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Large-scale coherence and "anomalous scaling" of high-order moments of velocity differences in strong turbulence

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The Hamiltonian formulation of hydrodynamics in Clebsch variables is used for construction of a statistical theory of turbulence. It is shown that the interaction of the random and large-scale coherent components of the Clebsch fields is responsible for generation of two energy spectra $E(k)\propto k^{-7/3}$ and $E(k)\propto k^{-2}$ at scales somewhat larger than those corresponding to the $-\frac{5}{3}$ inertial range. This interaction is also responsible for the experimentally observed Gaussian statistics of the velocity differences at large scales, and the nontrivial scaling behavior of their high-order moments for inertial-range values of the displacement r. The "anomalous scaling exponents" are derived and compared with experimental data.

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I. INTRODUCTION

A widely accepted formulation of turbulence theory deals with a flow governed by the Navier-Stokes equations (the density $\rho=1$):

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}=-\nabla p+\mathbf{F}+\nu_0\Delta\mathbf{v}\ ,$$

$$\nabla\cdot\mathbf{v}=0\ ,$$

where $\mathbf{F}$ is a forcing function having Fourier transform $F(k)\neq 0$ only in the interval $k<k_0\to 0$. This means that kinetic energy is pumped into the system at large scales only. At large enough Reynolds number Re the flow becomes turbulent and, introducing the Reynolds decomposition $\mathbf{v}=\mathbf{U}+\mathbf{u}$, where $\mathbf{U}$ and $\mathbf{u}$ describe the mean and fluctuating components of the velocity field, respectively, the equations of motion can be formally rewritten as

$$\frac{\partial \mathbf{U}}{\partial t}+\mathbf{U}\cdot\nabla\mathbf{U}+\mathbf{U}\cdot\nabla\mathbf{u}=-\nabla P+\mathbf{F}+\nu_0\Delta\mathbf{U}\ ,$$

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{U}=-\nabla p'+\nu_0\Delta\mathbf{u}\ .$$

If $p=P+p'$, these equations are equivalent to the original Navier-Stokes equations. The goal of the theory is to evaluate both $\mathbf{U}$ and $\mathbf{u}$.

Understanding the behavior of the small-scale velocity fluctuations is one of the main challenges of turbulence theory. It is usually assumed that the statistical characteristics of the flow at small enough scales are independent of the large-scale dynamics. This assumption is basic for comparison of the theoretical predictions with experimental data. However, as seen from the equation for $\mathbf{u}$, the small-scale velocity fluctuations interact with the "dressed" external field $\mathbf{U}$, responsible for turbulence production. This field must be taken into account when we are interested in such fine detail of the flow as deviations from the Kolmogorov theory observed in the high-order moments of the velocity differences, defined below. Moreover, at the not-too-small scales this field can dominate the dynamics, thus considerably simplifying the problem. Investigation of the possible effects of the large-scale, long-living structures on the small-scale properties of turbulence is the goal of this work.

Experimental data on the second-order structure function $S_2(x)$ measured in various high-Reynolds-number flows support the Kolmogorov prediction

$$S_2\equiv\overline{(\Delta u)^2}\equiv\overline{[u(X)-u(X+x)]^2}=C_K\bar{\epsilon}^{2/3}x^{2/3}\ , \qquad (1.1)$$

where $\bar{\epsilon}$ is the energy flux in wave-vector space. Relation (1.1), though not derived from mathematically rigorous theory, is readily obtained from various qualitative considerations. For example, at scales much smaller than the energy injection scale the following exact relation holds:

$$S_3=-\tfrac{4}{5}\bar{\epsilon}x\ .$$

Relation (1.1) is obtained immediately if the possibility of the "anomalous scaling" arising from the nontrivial dimensionality of $\bar{\epsilon}$ is disregarded.

However, dimensional arguments applied to the high-order moments of velocity differences give $S_n(x)\approx x^{(2/3)n}$, which contradicts the available experimental data when n is large. So far the observed scaling behavior of higher-order structure functions remains something of a mystery. Numerous experiments indicate that in the available range of displacements x the functions $S_n(x)$ seem to be rather well fitted by the power laws:

$$S_n=\overline{[u(X)-u(X+x)]^n}\propto x^{\kappa_n-\xi_n}\ , \qquad (1.2)$$

where the scaling exponents ξ_n describe deviations from the Kolmogorov values $\kappa_n=n/3$. Further, it was found that the larger the order n, the larger the deviation of ξ_n from the predictions of the Kolmogorov theory.

Another unresolved problem is the shape of the probability distribution function (PDF) of the velocity differences $P(\Delta u)$. It is well established that the single-point PDF is Gaussian. In other words,

$$P(\Delta u) \propto \exp\left[-\frac{(\Delta u)^2}{u_{\rm rms}^2}\right] \tag{1.3}$$

for $x \gg L$, where L is the integral scale of turbulence. Moreover, Gaussian statistics of velocity differences are observed for separations $x/L = O(1)$. For the scales x corresponding to the inertial range $l_d \ll x \ll L$, where l_d is the dissipation scale, the experimentally observed $P(\Delta u)$ seems to be close to exponential:

$$P(\Delta u) \propto \exp\left[-\alpha\frac{|\Delta u|}{u_{\rm rms}}\right], \tag{1.4}$$

with the dimensionless coefficient $\alpha = 0(1)$. Relations (1.3) and (1.4) represent the most dominant feature of the PDF of velocity differences in the inertial range.

No less interesting is the behavior of the fluctuations of the local value of the kinetic energy dissipation rate $\epsilon = \nu(\partial u_i/\partial x_j)^2$. The Kolmogorov theory (K41) simply neglected the ϵ fluctuations assuming $\epsilon(x) = \text{const}$. It was later suggested [1] that the ϵ fluctuations may be responsible for the experimentally observed deviations from Kolmogorov scaling. It is easy to show that if turbulent transport coefficients obtained from the K41 phenomenology are used for construction of a dimensional argument, the resulting correlation function is

$$S_2^\epsilon = \overline{[\epsilon(X) - \epsilon(X+x)]^2} \propto x^0 . \tag{1.5}$$

Relation (1.5), showing that the ϵ fluctuations are evenly distributed in space, has never been observed in numerical or physical experiments. Instead, observations suggest

$$S_2^\epsilon \propto x^{-\mu} , \tag{1.6}$$

with the "intermittency exponent" μ ranging from 0.1 to $\simeq 1$ depending on the experimental conditions, Reynolds numbers, etc. Relation (1.6) shows that the dissipation rate of kinetic energy is concentrated in the localized areas of space having a "spotty" nature. This is often interpreted as spatial intermittency of strong turbulence. Theoretical understanding of this behavior is a major challenge. Accurate experimental verification of (1.6) is very difficult and at the present time we cannot even be sure that a scaling relation of the type (1.6) exists at all, though it is clear that experimentally observed $S_2^\epsilon(x)$ decreases with x more slowly than predicted by (1.5).

Since the dissipation rate appears in the expression for the Kolmogorov energy spectrum, it is tempting to try to incorporate the ϵ fluctuations into Kolmogorov-like considerations and express the deviations from the K41 scaling observed in the high-order moments of velocity differences in terms of the intermittency exponent μ. This has been done in various models, often rather loosely related to the Navier-Stokes equation of motion. However, the Gaussianity of the large-scale velocity fluctuations has not been addressed by any model of intermittency known to me. It is remarkable that in various experimental situations, differing by geometry, production mechanisms, Reynolds number, etc., the observed PDF of velocity differences was so close to Gaussian at the scales $x/L \approx 1$. In my opinion the universality and robustness of this quantity is most surprising and very difficult to explain. In this paper I modify the Clebsch formulation of statistical theory of strong turbulence, developed in Ref. [3], to include the large-scale coherent structures, sometimes responsible for turbulence production. It will be shown that the interaction between coherent and random components of the Clebsch fields is responsible for the observed Gaussian statistics of the large-scale velocity differences and for the nontrivial behavior of the high-order moments. The results will be compared with experimental data. This paper is organized in the following way. In Secs. II and III, closely following Ref. [3], the formulation of hydrodynamics in a Clebsch variables is introduced and the Kolmogorov energy spectra corresponding to the constant fluxes of conserved quantities are derived. In Sec. IV the interaction of the large-scale coherent and random components is introduced, and it is shown that the theory reduces to the many-body problem in a strong external field. The interaction of the velocity fluctuations with coherent structures is responsible for two additional solutions at scales larger than those corresponding to the Kolmogorov range. The expressions for the probability distribution function of velocity differences, which is Gaussian at large scales and close to exponential at smaller scales, is derived in Sec. V. In Sec. VI the derivation of the dissipation rate correlation function, developed in Ref. [3], is presented. The discussion of experimental data and a detailed comparison with the outcome of this paper are presented in Sec. VII.

II. FORMULATION OF THE PROBLEM

We consider a fluid flow driven at the very large scales $l \gg L \to \infty$. Somewhere at the smallest scales $l \to 0$ an energy sink is assumed, so that a statistically steady state can be achieved. The flow is described by the Navier-Stokes equations:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \mathbf{F} + \nu_0 \Delta \mathbf{v} ,$$

$$\nabla\cdot\mathbf{v} = 0 ,$$

subject to initial and boundary conditions. We assume that the force $\mathbf{F}(\mathbf{x},\mathbf{t})$ is an arbitrary deterministic function of position and time acting at large scales only. In various flows $\mathbf{F}$ corresponds to the deterministic contribution to the pressure gradient. For example, the flow in a pipe driven by gravity is equivalent to the flow driven by a constant pressure gradient. When the Reynolds number $\text{Re} < \text{Re}_c$ this equation gives a laminar velocity profile $\mathbf{U}_L(\mathbf{x},\mathbf{t})$. At $\text{Re} > \text{Re}_c$ the laminar velocity profile is modified due to the interaction with turbulent velocity fluctuations.

