PHYSICAL REVIEW E VOLUME 49, NUMBER 4 APRIL 1994

Colored noise in spatially extended systems

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(Received 29 October 1993)

We study the effects of time and space correlations of an external additive colored noise on the steady-state behavior of a time-dependent Ginzburg-Landau model. Simulations show the existence of nonequilibrium phase transitions controlled by both the correlation time and length of the noise. A Fokker-Planck equation and the steady probability density of the process are obtained by means of a theoretical approximation.

PACS number(s): 05.40.+j, 02.50.−r

I. INTRODUCTION

A. General aspects

Langevin equations in spatially extended systems are accepted as a common reference frame in the study of those equilibrium and nonequilibrium phenomena in which fluctuations play a relevant role. This kind of equations has been used in the study of critical phenomena [1], phase-separation dynamics [2], instabilities in liquid crystals [3], and bistability in chemical reactions [4,5], among a large variety of systems. Fluctuations can have an internal (thermal) or external origin with respect to the system under study. Internal fluctuations have been considered in critical phenomena and phase separation, whereas external fluctuations have been mostly studied in relation to liquid crystals and chemical reactions. In these last cases internal fluctuations are also present, although they are much less relevant than the external ones.

Statistical mechanics shows that internal fluctuations produce phase transitions from an ordered to a disordered state as their intensity increases. These phenomena are described by means of a singular behavior of the relevant variable (order parameter) in the vicinity of the transition point. In this paper we will consider the situation in which fluctuations are external (not thermal) and study the possible existence of nonequilibrium transitions controlled by this sort of fluctuation.

A prototype Langevin equation can be written in the general form

$$\dot{\psi}(\mathbf{r},s) = f(\psi(\mathbf{r},s),\nabla,\alpha) + g(\psi(\mathbf{r},s),\nabla)\eta(\mathbf{r},s). \quad (1.1)$$

The deterministic force f depends in general on the field variable ψ, its spatial derivatives, and a set of control parameters α. This force comes either from reaction-diffusion terms or from a free-energy functional. The spatial derivatives model the coupling of the field at one given point with its value in the neighborhood. The existence of such a coupling implies that the Langevin equation (1.1) will not be an ordinary but a *partial* stochastic differential equation, whose rigorous mathematical study is nowadays under active research.

The stochastic force represents the influence of the surroundings, a heat bath, other internal degrees of freedom, or a stochastic external control parameter. It is usually supposed to be proportional to a noise term η. The form of g depends on the kind of coupling between the noise and the variable. When the noise is not coupled to the field, g takes a constant value and the noise is said to be *additive*. When some kind of coupling between the field and the noise exists, the function g depends on the field. This is the so-called *multiplicative* noise.

On behalf of the central limit theorem the random variable η can be supposed to be Gaussian distributed with zero mean. Its correlation at different points and instants of time will in general be given by

$$\langle\, \eta(\mathbf{r},s)\, \eta(\mathbf{r}',s')\,\rangle = D\, h\left(\frac{\mathbf{r}-\mathbf{r}'}{l_0};\frac{|\, s-s'\,|}{\sigma}\right), \quad (1.2)$$

where D is the intensity of the noise, σ is its correlation time and l_0 its correlation length. When there is no correlation either in space or in time the function h becomes the product of two δ functions: it is the *white*-noise case. In any other case the noise is said to be *colored.*

A noise accounting for fluctuations of internal (thermal) origin is supposed to be uncoupled from the system. Moreover one assumes that it is white in space and time. This is so because the noise represents many microscopic degrees of freedom which evolve in spatial and temporal scales much shorter than those of the relevant variables of the system. Hence internal noise is usually modeled by additive white noise.

When the origin of the noise is external there may exist a coupling between the system and the fluctuations. Besides, in general there is no difference between the time and length scales of the noise and the field and one has to take into account the fact that this noise could have

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some structure either in space or in time. Thus external noise can be colored and/or multiplicative.

We are going to focus on the case of a potential system affected by an external additive noise ($g = 1$). When this noise is white the steady probability distribution of the field can be obtained *exactly* for suitable boundary conditions. When it is colored this is not possible in general, and only *approximate* techniques can be used. Actually, difficulties here arise from a correlation of the noise in *time*, in which case the stochastic process defined by Eq. (1.1) is *non-Markovian.* Correlation in space does not present this type of mathematical difficulties.

