frac{\partial S_\nu^{(1)}}{\partial P_4}+\left[\left[\frac{\partial}{\partial\mathbf{x}}\frac{\partial H_\nu^{(0)}}{\partial\mathbf{x}}\right]\cdot\frac{\partial S_\nu^{(1)}}{\partial\mathbf{P}}\right]\cdot\frac{\partial S_\nu^{(1)}}{\partial\mathbf{P}} \tag{A29}$$

yields

$$\frac{\partial^2 H_\nu^{(0)}}{\partial q_i\partial q_\kappa}\frac{\partial S_\nu^{(1)}}{\partial P_\kappa}\frac{\partial S_\nu^{(1)}}{\partial P_i}=-m_\nu g'_\nu\frac{d}{dq_4}\left[\frac{q_4}{g'_\nu}\right]\xi_4^2+2m_\nu\frac{c}{e_\nu}\mu g'_\nu Y_{xz}\frac{(B^{(0)})'}{B_\nu^{*(0)}}\xi_y\xi_4$$

$$-\frac{e_\nu}{c}\frac{q_4}{g'_\nu}B_\nu^{*(0)}Y_{xz}\xi_y^2-\mu B_\nu^{*(0)}\left[\frac{(B^{(0)})'}{B_\nu^{*(0)}}\right]'\xi_y^2\ . \tag{A30}$$

On the basis of Eqs. (A3), (A4), (A19), (A20), (A25), and (A30), $\mathcal{A}$ is written in the form given by Eq. (73).

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