It is customary to describe the dynamics of the intermediate scales by the Euler equation (the density $\rho = 1$):

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p , \qquad \nabla\cdot\mathbf{v} = 0 . \tag{2.1}$$

The Clebsch variables are defined as

$$\mathbf{v}=\lambda\nabla\mu+\nabla\phi \; . \tag{2.2}$$

Using the incompressibility condition, the potential ϕ can be expressed through λ and μ:

$$\phi=-\nabla^{-2}\nabla\cdot(\lambda\nabla\mu)$$

and thus

$$\mathbf{v}=-\nabla^{-2}\nabla\times(\nabla\lambda\times\nabla\mu) \; , \tag{2.3}$$

$$\omega=\nabla\lambda\times\nabla\mu \; . \tag{2.4}$$

The Clebsch variables are transported by the flow and the Euler equation can be represented as

$$\mathcal{D}\mu=\frac{\partial\mu}{\partial t}+\mathbf{v}\cdot\nabla\mu=0, \quad \mathcal{D}\lambda=\frac{\partial\lambda}{\partial t}+\mathbf{v}\cdot\nabla\lambda=0 \; . \tag{2.5}$$

It follows from Eq. (2.3) that the velocity field does not uniquely define the Clebsch field $(\lambda(x,t),\mu(x,t))$. In fact, a set of pairs of the Clebsch variables $(\lambda_i(x,t),\mu_i(x,t))$ can be used to express the velocity $\mathbf{v}(\mathbf{x},\mathbf{t})$:

$$\mathbf{v}=\sum_{i=1}^{M}\lambda_i\nabla\mu_i+\nabla\phi \tag{2.6}$$

and

$$\omega=\sum_{i=1}^{M}\nabla\lambda_i\times\nabla\mu_i \; , \tag{2.7}$$

where M is the number of Clebsch pairs necessary for the complete representation of velocity field.

The equations of motion for each pair (λ_i,μ_i), given by (2.5) with the subscript i specifying the pair, can be written in a Hamiltonian form since λ and μ are canonical variables. The Hamiltonian and the corresponding equations of motion are given below. The minimal number of canonical pairs needed to describe an arbitrary flow depends on the topology of the field $\mathbf{v}$. It is a plausible conjecture that $M=2$ is sufficient to represent a wide class of turbulent flows. Indeed, the velocity field in a three-dimensional incompressible flow has two independent components. This field, however, cannot be described by one pair of Clebsch variables due to the constraint $\mathbf{v}\cdot\omega=0$ which tells us that, in fact, we have only one independent Clebsch variable. Introducing the second Clebsch pair we create two independent variables, sufficient for the description of the general velocity field with nonzero values of the local helicity. In this paper we will discuss only the case of $M=1$ which corresponds to zero helicity $\int \mathbf{v}\cdot\omega\, d\mathbf{x}\equiv 0$, following directly from the definition (2.4). This restricts applicability of the analysis to flows in which the vortex lines do not have any knots. However, the results of this work may be readily generalized to the case of $M=2$ corresponding to an arbitrary topology.

Introducing the complex variables $a(\mathbf{k})$ and $a^*(\mathbf{k})$,

$$\mu(\mathbf{k})=\frac{1}{\sqrt{2}}[a(\mathbf{k})+a^*(-\mathbf{k})] \; ,$$
$$\lambda(\mathbf{k})=\frac{i}{\sqrt{2}}[a(\mathbf{k})-a^*(-k)] \; , \tag{2.8}$$

the Euler equation can be written in a Hamiltonian form:

$$i\frac{\partial a(\mathbf{k})}{\partial t}=\frac{\delta H}{\delta a^*(\mathbf{k})} \; , \tag{2.9}$$

where the Hamiltonian H is

$$H=\tfrac{1}{4}\int T_{12,34}a^*(\mathbf{k}_1)a^*(\mathbf{k}_2)a(\mathbf{k}_3)a(\mathbf{k}_4) \times\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)d\mathbf{k}_1,d\mathbf{k}_2d\mathbf{k}_3d\mathbf{k}_4 \; . \tag{2.10}$$

The interaction potential is

$$T_{12,34}\equiv T(\mathbf{k}_1\mathbf{k}_2,\mathbf{k}_3\mathbf{k}_4)=\varphi_{13}\varphi_{24}+\varphi_{14}\varphi_{23} \; , \tag{2.11}$$

where

$$\varphi(\mathbf{k}_1,\mathbf{k}_2)\equiv\varphi_{12}=\mathbf{k}_1+\mathbf{k}_2-(\mathbf{k}_1-\mathbf{k}_2)\frac{k_1^2-k_2^2}{|\mathbf{k}_1-\mathbf{k}_2|^2} \; . \tag{2.12}$$

In these variables

$$\mathbf{v}=\int \varphi_{\mathbf{q},\mathbf{q}-\mathbf{k}}a_{\mathbf{q}}^*a_{\mathbf{q}-\mathbf{k}}d\mathbf{q} \; . \tag{2.13}$$

The function $\varphi(k_1,k_2)$ is a discontinuous function at $k_1=k_2$ since the diagonal elements of $\varphi(k,k)$ determine an arbitrary mean velocity in the flow $\mathbf{v}(\mathbf{k}=\mathbf{0})$. So, in what follows we set $\varphi(k,k)=0$.

Substituting (2.10)–(2.12) into (2.9) the equation of motion for the "creation-annihilation" operators $a(\mathbf{k})$ is readily derived:

$$i\frac{\partial a(\mathbf{k})}{\partial t}=\frac{1}{2}\int T(\mathbf{k}\mathbf{k}_2,\mathbf{k}_3\mathbf{k}_4)a^*(\mathbf{k}_2)a(\mathbf{k}_3)a(\mathbf{k}_4) \times\delta(\mathbf{k}+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)d\mathbf{k}_2d\mathbf{k}_3d\mathbf{k}_4 \; . \tag{2.14}$$

Equation (2.14) conserves the total energy, since it is a Hamiltonian equation of motion. In addition, it conserves an infinite number of integrals of motion $\int F(\lambda,\mu)d\mathbf{r}=\text{const}$. These integrals do not have simple interpretation in terms of the velocity field. In the present paper we concentrate only on one of the integrals of motion:

$$N_0=\tfrac{1}{2}\int(\lambda^2+\mu^2)d\mathbf{x}=\int a^*(\mathbf{k})a(\mathbf{k})d\mathbf{k}=\text{const} \; . \tag{2.15}$$

The parameter N has the dimensionality of action and can be called the "hydrodynamic action" or number of quasiparticles (elementary excitations) describing turbulent flow. The relation (2.15) has the most important impact on what follows, so the elucidation of the physical meaning of the "quasiparticles" or waves and of the topological consequences of this conservation law remains a very important task. Due to the negative sign of the flux of the number of particles, their source is expected to be at the small scales.

III. RANDOM-PHASE APPROXIMATION. KINETIC EQUATION

Let us single out the diagonal contributions to the equation of motion (2.14):

$$i\frac{\partial a(\mathbf{k})}{\partial t}-\omega(\mathbf{k})a(\mathbf{k})=\int' T(\mathbf{k},\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)a^*(\mathbf{k}_2)a(\mathbf{k}_3)\times a(\mathbf{k}_4)\delta(\mathbf{k}+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\times d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 \ , \tag{3.1}$$

where

$$\omega(\mathbf{k})=\int T(\mathbf{k}\mathbf{k}_2,\mathbf{k}\mathbf{k}_2)a^*(\mathbf{k}_2)a(\mathbf{k}_2)d\mathbf{k}_2 \tag{3.2}$$

and the symbol $'$ in the integral in (3.1) means that the diagonal contributions with $k=k_3$, $k_2=k_4$ are not included. It will be shown in what follows that the integral

$$\overline{\omega}(\mathbf{k})=\int T(k,k_2,k,k_2)n(\mathbf{k}_2)d\mathbf{k}_2 \ , \tag{3.3}$$

with

$$n(k)=\langle a^*(\mathbf{k})a(\mathbf{k})\rangle \tag{3.4}$$

converges when calculated on the solutions $n(\mathbf{k})$ of the kinetic equation derived below. This means that the main contribution to (3.3) comes from the region $k \simeq k_2$. In this work we are interested in statistically steady solutions $n(\mathbf{k})$, so $\overline{\omega(k)}=\text{const}$ is time independent. Thus we introduce the mean-field approximation [3]

$$i\frac{\partial a(\mathbf{k})}{\partial t}-\overline{\omega(\mathbf{k})}a(\mathbf{k})=S \ , \tag{3.5}$$

where the collision integral $S(\mathbf{k})$ is defined by the right side of Eq. (3.1). In the zeroth order of the expansion in powers of the nonlinear interaction S we have

$$a^0(\mathbf{k},t)=a(\mathbf{k})e^{-i\omega(\mathbf{k})t} \ . \tag{3.6}$$

The bar over $\overline{\omega(k)}$ defined by (3.3) is omitted in what follows. The statistical ensemble can be constructed by introducing an infinite set of realizations differing in the values of the initial phases $\varphi(\mathbf{k})$ in (3.6):

$$a^0(k,t)=|a(k)|e^{i\omega(\mathbf{k})t+i\varphi(\mathbf{k})} \ . \tag{3.7}$$

The key element of the theory of weak turbulence, adopted in Ref. [3] for consideration of strong turbulence, is the assumption that all phases $\varphi(\mathbf{k})$ are uncorrelated, i.e.,

$$\langle a(\mathbf{k})\rangle=\langle |a(\mathbf{k})|e^{i\varphi(\mathbf{k})}\rangle=0 \ ,$$

$$\langle a(\mathbf{k})a(\mathbf{k}')\rangle=\langle |a(\mathbf{k})||a(\mathbf{k}')|\exp[i\varphi(\mathbf{k})+\varphi(\mathbf{k}')]\rangle=0 \ ,$$

$$\langle a(\mathbf{k})a^*(\mathbf{k}')\rangle=\langle |a(\mathbf{k})||a(\mathbf{k}')|\exp[i\varphi(\mathbf{k})-\varphi(\mathbf{k}')]\rangle=\mathbf{n}(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}') \ .$$