Non-Markovian stochastic processes in zero dimensions [no gradient terms in Eq. (1.1)] have been thoroughly studied by now (see Ref. [6], Vol. 1, for a review). Among the interesting systems where colored-noise effects have been studied one finds lasers [7], chemical reactions [5], and liquid crystals [3]. Time correlations in the noise happen to induce transitions in such systems [5]. Due to the zero-dimensional character of these systems, these transitions can only be characterized by means of qualitative changes in the shape of the steady probability distribution of the variable (for instance, a change in the number of its minima and maxima). These effects have been observed in analog [8] and digital simulations [9] and have been explained analytically [9,10].

Stochastic processes in $d > 0$ dimensions affected by space-and-time colored noises have not been studied so extensively. In this case, and due to the spatial character of the system, an ergodicity-breaking effect permits a phase transition to be characterized by means of a singular behavior of an order parameter of the system, its relative fluctuations, and its statistical moments and correlations, such as is done in equilibrium phase transitions.

In this paper, our aim is to study the effects of the correlation time and length of an additive noise in the behavior of a two-dimensional stochastic model.

B. The time-dependent Ginzburg-Landau model with colored noise

We will study a model given by the following Langevin field equation:

$$\frac{\partial \phi(\mathbf{r},s)}{\partial s} = \Gamma\left(b\phi - u\phi^3 + k\nabla^2\phi\right) + \Gamma^{1/2}\eta(\mathbf{r},s) . \quad (1.3)$$

In the study of critical phenomena [1] and phase-separation dynamics [2], the noise term has been assumed to be Gaussian and white, with zero mean and correlation

$$\langle\, \eta(\mathbf{r},s)\,\eta(\mathbf{r}',s')\,\rangle = 2D\,\delta(\mathbf{r}-\mathbf{r}')\,\delta(s-s') , \quad (1.4)$$

with $D = k_B T$ (the so-called *intensity* of the noise). In this case the model is known as the nonconserved time-dependent Ginzburg-Landau model (model A in the notation of Ref. [1]). Here a fluctuation-dissipation relation ensures that the steady-state probability distribution of ϕ is given by the Boltzmann expression in terms of the following Ginzburg-Landau free energy:

$$F[\phi(\mathbf{r},s)] = \int \left\{ -\frac{1}{2}b\phi^2 + \frac{1}{4}u\phi^4 + \frac{1}{2}k\left(\nabla\phi\right)^2 \right\} d\mathbf{r} . \quad (1.5)$$

When $b > 0$, this function has a double-well structure which can be overridden by a high-intensity noise. Thus a phase transition will be observed from an ordered towards a disordered state when the intensity of the noise increases beyond a critical value [11].

In this paper we are interested in the effects of a correlated noise. Hence we will assume that the stochastic force in Eq. (1.3) is also Gaussian distributed with zero mean but with a correlation given by the general expresion of Eq. (1.2). Now there is no fluctuation-dissipation relation, so that one cannot expect to reach an equilibrium steady state such as the one given by the free energy (1.5).

In order to analyze the effects of the noise correlations in the dynamics of the stochastic process governed by Eq. (1.3) we start by removing all unnecessary parameters. This will be done by means of an adimensionalization of the equation through the following change of variables:

$$\phi = \sqrt{\frac{b}{u}}\,\psi, \qquad s = \frac{1}{2\Gamma b}\,t, \qquad \mathbf{r} = \sqrt{\frac{k}{b}}\,\mathbf{x} . \quad (1.6)$$

Now the Langevin equation corresponding to our model is

$$\frac{\partial\psi(\mathbf{x},t)}{\partial t} = \frac{1}{2}\left(\psi - \psi^3 + \nabla^2\psi\right) + \xi(\mathbf{x},t) , \quad (1.7)$$

$$\langle\, \xi(\mathbf{x},t)\xi(\mathbf{x}',t')\,\rangle = \varepsilon\, h\left(\frac{\mathbf{x}-\mathbf{x}'}{\lambda}; \frac{|\,t-t'\,|}{\tau}\right) . \quad (1.8)$$

Thus we now have only three dimensionless parameters

$$\varepsilon = \frac{u}{2b^{2-d/2}k^{d/2}}\,D, \qquad \tau = 2\Gamma b\,\sigma, \qquad \lambda = \sqrt{\frac{b}{k}}\,l_0. \quad (1.9)$$

The rest of the paper is organized as follows. In Sec. II the simulation procedure and its results are presented and in Sec. III a theoretical approximate approach to the problem is developed. Finally some conclusions and comments are stated.

II. A NUMERICAL APPROACH

Due to the nonlinear character of Eq. (1.7) an exact theoretical treatment of the problem cannot be done, and therefore a numerical approach is required. Simulations are performed on a Cray YMP computer where vectorization, but not parallelization, is used. Equations (1.7) and (1.8) have been simulated on a regular two-dimensional lattice with $L \times L$ square cells of size $\Delta x = 1$. In this space, the Langevin equation has the form

$$\frac{\partial\psi_i}{\partial t} = f_i(\boldsymbol{\psi}) + \xi_i(t) , \quad (2.1)$$

where the cells have been named with one index independently of the dimension of the discrete space. And the correlation (1.8) of the noise in this discrete space is

$$\langle \xi_i(t)\, \xi_j(t') \rangle = \varepsilon h_{ij} \left(\frac{|\, t - t' \,|}{\tau} \right) . \tag{2.2}$$

The force is given by

$$f_i(\psi) = \frac{1}{2} \left[\psi_i - \psi_i^3 + \left(\nabla^2 \psi \right)_i \right] , \tag{2.3}$$

where $\left(\nabla^2 \psi\right)_i = \nabla^2_{ik} \psi_k$ and ∇^2_{ik} is a discretized version of the Laplacian operator [12] (repeated indexes are summed up).

The stochastic correlated force $\xi_i(t)$ is simulated by means of the following Langevin equation [13]:

$$\dot{\xi}_i(t) = -\frac{1}{\tau} \left(\delta_{ij} - \lambda^2 \nabla^2_{ij} \right) \xi_j + \frac{1}{\tau}\, \mu_i(t) , \tag{2.4}$$

which is a generalization of the evolution equation of an Ornstein-Uhlenbeck process [9]. $\mu_i(t)$ is a Gaussian white noise with zero mean and intensity equal to ε, which can be efficiently generated by means of a vectorizable approximate algorithm implementing a numerical inversion method [14]. The Laplacian term ensures a correlation in space of order λ and τ is the correlation time. Figure 1 shows the spatial decay of the spherically averaged correlation function of the noise for several values of λ. There one can see that λ is indeed a measure of the correlation length of the stochastic force. A similar picture is obtained if the correlation is plotted versus $t - t'$ [13].

Equation (2.4) is linear, so that it can be simulated *exactly* in Fourier space [13] and antitransformed to real space along with the integration of Eq. (2.1). This last numerical integration cannot be exact [$f_i(\psi)$ is not linear], so that a second order Runge-Kutta algorithm has been used. This type of algorithm allows us to take a relatively large integration time step ($\Delta t = 0.05$) with no loss of stability.

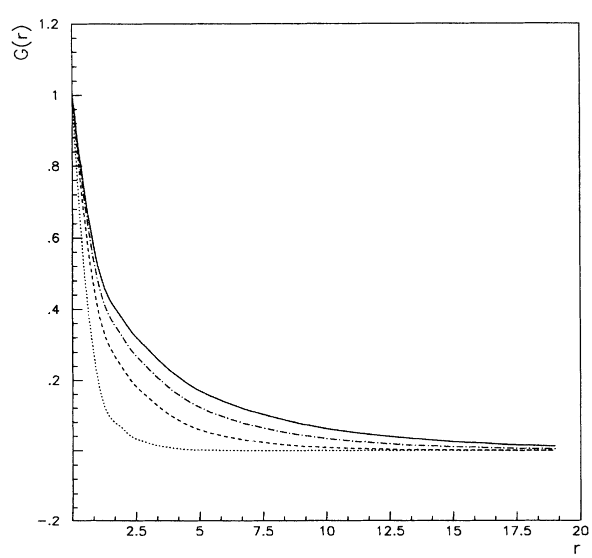

FIG. 1. Spherically averaged spatial correlation function for the noise driven by Eq. (2.4) for four different values of the correlation length. The solid line corresponds to $\lambda = 7$, the dashed-dotted line to $\lambda = 5$, the dashed line to $\lambda = 3$, and the dotted line to $\lambda = 1$.