All odd-order correlation functions of the fields $a(\mathbf{k})$ are equal to zero in this random-phase approximation (RPA). As was mentioned above the averaging is performed over the ensemble of initial phases $\phi(k)$.

To derive equations of motion for the "occupation numbers" $n(\mathbf{k})$, let us multiply (3.5) and the corresponding equation for $a^*(\mathbf{k})$ by $a^*(\mathbf{k})$ and by $a(\mathbf{k})$, respectively. Then, the equation of motion for $n(\mathbf{k})$ reads

$$\frac{\partial n(k,t)}{\partial t}=\text{Im}\int T_{kk_2,k_3k_4}J_{kk_2,k_3k_4}\times\delta(\mathbf{k}+\mathbf{k}_1,-\mathbf{k}_2-\mathbf{k}_3)d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 \ , \tag{3.8}$$

where

$$J_4=J_{kk_2,k_3k_4}=\langle a^*(\mathbf{k})a^*(\mathbf{k}_2)a(\mathbf{k}_3)a(\mathbf{k}_4)\rangle \ . \tag{3.9}$$

Writing the equation of motion for J_4 as

$$\frac{\partial J_4}{\partial t}=\left\langle\frac{\partial}{\partial t}[a^*(\mathbf{k})a^*(\mathbf{k}_2)a(\mathbf{k}_3)a(\mathbf{k}_4)]\right\rangle \tag{3.10}$$

and expressing the time derivatives in (3.10) using (3.6) we obtain in the random-phase approximation in the long time limit $t\to\infty$

$$\frac{\partial n(k,t)}{\partial t}=\frac{\pi}{2}\int |T_{k_2,k_3k_4}|^2 J_4\delta(\mathbf{k}+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_3)\times\delta(\omega(k)+\omega(k_2)-\omega(k_3)-\omega(k_4))\times d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 \ , \tag{3.11}$$

with

$$J_4=n_3n_4(n_2+n_k)-n_2n_k(n_3+n_4) \ . \tag{3.12}$$

Here $n(\mathbf{k}_i)=\mathbf{n}_i$. The δ function in the collision integral in (3.11), describing energy conservation per collision, appears in the equation of motion as a result of iteration:

$$J_4=\langle a^*(k_1)a^*(k_2)a(k_3)a(k_4)\rangle\times\int_0^t dt\,\exp\{it[\omega(k_1)+\omega(k_2)-\omega(k_3)-\omega(k_4)]\}$$

in the limit of large time t. The kinetic Eq. (3.10) and (3.11) has been analyzed and solved by Zakharov in the context of the weak turbulence theory (see the excellent review in [2], and references therein). It has been shown that if

$$\omega(\mathbf{k})\propto k^{\alpha} \tag{3.13}$$

there exist four scaling solutions:

$$n(k)=\text{const}, \quad n(k)\propto\frac{1}{\omega(k)} \ , \tag{3.14}$$

and

$$n(k)\propto k^{-x} \ , \tag{3.15}$$

with

$$x_1=\frac{4}{3}+d, \quad x_2=\frac{4-\alpha}{3}+d \ . \tag{3.16}$$

The solutions (3.14) correspond to a fluid in thermodynamic equilibrium, while the relations (3.15), (3.16) describe a nonequilibrium flow. From the definition of $\omega(\mathbf{k})$ given by (3.3) we find readily

$$\alpha=-x+d+2=-x+5 \quad (d=3) \tag{3.17}$$

and the expressions for $n(\mathbf{k})$ can be obtained in a closed form:

$$n\propto k^{-13/3}, \quad \alpha=\tfrac{2}{3} \ , \tag{3.18}$$

and

$$n\propto k^{-4}, \quad \alpha=1 \ . \tag{3.19}$$

It can be checked easily that the total energy can be evaluated from the following relation:

$$E=\int \omega(\mathbf{k})n(\mathbf{k})d\mathbf{k} , \quad (3.20)$$

which defines the energy spectra in terms of the Clebsch variables:

$$E(k)=4\pi k^{d-1}\omega(k)n(k) .$$

The relations (3.18) and (3.19) generate two solutions,

$$E(k)\propto \epsilon^{2/3}k^{-5/3} \quad (3.21)$$

and

$$E_n(k)\propto Pk^{-1} , \quad (3.22)$$

where P denotes the "particle" flux in the wave-number space. It has also been shown [2] that, while the energy flux is positive, i.e., the energy is cascading from the largest to the smallest scales, the flux of particles is in the opposite direction: from small to large scales (inverse cascade). The importance of this fact will be discussed below. Thus, as follows from (3.21) and (3.22), the small- and large-scale dynamics in turbulent flows are characterized by two different energy spectra. This is a completely new development which is discovered due to our use of the Clebsch variables. It is clear that the energy spectra (3.21) and (3.22) can readily be obtained from dimensional considerations which do not require one-loop approximation and kinetic Eqs. (3.11) and (3.12).

Strictly speaking, Clebsch variables are formally defined for the inviscid Euler equation. However, it has been shown [3] that using the definitions (2.3), (2.4), the viscous hydrodynamics can be described by Eqs. (2.5) with the source term

$$I_c=O(\xi\nu\Delta\lambda) ,$$

where $\xi=O(\Delta^{-2}\Delta\omega/\omega)$ is a complicated functional of the vorticity field ω. If ξ can be considered as a Gaussian random function, then this term represents the small-scale source of the "quasiparticles," consistent with the results presented above. In the Fourier space $I_c=O(\nu k^2a(k))$, which can be assumed small in the inertial range. This fact is important for the application of the Clebsch formulation of hydrodynamics for investigation of the turbulence problem.

Another problem with the Clebsch formulation is its nonuniqueness: any velocity field $\mathbf{v}$ can be represented by an infinite number of different Clebsch fields (λ,μ). At the present time we do not know how to solve this problem, and therefore we neglect it for the time being and hope that one day the way to calibrate the Clebsch representation will be found.

IV. LARGE-SCALE COHERENT STATE AND EQUATIONS OF MOTION

All results described in the previous sections were derived in the RPA combined with the mean-field approximation introduced in the work of Yakhot and Zakharov [3]. This theory used the unjustified assumption of the close-to-Gaussian statistics of the Clebsch variables and led to the explanation of various experimental observations. For example, the exponential distribution of the velocity differences is readily understood: The assumed PDF of the Clebsch field $P(a;a^*)\approx e^{-aa^*/N}$. Since $u=O(aa^*)$ then $P(U)\approx \exp(-U/U_{\rm rms})$. The actual derivation is rather difficult due to the complex form of the matrix element φ_{12} entering the expression of the velocity field in terms of Clebsch variables but the origin of the exponential distribution is clear from the above dimensional considerations. This theory failed to explain the experimentally observed Gaussian statistics of velocity differences at the large scales and the "anomalous scaling" of the high-order moments S_n.

Let us reflect on the approximations involved in the derivation of the spectra in the preceding section. The close-to-Gaussian probability distribution of the Clebsch field is, at first glance, the crudest of all since it assumes weak interaction between modes $a(k)$. Still, this approximation qualitatively explained the experimentally observed close-to-exponential distribution of the velocity differences at the small scales.