The quantities which are computed and analyzed in the simulation are the following [15,16]: (1) the steady mean density of the absolute value of the field:

$$M_1 = \frac{\langle m \rangle}{L^2} , \tag{2.5}$$

where $m = |\, \sum_{i,j} \psi_{ij} \,|$, and the average is made over the time evolution of the process in the steady state and over different realizations of the noise, (2) the relative fluctuations of the field:

$$M_2 = \frac{\langle m^2 \rangle - \langle m \rangle^2}{L^2 \varepsilon} , \tag{2.6}$$

and (3) the linear relaxation time of the process, defined as

$$\tau_R^{-1} = - \left. \frac{d}{dt} C(t) \right|_{t=0} , \tag{2.7}$$

where the correlation function $C(t)$ is

$$C(t) = \frac{\langle m(t_0) m(t_0 + t) \rangle_{t_0} - \langle m \rangle^2}{\langle m^2 \rangle - \langle m \rangle^2} . \tag{2.8}$$

The system is made to evolve from an initially ordered state, $\psi = 1$, and we let it relax for an interval of time large enough to be close to the steady state, before performing the time averages. On the other hand, the number of samples in the collectivity averages is between 20 and 40. Periodic boundary conditions are considered in all cases and a finite-size analysis is performed by considering five different system sizes. It is worth noting that in this kind of numerical analysis the space discretization Δx is an independent parameter affecting the steady behavior of the system [11], but we have not explored this situation and have instead kept Δx fixed and equal to 1.

In a previous work [16] we reported results obtained for the case of a noise correlated in time ($\tau \neq 0$) but uncorrelated in space ($\lambda = 0$). In this case the algorithm for the generation of the noise is simpler than the Fourier algorithm mentioned above. Figure 2(a) shows clearly the effect of τ in the transition, as obtained in the simulations. When $\tau = 0$ (the standard white-noise case) a transition towards disorder is found at a critical noise intensity $\varepsilon_c = 0.38$. This result is in agreement with a study of the ϕ^4 model in the case of white noise [11]. As τ increases the critical intensity is shifted towards higher values (the peak of the relative fluctuations indicating the transition moves to the right). This means that τ somehow "softens" the effect of the noise, whose intensity needs to be higher to destroy the initial ordered state. Figure 2(b) shows the phase diagram of this system, where a curve in the (ε, τ) plane divides regions where the system is ordered and disordered. The sign of the slope of this curve is another indication of the "softening" effect of the correlation time of the noise.

Figure 3 shows the effects of τ on the three quantities defined above for a fixed value of ε and still for $\lambda = 0$.

Peaks for the relative fluctuations and the relaxation time are obtained for approximately the same value of $1/\tau$ and a decay of the mean value of the field shows that there is a transition from an ordered to a disordered state as τ decreases. We have thus found a phase transition controlled by the correlation time of the noise.

In Fig. 4 results are presented for fixed values of ε and τ against a decreasing value of λ. A similar behavior is obtained, showing the existence of a phase transition controlled by the correlation length of the noise. The role of the correlation length is also a "softening" one.

A finite-size scaling analysis applied to the two transitions found above allows us to evaluate the position of the critical point in both cases and to give an estimation

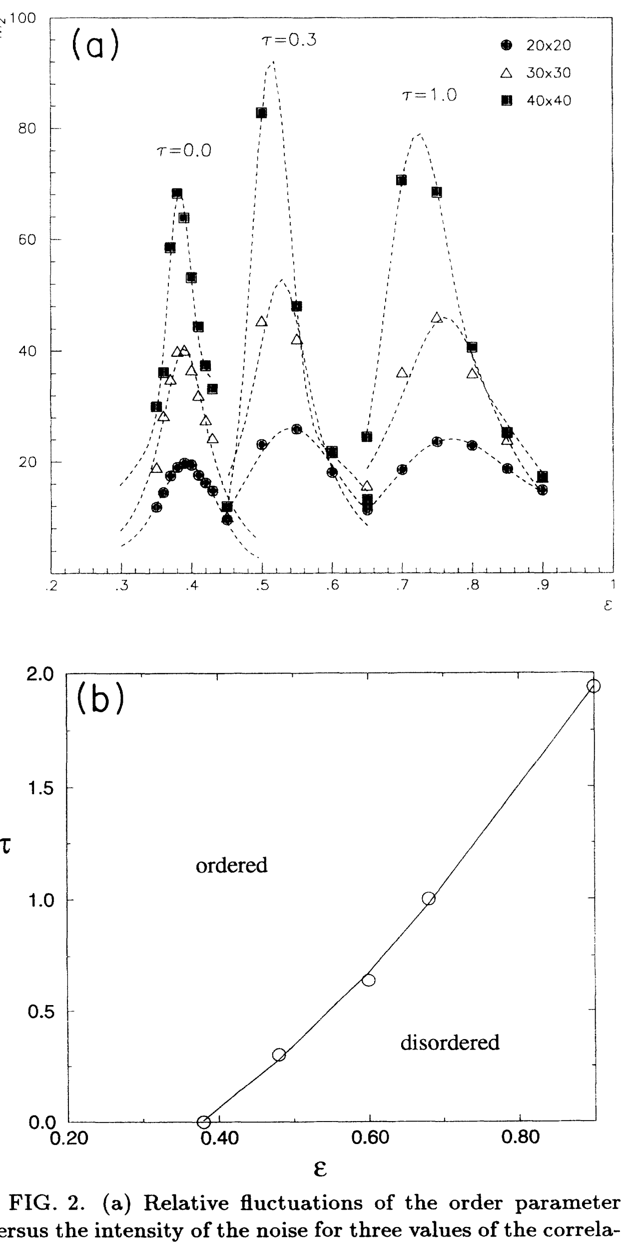

FIG. 2. (a) Relative fluctuations of the order parameter versus the intensity of the noise for three values of the correlation time; (b) phase diagram of the model. Noise is correlated only in time ($\lambda = 0$).

of the values of the critical exponents associated to them. Concerning the τ-controlled transition (Fig. 3), the critical point is found to be located at $\tau_c^{-1} = 1.0$, whereas in the case of the λ-controlled transition (Fig. 4) we find $\lambda_c^{-1} = 1.9$. These transition points are determined by extrapolating to infinite size the position of the maximum of both the relative fluctuations M_2 and the linear relaxation time τ_R, and the results obtained by the two methods coincide within the estimated numerical error ($\sim 10\%$). Static critical exponents can be calculated by means of the following finite-size scaling relations [11,18]:

$$\begin{aligned} &\mid \alpha_c(L) - \alpha_c \mid \sim L^{-1/\nu} , \\ &M_1 \sim \mid \alpha_c - \alpha \mid^{\beta} , \\ &M_2^{\max} \sim L^{\gamma/\nu} , \end{aligned} \tag{2.9}$$

where $\alpha = \tau^{-1}$ or λ^{-1}. The dynamic critical exponent z can be found with the relation [19]

$$\tau_R^{\max} \sim L^z . \tag{2.10}$$

The numerical results obtained for these exponents for the τ transition (Fig. 3) are $\beta \simeq 0.19$, $\nu \simeq 0.78$, $\gamma/\nu \simeq 1.6$, and $z \simeq 1.7$. For the λ transition (Fig. 4) we obtain $\beta \simeq 0.14$, $\nu \simeq 0.99$, $\gamma/\nu \simeq 1.6$, and $z \simeq 1.7$. The comparison between the nonequilibrium static exponents obtained here and the exact values corresponding to the equilibrium Ising model is not straightforward. This is due to several reasons: first, ours is a nonequilibrium model; second, numerical errors involved are important; and third, one can expect crossover effects coming from the fact that Δx is not zero [20]. Concerning our results for the dynamical exponent z, they are in accordance with recent estimates for the equilibrium version of this model [21]. Furthermore, a simple dynamical scaling analysis such as the one performed in Ref. [22] gives no changes of the colored-noise exponents with respect to the white-noise case. This is due to the fact that correlation of the noise (1.8) is not a power law either in space or in time. Hence in principle one cannot expect large changes in our exponents as compared to the values of the equilibrium ϕ^4 model.