Random-phase approximation was a key element of the theory developed in Ref. [3]. This approximation, which disregards the possibility of generation of the large-scale coherent structures, can serve only as a crude model for the small-scale dynamics. In reality turbulence is produced as a result of hydrodynamic instabilities of the large-scale ordered flows (structures) characterized by very strong phase correlation. The well-known examples include traveling waves and streaks in wall flows, convection rolls, Karman vortex streets in jets and mixing layers. Taylor vortices etc. In all these cases the structures, though strongly interacting with the random component of the velocity field, preserve the phase correlation even in high-Reynolds-number flows. The organized motions, reflecting the physical mechanisms of turbulence generation, create large-scale shear acting on the small-scale component of velocity fluctuations. This shear is not a result of pure nonlinear interaction and can be considered as a "dressed" external field. The dynamical consequences of the large-scale coherent component are not taken into account in the theories based on the random-phase approximation. Let $\mathbf{k}_0^i$ denote the wave vector or set of wave vectors corresponding to the basic flow structure. Then, the random-phase approximation cannot be valid for $k\to k_0^i\to 0$. To incorporate coherent structures in the theory we relax the random-phase approximation and introduce

$$\langle a(k)\rangle=\sum_i A(k)\delta(\mathbf{k}-\mathbf{k}_0^i) ,$$

$$\langle a^*(k)\rangle=\sum_i A^*(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}_0^i) ,$$

and

$$\langle a(k)a^*(k)\rangle=\sum_i N_0^i\delta(\mathbf{k}-\mathbf{k}_0^i)+n(\mathbf{k}) ,$$

where $N_0^i=\langle A(k_0^i)A^*(k_0^i)\rangle$ correspond to the coherent components of the Clebsch field while the modes $a(k)$ with $k>k_0^i$ describe fluctuations and can be considered in the random-phase approximation so that $\langle a(k)\rangle=0$. This will be discussed in what follows. Although the detailed structure of the ordered component can be very involved and vary from flow to flow, here, as a first step, I consider a simplified model and investigate consequences of the very existence of the large-scale coherent mode. Taking into account that

$$\varphi(\mathbf{k},\mathbf{k}_1)\approx\mathbf{k}-\frac{\mathbf{k}_1(\mathbf{k}\cdot\mathbf{k}_1)}{\mathbf{k}_1^2}\approx\mathbf{k}$$

when $k \ll k_1$, the velocity definition in terms of the Clebsch field reads

$$\mathbf{v}(\mathbf{k})\approx\mathbf{V}_0+[\,\mathbf{A}^*(\mathbf{k}_0,\mathbf{k})a(-\mathbf{k})+\mathbf{A}(-\mathbf{k}_0,\mathbf{k})a^*(\mathbf{k})\,] + \int \varphi_{\mathbf{q},\mathbf{q}-\mathbf{k}}a_{\mathbf{q}}^*\mathbf{a}_{\mathbf{q}-\mathbf{k}}d\mathbf{k}\ , \tag{4.1}$$

where $k_0^2=O(L^{-2})$ and $\mathbf{k}_0$ can, in principle, slowly vary in time so that $\langle \mathbf{k}_0\rangle=\mathbf{0}$. The coherent component of the velocity field V_0, derived from the definition (2.13), calculated on the coherent contributions to the Clebsch field $A(k_0)$, is

$$\mathbf{V}_0=\sum\varphi_{\mathbf{k}_0^i,\mathbf{k}_0^i}A(k_0^i)A^*(k_0^i)\ ,$$

where the summation is carried out over the wave vectors $\mathbf{k}_0^m$ forming the coherent state. The vector

$$\mathbf{A}(\mathbf{k}_0,\mathbf{k})=\sum_i\left[\mathbf{k}_0^i-\frac{\mathbf{k}(\mathbf{k}\cdot\mathbf{k}_0^i)}{\mathbf{k}^2}\right]A(\mathbf{k}_0^i)\ .$$

This relation introduces some violation of both isotropy and homogeneity, always present at the large scales in real-life flows. This anisotropy is not essential for the results derived below, since one can introduce an ensemble of the large-scale coherent structures and average the results derived for each realization over directions of $\mathbf{k}_0^i$ [4]. In many flows the large-scale structure has rather complicated topology and the anisotropy introduced by it is small. To proceed further we need information about physical properties of the large-scale coherent structures which has to emerge as a result of solution of the entire dynamical problem. Since at the present time we cannot develop a comprehensive theory, approximations based on a sensible physical picture are to be invoked. The coherent structures, often generated at the transition to turbulence, are long-living solutions of the dynamical problem and thus it is reasonable to assume that they are stable or marginally stable in the limit $a(k)\rightarrow 0$. In other words,

$$\frac{\delta H}{\delta a(k)}=0$$

when $a(k)\rightarrow 0$. This means that the linear in $a(k)$ contributions to the Hamiltonian of the kind $\int d\mathbf{k}\,O(V_0\cdot\mathbf{A}(\mathbf{k}_0,\mathbf{k})\mathbf{a}(\mathbf{k}))=0$ can be neglected. Thus the Hamiltonian, which does not include linear in $a(k)$ contributions, can be written as

$$H=H_0+H_2+H_3+H_4\ ,$$

$$H_0=\sum\varphi_{\mathbf{k}_0^i,\mathbf{k}_0^j}\cdot\varphi_{\mathbf{k}_0^p,\mathbf{k}_0^q}A^*(k_0^i)A^*(k_0^p)A(k_0^j)A(k_0^q)\ ,$$

$$H_2=\int dk(|A(k_0,k)|^2a(k)a^*(k) + \tfrac{1}{2}\{A^2(k_0,k)a(k)a(-k) + [A^*(k_0,k)]^2a^*(k)a^*(-k)\})\ ,$$

$$H_3=\int dk\,O(\,\mathbf{A}(\mathbf{k}_0,\mathbf{k})\cdot\phi_{\mathbf{q},\mathbf{q}-\mathbf{k}}[a^*(q)a(q-k)a(k) + a^*a^*a+\cdots])\ ,$$

$$H_4=S\ .$$

The same result is derived directly from the Hamiltonian (2.10): as in the Bogolubov theory of weakly interacting Bose gas, the linear in $a(k)$ contributions are absent when $k_0^i \ll 1$ and $k=O(1)$ due to the impossibility to satisfy momentum conservation.

In general the large-scale field $\mathbf{A}(\mathbf{k}_0,\mathbf{k})$, which is to be obtained as a solution of the full dynamical problem, is a functional of $a(k)$ and the derivation of $A(k;a(k))$ is a very difficult task. First, we consider the simpler case of low-Reynolds-number flow, in which the coherent component of the velocity field does not strongly differ from the laminar velocity profile $\mathbf{U}_L(\mathbf{x},t)$. We can neglect the dependence of $\mathbf{A}(\mathbf{k}_0,\mathbf{k})$ on $a(k)$ and treat the large-scale coherent component as an external field.

Due to quadratic in $a(k)$ contributions to the Hamiltonian, the equations of motion have linear terms:

$$i\frac{\partial a(k)}{\partial t}=|A(k_0,k)|^2a(k)+\frac{1}{2}[A^*(k_0,k)]^2a^*(-k) \equiv I_L$$

and

$$-i\frac{\partial a^*(k)}{\partial t}=|A(k_0,k)|^2a^*(k)+\frac{1}{2}A^2(k_0,k)a(-k) \equiv I_L^*\ .$$

The physical meaning of I_L can be understood readily. Let us introduce the Reynolds decomposition of the velocity field:

$$\mathbf{v}=\mathbf{U}+\mathbf{v}'\ ,$$

where $\mathbf{U}$ and $\mathbf{u}'$ correspond to the mean (coherent) and fluctuating velocity fields, respectively. This decomposition is difficult to realize in unsteady flows and it is used here only to illustrate the physical origin of the external field in the Clebsch formulation of hydrodynamics. Then, the Euler equation (2.1) can formally be rewritten

$$\frac{\partial\mathbf{v}'}{\partial t}+\mathbf{v}'\cdot\nabla\mathbf{v}'+I_v=-\nabla p'$$

and

$$\frac{\partial\mathbf{U}}{\partial t}+\mathbf{U}\cdot\nabla\mathbf{U}+I_U=-\nabla P+\mathbf{F}\ ,$$

where

$$I_v+I_U=\mathbf{U}\cdot\nabla\mathbf{v}'+\mathbf{v}'\cdot\nabla\mathbf{U}$$

and $p'+P=p$. It is clear that these two equations are equivalent to the Navier-Stokes equations at the scales $l \gg l_d$.

Taking into account that

$$A^2(k_0,k)=\sum_{i,j}\left[\mathbf{k}_0^i\cdot\mathbf{k}_0^j-\frac{(\mathbf{k}\cdot\mathbf{k}_0^j)(\mathbf{k}\cdot\mathbf{k}_0^i)}{\mathbf{k}^2}\right]\mathbf{A}(\mathbf{k}_0^i)\,\mathbf{A}(\mathbf{k}_0^i)$$
$$=O\left[\sum_{i,j}\nabla_i\mathbf{U}_j\right]$$

we come to the conclusion that the linear contribution to the equation of motion for $a(k)$ comes from the large-scale shear. This result is expected, since it is this shear which strongly contributes to the turbulence production in the Navier-Stokes equations. Thus the introduction of coherent and fluctuating components of the Clebsch field is equivalent to the Reynolds decomposition of the velocity field. It is clear from these relations that $\mathbf{A}$ can be treated as external field only if I_U is small enough to introduce large deviations from the laminar solution U_L.

If the amplitude of $A(k)$ is large the interaction Hamiltonians H_3 and H_4 can be taken into account perturbatively. The small coupling constant will be introduced below. Using the Bogolubov transformation [1,5]

$$a(k)=uc(k)+vc^*(-k)$$

and

$$a^*(k)=u^*c^*(k)+v^*c^*(-k)$$

the expression for H_2 can be diagonalized:

$$H_2=\int e(k)c(k)c^*(k)dk\ ,$$

with

$$e^2(k)=u^2-vv^*\ ,$$
$$u^2=[|A(k_0,k)|^2]^2\ ,$$
$$v=A^2(k_0,k)\ ,$$

where $|A|^2=|A_1|^2+|A_2|^2+|A_3|^2$ and $A^2=A_1^2+A_2^2+A_3^2$. From these expressions we can see that for the general structure of the field $\mathbf{A}(\mathbf{k}_0,\mathbf{k})$, the dispersion relation $e^2(k)>0$ and thus the coherent state is linearly stable as was assumed when the linear contributions to the Hamiltonian were neglected.