III. AN APPROXIMATE THEORETICAL APPROACH

The steady probability density of the non-Markovian stochastic process defined by Eq. (1.7) can be found by discretizing space in a regular d-dimensional lattice with spacing Δx. In this lattice, Eq. (1.7) has the form (2.1). The approximate Fokker-Planck equation corresponding to this discrete Langevin equation for small λ and τ can be shown to be

$$\frac{\partial P}{\partial t} = \left(- \frac{\partial}{\partial \psi_i} f_i + \varepsilon \frac{\partial^2}{\partial \psi_i^2} + \varepsilon \lambda^2 \nabla_{ik}^2 \frac{\partial^2}{\partial \psi_i \partial \psi_k} + \varepsilon \tau \frac{\partial^2}{\partial \psi_i \partial \psi_k} \frac{\partial f_k}{\partial \psi_i} \right) P . \tag{3.1}$$

Details of the derivation of this equation are presented in Appendix A. The discretized version of the Laplacian operator can be written in terms of forward and backward finite differences [12]:

$$\nabla^2_{il} = \nabla^+_{ik}\nabla^-_{kl} \, ,$$

$$\text{with } \nabla^+_{ij} \equiv \delta_{i+1,j} - \delta_{ij} \, , \; \nabla^-_{ij} \equiv \delta_{i,j} - \delta_{i-1,j} \, . \quad (3.2)$$

It can easily be seen from these definitions that $\nabla^+_{ij} = -\nabla^-_{ji}$. Moreover, the following relations hold:

$$\frac{\partial}{\partial\psi_l}\sum_j \left(\nabla^+\psi\right)^2_j = -2\left(\nabla^2\psi\right)_l \, , \quad (3.3a)$$

$$\frac{\partial}{\partial\psi_l}\sum_j \left(\nabla^2\psi\right)^2_j = 2\left(\nabla^4\psi\right)_l \, , \quad (3.3b)$$

$$\frac{\partial}{\partial\psi_l}\sum_j \psi^3_j\left(\nabla^2\psi\right)_j = \left(\nabla^2\psi^3\right)_l + 3\psi^2_l\left(\nabla^2\psi\right)_l \, , \quad (3.3c)$$