In a general case of high-Reynolds-number flow $\mathbf{A}(\mathbf{k}_0,\mathbf{k})=\mathbf{A}(\mathbf{k}_0,\mathbf{k},\mathbf{a}(\mathbf{k}))$ and the equations of motion, derived above, are incorrect. Following the theory of superfluidity, let us use the particle conservation law (2.15) and express $A(k_0^i)$ in terms of the external parameters of the problem and fluctuation $a(k)$:

$$\sum_{i,j}A(k_0^i)A(k_0^j)=N_0-\int a^*(k)a(k)d^dk\ ,$$

where $N_0=\text{const}$, independent of $a(k)$. Due to the complexity of the matrix elements $\varphi_{k,k'}$ we cannot exactly express the Hamiltonian H in terms of N_0 and $a(k)$. However, the particle conservation relation can be used to estimate various contributions. The dangers of this approach are clear but at the present time this is all we can do. Thus

$$H_2\approx\int dk\,(Na(k)a^*(k)$$
$$+\tfrac{1}{2}\{A^2(k_0,k)a(k)a(-k)$$
$$+[A^*(k_0,k)]^2a^*(k)a^*(-k)\})\ ,$$

where

$$N=O(k_0^2N_0)\ .$$

The corresponding $O((k_0/k)^2)$ correction to H_4 is small in the limit $k_0\to 0$. Using the same estimate we can express the operators $A(k)$ in H_0 in terms of N_0 and $A(k)$. The resulting Hamiltonian has constant $O(k_0^2N_0^2)$ term and the $a(k)$-dependent contributions which simply modify all factors in H_2. The magnitudes of these factors can be important for the results of this work since, in principle, they can modify the frequency shift we are trying to calculate.

In the theory of weakly interacting Bose gas the condensate is considered at $k_0=0$ and the corresponding operators are taken as c numbers, so that $A=A^*=\sqrt{N}$. In this case the frequency shift is equal to zero. This gives rise to phonons, characterized by linear in wave-number energy spectrum $\omega(k)\propto k$. In the theory of turbulence the coherent state cannot occupy $k_0=0$ since the finite shear is necessary for the turbulence production. Thus some violation of translational invariance is always expected at the large scales.

As was shown above, the particle conservation law gives some information about the product $A(k)A^*(k)$. In order to say anything about A^2 we have to solve the corresponding dynamic equations describing the large-scale coherent component of the Clebsch field. At the present time we cannot do it. However, it is reasonable to assume that

$$N^2\neq|A^2(k_0,k)|^2\ .$$

Based on the particle conservation law, we estimate

$$e^2(k)\approx N^2-|A^2(k_0,k,a(k))|^2=O(k_0^4N_0^2)>0\ .$$

If the large-scale flow is produced by the nonlinear interactions without the large-scale symmetry breaking external field, then it is possible that $e(k)=0$ and the results of this work should be modified.

In what follows we take the frequency shift $\Omega\equiv e(k)\approx k_0^2N_0$ and, to avoid introduction of a new notation, set $a(k)\equiv c(k)$. This approximation totally neglects the geometric structure of the coherent state and the fact that it depends not only on the wave vectors $\mathbf{k}_0^i$ but also on the angles between vectors $\mathbf{k}$ and $\mathbf{k}_0^i$. So, in what follows the dispersion relation $e(k)$ is taken into account in an average way, which is sufficient for the qualitative theory of "weakly interacting Clebsch gas" developed in this paper.

Since the fluctuations from the coherent state $\langle a(k)\rangle=\langle a^*(k)\rangle=0$ the Fourier component of velocity $\langle v(k)\rangle=0$ for k large enough. Thus I assume that the field $a(k)$ is isotropic and homogeneous. This assumption should come out as a result of a dynamical theory showing that anisotropic perturbations of the coherent state decay while the isotropic ones survive. This can be done easily considering linear stability of a basic flow of a given structure in Clebsch variables. However, in fully developed turbulence, the coherent and random components strongly interact: the large-scale turbulence-producing eddies are influenced by the small-scale velocity fluctuations and the resulting "turbulent profiles" arise from the complicated interactions. So, in this work I will not consider the nature of the coherent state and simply postulate its existence. As will be clear from what follows some of the consequences of the interaction between organized and random motions are of a general nature independent of the detailed structure of large-scale eddies.

The mean-field equations of motion for the Clebsch variables are derived from the approximation Hamiltonian introduced above:

$$i\frac{\partial \mathbf{a}(\mathbf{k})}{\partial t}-[\omega(\mathbf{k})+2N_0k_0^2]\mathbf{a}(\mathbf{k})=S\ . \tag{4.2}$$

The $O(aa)$ terms on the right side of the equation of motion coming from H_3, violating particle conservation, are neglected in (4.2). However, they are very important since it is only due to these terms the constant nonlinear frequency shift in (4.2) cannot be removed by the trivial transformation

$$b(k)=a(k)e^{-iN_0k_0^2t}\ .$$

Let us show that in the kinetic equation approximation, considered in this work, the $O(aa)$ contributions are negligibly small due to the absence of the resonant interaction on the dispersion relations $\omega(k)+\text{const}$. Indeed, repeating the derivation of the kinetic equation we find an additional contribution to (3.8) of the type

$$\frac{\partial n(k,t)}{\partial t}=\text{Im}\int T_{k,k_3k_4}J_{k,k_3k_4}\delta(\mathbf{k}-\mathbf{k}_4-\mathbf{k}_3)d\mathbf{k}_3d\mathbf{k}_4\ , \tag{4.3}$$

with

$$J_{k,k_3,k_4}=J_3=O(\langle a^*(k)a(k_3)a(k_4)\rangle)\ .$$

Using the results of the preceding section we derive after the iteration

$$J_3\propto\int_0^t dt\,\exp\{it[\omega(k_3+k_4)-\omega(k_3)-\omega(k_4)-2N_0k_0^2]\}\ ,$$

which disappears in the limit $t\to\infty$ due to the absence of the resonances. It can be shown [3] that elimination of the nonresonant terms can be done rigorously by introduction of the new operators $\alpha(k)=a(k)+s$ where the linear shift s is chosen in such a way that the third-order contribution disappears from the resulting equation for $a(k)$. In the case considered in this work this leads to a simple multiplication of the matrix element (2.11) by a constant factor which does not modify the scaling relations derived in this paper.

Thus the kinetic equation corresponding to (4.2) is exactly the same as one derived in the preceding section since the constant nonlinear frequency shift disappears from the conservation laws. Solutions (3.18) and (3.19) stay intact but in addition to the energy spectra (3.21) and (3.22) we also have

$$E_3(k)\propto k^{-7/3} \tag{4.4}$$

and

$$E_4(k)\propto k^{-2} \tag{4.5}$$

corresponding to energy and particle conservation laws, respectively. The origin of these spectra is clear: in general $E(k)\approx F^{1/3}n(k)/\tau(k)$ where F is the corresponding flux and τ is the characteristic time scale. Due to the interaction with the "external field," we now have an additional wave-number-independent time $\tau=\text{const}$, leading to the energy spectra (4.4) and (4.5). We will see below that the $\frac{7}{3}$ spectrum is indeed observed in shear flows at scales larger than those corresponding to the $\frac{5}{3}$ Kolmogorov range. Since the limit $k\to 0$ is dominated by the $-\frac{7}{3}$ spectrum the second $E(k)\approx k^{-2}$ spectrum will not be considered in what follows. If $\Omega=N_0k_0^2$ is not small, then the expansion parameter of the theory $\eta=\omega(k)/\Omega\to 0$ when $k\to 0$ since the dispersion relations corresponding to both spectra E_3 and E_4 are $\omega(k)\propto k^{\alpha}$ with $\alpha>0$. In this case the solutions E_3 and E_4 are asymptotically exact. Let us estimate η for some well-known cases of turbulent flows. The mean dissipation rate $\bar{\epsilon}\approx u_{\text{rms}}^3/L$ where u_{rms} is the root-mean-square velocity of turbulent fluctuations and L is the integral scale. The characteristic frequency of the coherent structure is simply given by the large-scale shear $\Omega\approx S\approx U/L$ where U is the characteristic velocity of coherent motion. Using these estimates we obtain $\eta\approx u_{\text{rms}}/U$. In wall flows $\eta\approx 10^{-1}-10^{-2}$ in typical laboratory situations and, according to the data $\eta\to 0$ when the Reynolds number $\text{Re}\to\infty$. The same estimate is applied to other wall flows and wakes behind bluff bodies. In Bernard convection $\eta\approx\text{Ra}^{-1/14}$ in the experimentally covered range of variation of the Rayleigh number $10^7<\text{Ra}<10^{14}$. In this flow U is the mean velocity of the coherent vortex ("wind") observed in high Ra flows. According to theoretical arguments in this case too, $\eta\to 0$ when $\text{Ra}\to\infty$. The existence of a small parameter in this formulation of the theory of turbulence means that the fields $a(k)$ and $a^*(k)$ are close to Gaussian when $k\to k_0\to 0$. However, in jets the parameter $\eta\approx 10^{-1}$ and does not seem to decrease with increase of the Reynolds number. The experimentally observed close-to-Gaussian large-scale statistics of velocity field in this kind of flows should be related to numerical smallness of η and, unlike the wall flows, the extent of the range where the Gaussian statistics is observed should not grow with Reynolds number.