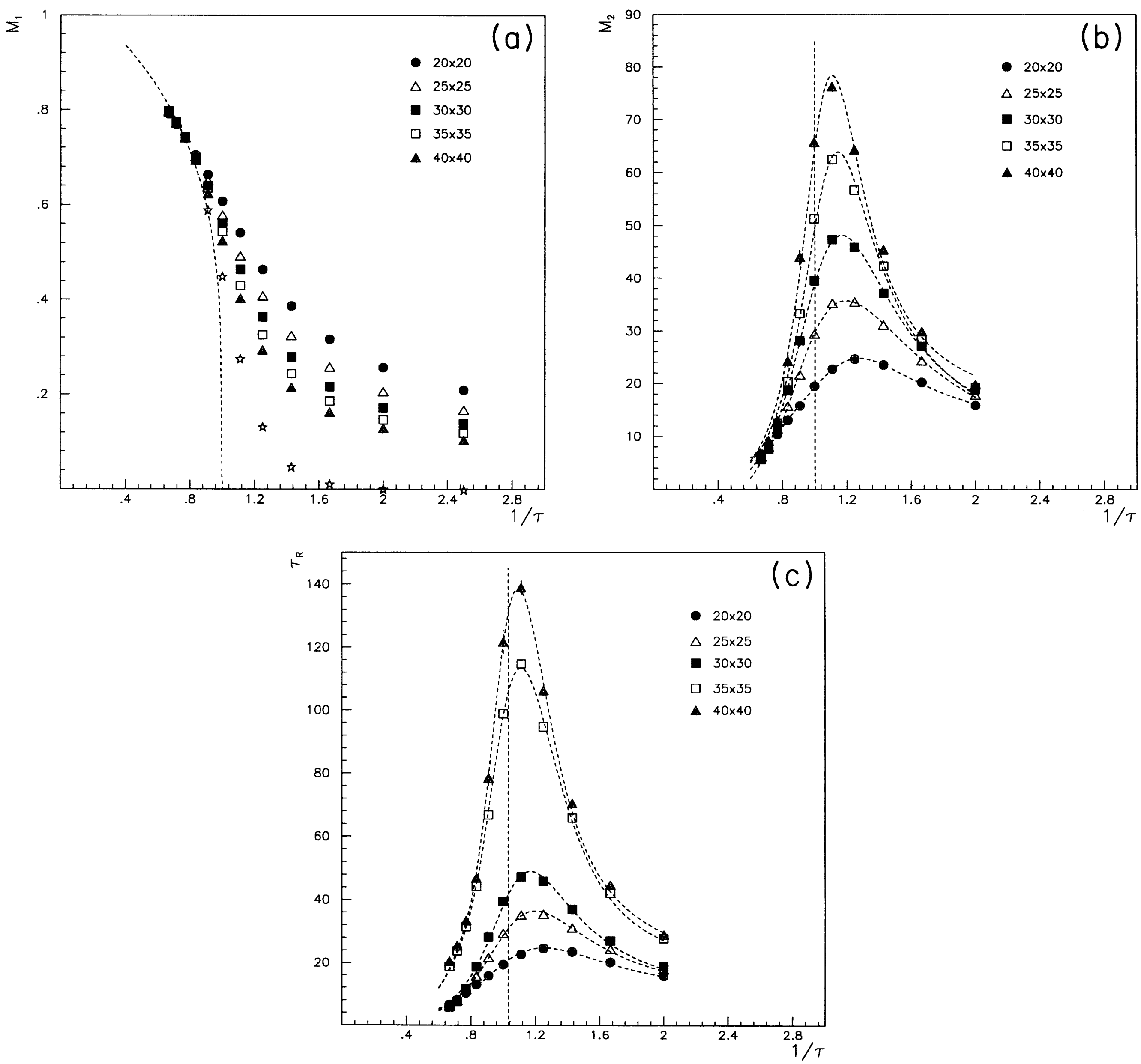

FIG. 3. Numerical results for $\varepsilon = 0.7$ and $\lambda = 0$. The order parameter M_1 (a), the relative fluctuations of the field M_2 (b), and the linear relaxation time τ_R (c) are shown against the inverse of the correlation time of the noise for different sizes of the system. The broken lines are a guide to the eye. In (a) empty stars correspond to an extrapolation to infinite size. In (b) and (c) the vertical dashed lines denote the position of the transition point also extrapolated to the thermodynamic limit.

$$\frac{\partial}{\partial\psi_l}\sum_j \psi_j^2\left(\nabla^2\psi^2\right)_j = 4\psi_l^2\left(\nabla^2\psi\right)_l \,. \tag{3.3d}$$

The equation that the steady solution of (3.1) must verify is

$$\left(-f_i + \varepsilon\frac{\partial}{\partial\psi_i} + \varepsilon\lambda^2\nabla^2_{ik}\frac{\partial}{\partial\psi_k} + \varepsilon\tau\frac{\partial}{\partial\psi_k}\frac{\partial f_k}{\partial\psi_i}\right) P_{st} = 0, \tag{3.4}$$

where the probability flux has been taken to be zero, as usual. We will assume now that the solution of this equation has the form [17]

$$P_{st} \sim e^{-(F_0 + F_1\tau + F_2\lambda^2)/\varepsilon} \,. \tag{3.5}$$

It is evident that F_0 corresponds to the solution of the Fokker-Planck equation for the white-noise case ($\tau = 0$, $\lambda = 0$):