The expansion parameter $\eta(k)=O(1)$ when $k\to\infty$. Still, this parameter will be considered small in what follows. This restricts the results of this paper to not-too-large wave numbers. The limits of validity of the theory will be discussed below.

V. PROBABILITY DISTRIBUTION OF THE VELOCITY DIFFERENCES

Let us consider the probability density of the velocity difference

$$U(X,x)=u(X+x)-u(X)$$
$$=k_0\sqrt{N_0}\int[a(k)+a^*(k)](e^{ikx}-1)d\mathbf{k}$$
$$+\int\varphi(k_1,k_2)a^*(k_1)a(k_2)(e^{i(k_1-k_2)x}-1)$$
$$\times d\mathbf{k}_1 d\mathbf{k}_2\ , \tag{5.1}$$

where we have omitted the subscript x denoting the x components of the velocity and wave vectors. It is well known that the odd moments $\overline{U^{2n+1}}$ are not equal to zero and that the PDF $P(U)\neq P(-U)$. However, this asymmetry is not very strong and it is absent when x denotes displacement in the direction perpendicular to the x axis. In this work we shall discuss the behavior of even moments only, leaving investigation of the odd-order moments to future publications. It is clear from the above definitions that

$$\overline{U^2}=U_R^2+U_c^2\ ,$$

with

$$U_R^2=\int\varphi_{k_1,k_2}^2 n(\mathbf{k}_1)n(\mathbf{k}_2)\{2-2\cos[(k_1-k_2)x]\}d\mathbf{k}_1 d\mathbf{k}_2$$
$$=O(x^{2/3})\equiv\beta x^{2/3} \tag{5.2}$$

and

$$U_c^2=4N_0k_0^2\int n(k)(\cos kx-1)d\mathbf{k}$$
$$=O(x^{4/3})\equiv\gamma\left[\frac{x}{L}\right]^{4/3}\ . \tag{5.3}$$

The proportionality coefficients β and γ are given by $\beta\approx\bar{\epsilon}^{2/3}$ and $\gamma\approx N_0k_0^2\bar{\epsilon}^{1/3}$. It will become clear below that in many cases $N_0k_0^2\approx S$, where S is the shear due to the coherent state which dominates the large-scale dynamics. The probability distribution function $P(U)$ is defined as

$$P(U)\propto\int\delta[U-u(x)+u(x+r)]d\mathbf{X}$$

or, using the integral representation of the δ function

$$P(U)\propto\int_{-\infty}^{\infty}d\alpha\, e^{i\alpha U}\int e^{-i\alpha U(x,r)}d\mathbf{x}\ .$$

Introducing the ensemble of fluctuating Clebsch fields and assuming that the space and ensemble averaging procedures are equivalent leads to

$$P(U)=\int_{-\infty}^{\infty}d\alpha\, e^{i\alpha U}\langle e^{-i\alpha U(0,x)}\rangle\ . \tag{5.4}$$

Our goal is to evaluate

$$I=\langle e^{-i\alpha U(0,x)}\rangle\ .$$

Assuming further that the statistics of the Clebsch fields $a(k)$ and $a^*(k)$ are near Gaussian for $k>k_0$, we can analyze the expression for I with $U(x)$ defined by (5.1). In principle the result of Gaussian averaging can be formally expressed in terms of the corresponding determinants. The resulting formulas are very involved and it is difficult to extract any useful information from them. Here I will expand I in powers of $i\alpha U(x)$ and evaluate the outcome of Gaussian averaging. First of all it is clear that if the quadratic contribution to (5.1) is small, the Gaussian averaging can be done exactly with the result

$$I=\exp(-\alpha^2U_c^2/2)\ ,$$

which after substitution into (5.4) yields the experimentally observed Gaussian distribution function

$$P(U)\propto\exp(-U^2/8U_c^2)$$

for large values of the displacement $x/L\to 1$. Now let us set for the time being $N_0=0$ and try to evaluate I. Due to the complicated functional shape of the matrix elements $\varphi(k_1,k_2)$, this is a rather difficult task. Expanding the exponent in I and examining the series in powers of α we find that the second-order term is equal to

$$I_2=-\tfrac{1}{2}\alpha^2U_R^2\ .$$

The fourth-order contribution is

$$I_4=\frac{\alpha^4}{24}(3U_R^4+6\mathcal{N}_4)\ ,$$

where $6\mathcal{N}_4$ denotes six off-diagonal contributions to I of the kind

$$\mathcal{N}_4=\int\varphi_{1,2}^0\varphi_{3,1}^0\varphi_{2,6}^0\varphi_{3,6}^0 n_1n_2n_3n_6\,d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3d\mathbf{k}_6\ ,$$

$$\varphi_{i,j}^0=\varphi(k_i,k_j)(e^{i(k_i-k_j)x}-1)\ .$$

All others are obtained from this one by permutation of the subscripts. It can be shown easily that the $2n$th-order term in the expansion has the form

$$I_{2n}=\frac{(-1)^n}{2n!}\left[B_{2n-2}\frac{2n(2n-1)}{2}U_R^{2n}+G_{2n-2}\frac{2n(2n-1)}{2}\mathcal{N}_{2n}\right],$$

where B_{2n-2} and G_{2n-2} denote the number of the corresponding contributions to I_{2n-2}. Neglecting the off-diagonal contributions we can sum up the remaining series with the result

$$I=\frac{1}{2}\left[1+\frac{\alpha^2U_R^2}{2}\right]^{-1}\ .$$

If all terms in the expansion are assumed to be equal, then we have the same result with the coefficient $\frac{3}{2}$ replacing $\frac{1}{2}$ in the above relation. Substituting the expression for I into (5.4) and evaluating a simple integral yields the exponential probability distribution function

$$P(U)\propto\exp\left[-\frac{\sqrt{2}U}{U_R}\right]\ .$$

This relation resembles the experimentally observed $P(U)$ for small separations $x/L\ll 1$. The exact evaluation of the function I is a difficult problem because the number of contributions to I_n involving various combinations of wave vectors grows very rapidly with n. This indicates the possibility to evaluate I_n for $n\gg 1$ using statistical methods of diagram calculations. Here I assume that when n is large the contribution of the off-diagonal terms is small due to the cancellations stemming from proliferation of $\cos[(k_i-k_j)x]$. So, I select an infinite subset of contributions to I with the correct asymptotic properties:

$$I=\frac{e^{-(1/2)\alpha^2 U_c^2}}{1+\alpha^2 U_R^2/2} . \tag{5.5}$$

The relation for $P(U)$ reads

$$P(U)=\int_{-\infty}^{\infty} d\alpha\, e^{i\alpha U}\frac{e^{-(1/2)\alpha^2 U_c^2}}{1+\alpha^2 U_R^2/2} .$$

It follows from the above considerations that this expression should be used only for the calculation of the high-order moments of velocity differences $\overline{U^{2n}}$ with $n \gg 1$. Substituting (5.5) into the expression for $P(U)$ gives $\int P(U)dU=1$. Evaluation of the moments of velocity differences

$$\overline{U^{2n}}=\int_{-\infty}^{\infty} P(U)U^{2n}dU$$

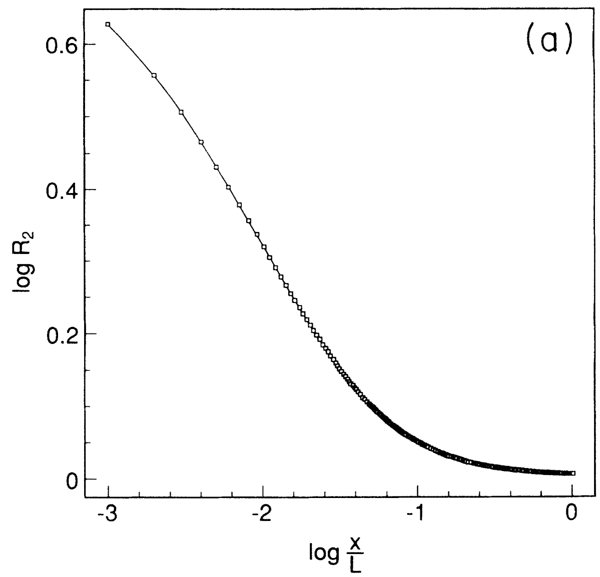
using (5.4) and (5.5) is done readily. The result for the normalized moments R_n is

$$R_{2n}=(2n-1)!!\frac{P^n}{P_n} , \tag{5.6}$$

where

$$P^n=\int_0^{\infty} dt\, e^{-t}\left[\left[\frac{x}{L}\right]^{2/3}+\frac{t}{C_0}\right]^n . \tag{5.7}$$

The parameter $C_0=2\gamma/2\beta$ and $P_n=(P^1)^n$. In the limit $x/L \geq 1$ and if the constant C_0 is large, this expression gives $R_n \approx (2n-1)!!$ corresponding to the close-to-Gaussian statistics of the velocity differences. When $x/L \ll 1$, $R_{2n}=(2n-1)!!n!$ indicating strong deviations of $P(U)$ from Gaussian. In wall flows, $\gamma \approx S$ and $\beta \approx 1/2\epsilon^{1/3}L^{2/3}$ where the shear $S \approx U/L$ and $\epsilon \approx u^3/L$ with U and u denoting the mean and rms fluctuating velocities, respectively. Not too far from the wall region of a boundary layer flow, $U \approx (10-20)u$ and $C_0 \approx 10-20$. Some of the normalized moments R_{2n}, given by (5.6) and (5.7), are plotted in Fig. 1 ($C_0=20$). The most striking feature of the plots is that the transition between these two asymptotic values of R_n is very slow, covering almost two decades of the displacement x variation. Moreover, the transitional region can be accurately represented by the power law $R_n \approx x^{-\xi_n}$ with the "scaling exponents" ξ_n. This power law is compared with the experimental data in Table I where the experimentally observed [6] values of the "exponents" ξ_n are presented. Relations (5.6) and (5.7) show that intermittency is the consequence of interaction of coherent and random components of the Clebsch field. With decrease of the length scale, the role of this interaction diminishes and PDF undergoes transition from Gaussian to close to exponential. At the very small scales $x/L \to 0$, the normalized moments $R_n(x) \to \text{const} \approx (2n-1)!!n!$. However, this limit might be beyond experimental reach since proximity to the dis-



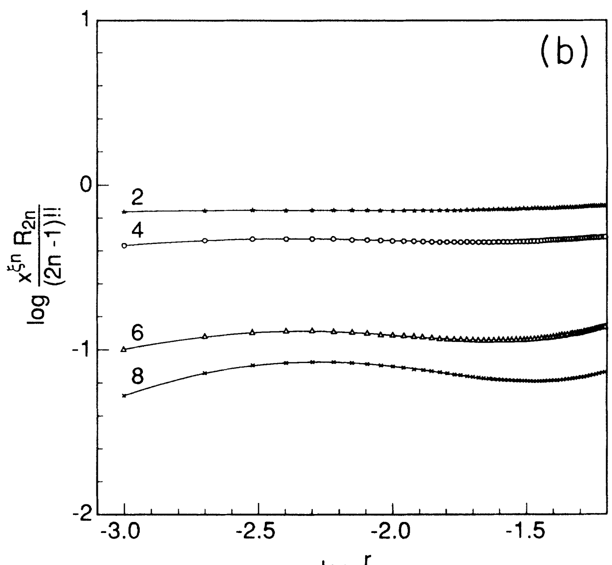


FIG. 1. (a) Typical form of the normalized moment $R_{2n}(x)$ plotted for $n=3$. (b) Compensated moments $x^{\xi_{2n}}R_{2n}(x)/(2n-1)!!$. The values of ξ_{2n} for $n=2$, 3, 4, and 5 are given in Table I.

TABLE I. Deviations of the scaling exponents of the moments of velocity differences from predictions of the Kolmogorov theory.

	Exponents ξ_{2n}		
$2n$	Experiment [5]	Present	Extrapolation
4	0.14±0.038	0.13	0.15
6	0.34±0.042	0.33	0.39
8	0.60±0.037	0.70	0.69
10	0.97±0.037	1.10	1.04
12	1.15±0.13		1.42

sipation range, not considered in this work, can strongly influence the small-scale behavior of the moments R_n. It follows from (5.7) that even in the same flow the measured deviations from the Kolmogorov scaling can vary from point to point depending on the ratio $S/\epsilon^{1/3}$ if the displacement is measured in the units of the integral scale $X=x/L$.

The results of this work can be explained in a very simple way. Let us consider, for example, the familiar case of a shear flow. At the large scales the dynamics are dominated by the shear S and the cospectrum of the anisotropic state $k^2\overline{v_i(k)v_j(k)}=e^{1/3}k^2n(k)\Omega$ where the characteristic frequency is $\Omega=S$. With the expression for $n(k)$ derived in this work, the $\frac{7}{3}$ energy spectrum is readily obtained in the interval $k_0<k<S^{3/2}\epsilon^{-1/2}$. The anisotropy-generated $\frac{7}{3}$ scaling of the cospectrum does not enter the expression for the energy spectrum. However, it does not disappear from the expressions for the high-order moments leading to the nontrivial behavior of $R_{2n}(x)$. I believe that this mechanism is quite general and the long-living anisotropic large-scale fluctuations can produce similar effects in statistically isotropic flows. Indeed, if the large-scale shear has complex topology, the arguments leading to the $-\frac{7}{3}$ cospectrum $E_{12}(k)\approx k^{-7/3}$ do not work since the problem loses strong anisotropy, which is the essential part of the derivation. In addition, the introduction of the large-scale superensemble, discussed in Sec. IV, makes the results of this work applicable to the statistically isotropic flows which are slightly anisotropic in each realization. These physical assumptions are reflected in the scalar expression for the condensate contribution to the dispersion relation in the equation of motion (4.2).

According to this theory, anomalous scaling is not necessary to explain the experimental data and the measured "scaling exponents" are the artifacts of signal processing. I illustrate how deceiving the data can be with the following consideration: The moments of the velocity differences grow by factor $\approx n!$ when x/L decreases by two orders of magnitude. Using a familiar extrapolation, the "exponents" ξ_n are found from the trivial relation $100^{\xi_n}=n!$, so that

$$\xi_n=\frac{1}{2}\log_{10}n!\ . \tag{5.8}$$

This formula is compared with the experimental data in Table I.

VI. DISCUSSION OF THE EXPERIMENTAL DATA

According to the theory presented in this work the large-scale coherent state, contributing to turbulence production in some flows, is responsible for the observed close-to-Gaussian probability distribution of velocity differences at large scales and the nontrivial behavior of high-order moments S_n. To begin to test the main predictions of the theory we have to look for the energy spectrum $E(k)\propto k^{-7/3}$ at scales somewhat larger than those corresponding to the $\frac{5}{3}$ inertial range spectrum. Experimental investigation of the large-scale dynamics is not a simple task. In a typical experiment the signal is acquired as a time sequence $v(x,t)$ at a given point x. The correlation functions are measured in the frequency domain:

$$F(\omega)=\int_{k_0}F(k,\omega+\mathbf{k}\cdot\mathbf{U})dk\ ,$$

where the integration domain is limited by the box size $O(k_0^{-1})$ and U is the mean velocity. The function $F(k,\omega)\approx k^xF(\omega/k^z)$ where the scaling exponents x and z differ from flow to flow (the dynamic exponent $z=\frac{2}{3}$ for Kolmogorov turbulence). The spectra correlation function is given by

$$F(k)=\int_{-\infty}^{\infty}F(k,\omega)d\omega\ ,$$

with the integration domain limited only by the data acquisition time which often can be made as long as desired. It is clear from these two expressions that when $\omega\gg O(k_0U)$ and the integrals converge, the expressions for $F(\omega)$ and $F(k)$ are simply related since $\omega\approx k$. This is called the Taylor hypothesis, which is very often used for interpretation of the data. However, this hypothesis fails at the large scales where $\omega\leq k_0U$ since in this case ω disappears from the problem, giving $F(\omega)=\text{const}$, observed in all experiments in the frequency domain. A few experiments conducted in both space and time domains revealed substantial differences in the large-scale behavior of the correlation functions.

The energy spectra, measured by high-Reynolds-number flow generated in the large NASA Ames wind tunnel [7], showed Kolmogorov $\frac{5}{3}$ energy spectrum over a decade of the wave-number variation. However, the cospectra $E_{1,2}(k)$ revealed a well-resolved range of the $\frac{7}{3}$ spectrum at the larger scales, in accord with the prediction in this work. As was mentioned above, the appearance of this spectrum in a shear flow is not surprising due to an additional scale-independent time constant. In another experiment [9] the second-order moment $S_2(x)$, measured in physical space using a particle-tracking technique, was bet fitted by the $\frac{2}{3}$ power law at the small scales and by $\frac{4}{3}$ exponent at the large values of the separation x. The results are shown in Fig. 2. We can see that when fitted by the Kolmogorov relation $S_2=O(x^{2/3})$ the experimental and fitting curves barely touched each other. However, the data were well represented by the dependence $S_2=ax^{2/3}+bx^{4/3}$, exactly as predicted in this paper. Even more striking confirmation of the predictions derived in this work can be found in the state-of-the-art (864^3 resolved Fourier modes) direct numerical simulations of the Taylor-Green vortex conducted by Brachet [8]. The calculated energy spectrum is presented in Fig. 3. We can see that the large-scale part of $E(k)$ is dominated by the $\frac{7}{3}$ scaling regime while the $\frac{5}{3}$ Kolmogorov spectrum can be found at the larger values of wave numbers.

In the Navier-Stokes equations different components of the velocity field are coupled via nonlinear interaction. It is a small miracle that the third-order structure function of the x component of velocity field can be expressed in terms of the x components only. This miracle is unlikely

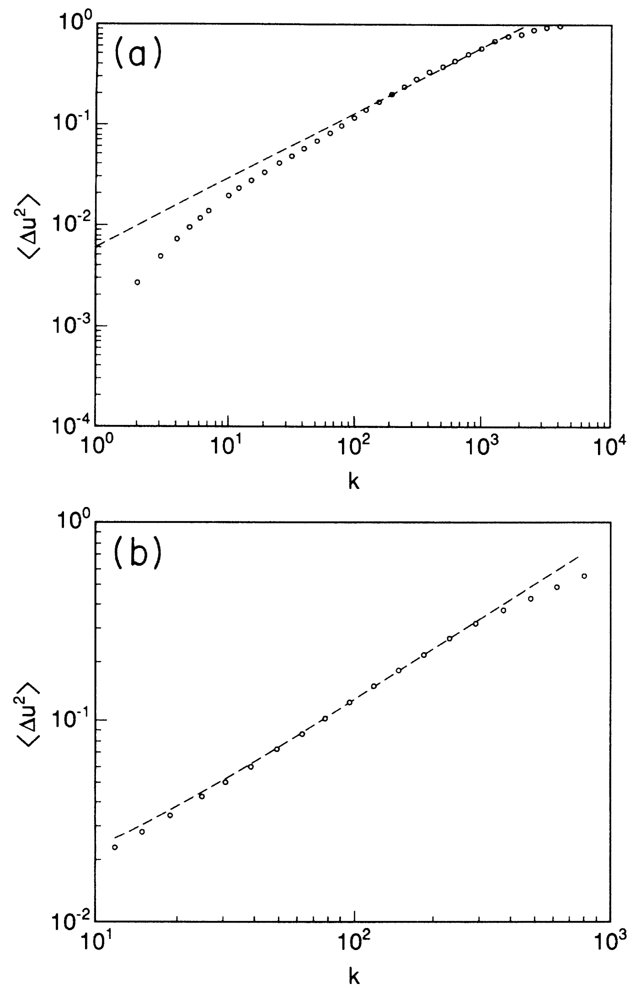


FIG. 2. Second-order structure function $S_2(x)$ Ref. [9] measured in the laboratory boundary layer. (a) Fitted by the Kolmogorov scaling; (b) the same data fitted by $S_2 = ax^{2/3} + bx^{4/3}$.

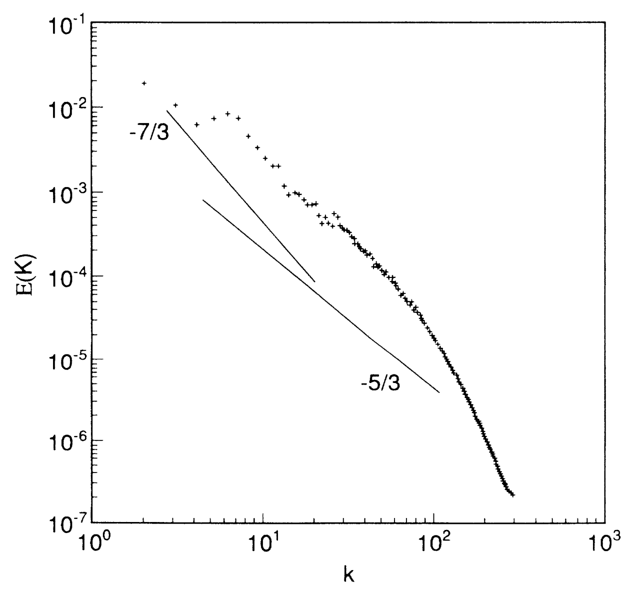


FIG. 3. The energy spectrum $E(k)$ from direct numerical simulations of the Taylor-Green vortex by Brachet [8].

to happen in the high-order correlation functions which means that the S_n's with large n are composed of contributions obeying different scaling laws. Examples of the experimentally observed compensated high-order moments are presented in Refs. [10–12]. From comparison of the graphs of Fig. 1 and those shown in Refs. [10–12] we conclude that the high-order moments, derived in this paper, though not obeying any real scaling laws, exhibit much more of a "scaling range" than the experimentally observed ones.

VII. CONCLUSIONS

The large-scale processes in all real-life turbulent flows are dominated by powerful, well-organized, long-living coherent structures. These structures participate in turbulence production and influence both global and small-scale features of the flow. It is always assumed in developing the theory of turbulence that in the so-called inertial range the large-scale processes and those happening in the viscosity-dominated dissipation range can be safely neglected. The role of the large-scale coherent state in the statistical properties of velocity functions has been analyzed in this work and it has been shown that, indeed, the influence of the coherent state diminishes with decrease of the length scale. However, the transition is very slow and experimentally observed behavior of the high-order structure functions can be mistakenly perceived as obeying nontrivial scaling laws.

The theory presented here is based on the assumption that even at the relatively high Reynolds numbers the coherent state in the external-field-driven flow reflects properties of this field. In principle, it is plausible to assume that the instability of the laminar velocity profile leads to a mean field having nothing in common with the laminar state of the flow due to the strong nonlinear interaction between coherent and fluctuating components. In this case the details of the external field are forgotten even in moderate-Reynolds-number situations. In my opinion, in the real-life flows this is not the case. The coherent structures, emerging at the transition to turbulence when $\mathrm{Re} \approx \mathrm{Re}_c$, persist even when $\mathrm{Re} \gg \mathrm{Re}_c$. This fact is readily understood: If the effect of the small-scale velocity fluctuations on the large-scale flow features can be described in terms of an eddy viscosity, then it is possible to show that the relevant effective Reynolds number $\mathrm{Re} = v_{\mathrm{rms}} L / \nu_{\mathrm{eff}} = O(1) \approx \mathrm{Re}_c$. Thus the large-scale flow is always transitional and is strongly influenced by the driving mechanism. This fact can be easily verified for the wall-shear flows, convection, jets, and mixing layers by simply using the data on the turbulent intensities (u_{rms}) in the definition of the turbulent viscosity $\nu_{\mathrm{eff}} = \overline{v_i v_j} / (\partial U_i / \partial x_j)$. In all these cases the effective Reynolds number is close to critical and that is why the large-scale flow features resemble organized structures characteristic of transitional flows. Experimental manifestation of this fact can also be found in the data on the energy spectra as a function of dimensionless frequency f. For example, in the wake behind the cylinder the spectrum $E(f)$ has a very sharp maximum at $f \approx 0.2$ independent of the Reynolds number in a very wide range of Re variation. A similar feature can be found in Benard convection where the characteristic velocity of the large-scale coherent vortex (wind) scales with the Rayleigh number as $U \approx \mathrm{Ra}^{1/2}$ which corresponds to the simple relation $U \propto \sqrt{gL}$ where g is the gravitational acceleration driving the flow. As we see, the external field g plays a very important part even at large Ra. These facts, in my opinion, support the main assumption that the coherent state can be treated as an external field.

According to the theory developed here, the existence of the large-scale coherent state explains the experimentally observed Gaussian statistics of the velocity fluctuations. Indeed, the Gaussian field cannot participate in the energy transfer from large- to small-scale fluctuations. This means that only the fluctuations $v(k)$ with $k > (S^3/\bar{\epsilon})^{1/2} \gg k_0$ can give substantial contribution to the effective transport coefficient. The same estimate holds for the rate of diminishing of the large-scale generated anisotropy with decrease of the wave number. The anisotropy of the velocity fluctuations makes the energy transfer from large to small scales more difficult, which is another reason for the relatively weak contribution of the large-scale velocity fluctuations to the effective transport. Thus there exists some scale separation, characterized by the small parameter $u_{\mathrm{rms}}/U \ll 1$ justifying the concept of turbulent viscosity which is so successful in the modeling of the large-scale features of turbulent flows.

The main unresolved problem with the Clebsch formulation of statistical hydrodynamics is nonuniqueness: it follows from (2.4) that any transformation $\lambda \to f(\lambda,\mu), \mu \to \phi(\lambda,\mu)$ with

$$\frac{\partial f}{\partial \lambda}\frac{\partial \phi}{\partial \mu} = 1$$

does not change the velocity field. The parameter

$$N' = \int (f^2 + \phi^2) d^3x \neq N_0 \ .$$

This is a problem since N_0 is an important parameter of the theory determining the constant frequency shift Ω. At the present time we do not have a rigorous way to gauge the Clebsch field and determine N_0. However, the only large-scale physically relevant frequency $\Omega = N_0 k_0^2 \approx S$ which can be serve as an estimate for N_0 since $k_0 = O(1)$. Thus the large-scale dynamics are strongly influenced by the interactions of the fluctuating and coherent components of the Clebsch field.

It is possible that this is only a part of the picture. The viscosity-dominated small-scale dynamics can produce another mechanism of deviation from Kolmogorov scaling. The physical reasons for this can be illustrated by the definition of the high-order moments of velocity differences:

$$S_{2n} = \overline{\Pi[u(X_i) - u(X_i + x)]}$$

in the limit when all $X_i \to X_j$ and $2 < i < 2n$. We can see that S_{2n} are mixed quantities, involving not only inertial range separation x but also dissipation range displacements $X_i - X_j \to 0$. Thus S_{2n} are not pure inertial range properties of the flow and corrections to the Kolmogorov scaling are expected. If this is so, then we can predict a crossover from the large-scale "scaling," dominated by the coherent structures, to another one reflecting intermittency of fully developed turbulence due to the viscous contribution to the equations of motion.

To conclude this paper I would like to mention that the $E_{1,2} \propto k^{-7/3}$ has been obtained by Lumley [13] and by Nelkin and Nakano [14] on the basis of dimensional considerations applied to the boundary layer flows. It has also been demonstrated by Katz and Kontorovich (see Ref. [3]) that a similar solution arises as a small perturbation to the Kolmogorov spectrum caused by weak linear anisotropy. It is shown in this work that the long-living large-scale coherent structures should lead to the $\frac{7}{3}$ energy spectrum at large enough scales even in statistically isotropic flows. Moreover, interaction of coherent and random components is responsible for the close-to-Gaussian statistics of the velocity field at large scales and for the deviations from the Kolmogorov scaling of the high-order moments of velocity differences.

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