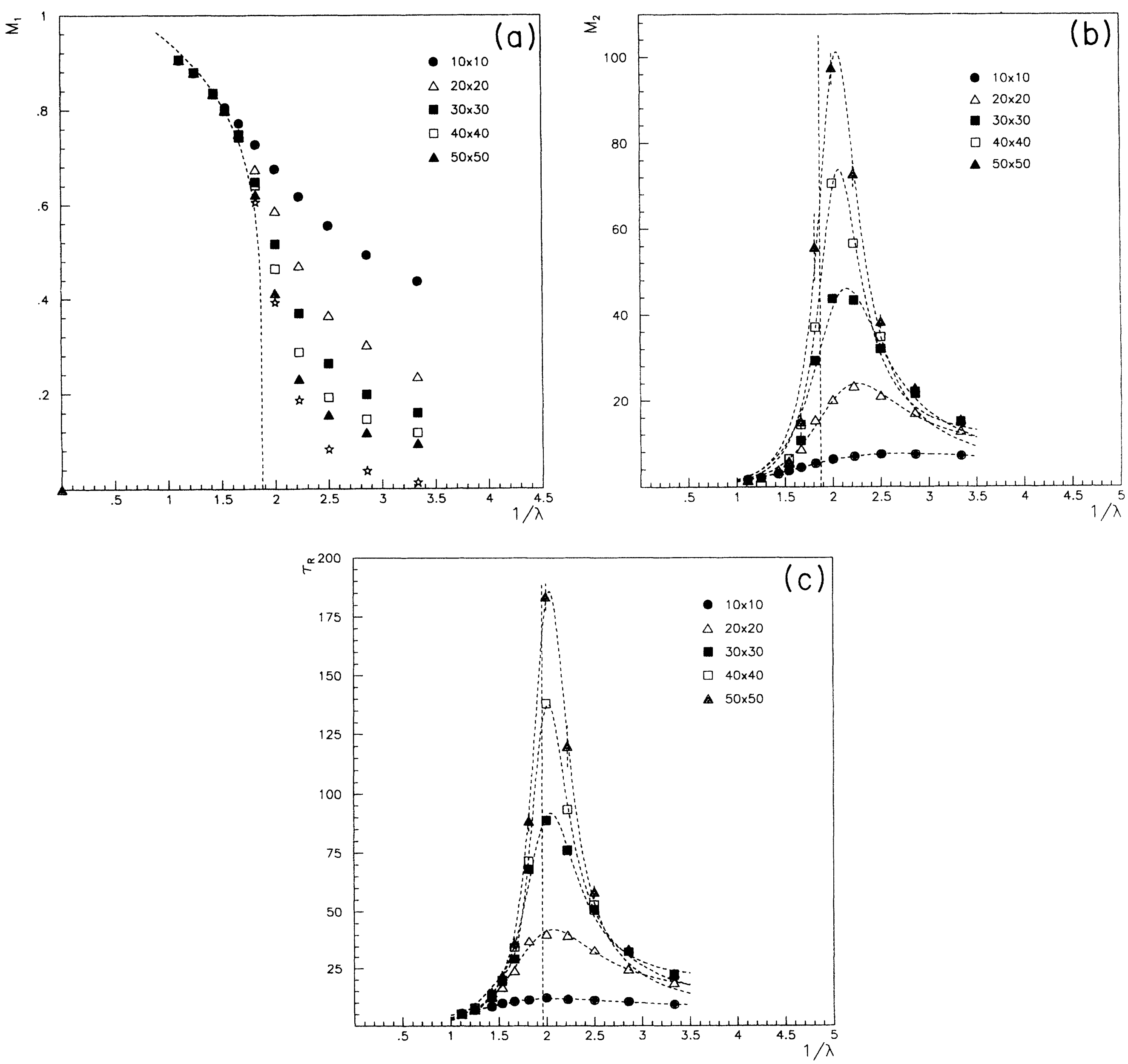

FIG. 4. Numerical results for $\varepsilon = 0.8$ and $\tau = 0.3$. The order parameter M_1 (a), the relative fluctuations of the field M_2 (b), and the linear relaxation time τ_R (c) are shown against the inverse of the correlation length of the noise for different sizes of the system. The broken lines are a guide to the eye. In (a) empty stars correspond to an extrapolation to infinite size. In (b) and (c) the vertical dashed lines denote the position of the transition point also extrapolated to the thermodynamic limit.

$$\left(-f_i + \varepsilon \frac{\partial}{\partial \psi_i}\right) P_{\mathrm{st}} = 0 . \qquad (3.6)$$

Introduction of $P_0 \sim \exp(-F_0/\varepsilon)$ leads to the following differential equation for F_0:

$$\frac{\partial F_0}{\partial \psi_l} = -f_l , \qquad (3.7)$$

which can be immediately solved after an inspection of relation (3.3a), leading to

$$F_0 = \frac{1}{4} \sum_k \left(-\psi_k^2 + \frac{1}{2}\psi_k^4 + \left(\nabla^+\psi\right)_k^2\right) . \qquad (3.8)$$

The next step consists of introducing the ansatz (3.5) into Eq. (3.4). When only the first nonzero orders in τ and λ^2 are considered and Eq. (3.7) is taken into account the following equality is obtained:

$$\left(-\tau \frac{\partial F_1}{\partial \psi_i} - \lambda^2 \frac{\partial F_2}{\partial \psi_i} + \lambda^2 \nabla_{ik}^2 f_k + \tau\varepsilon \frac{\partial^2 f_k}{\partial \psi_k \partial \psi_i} + \tau f_k \frac{\partial f_k}{\partial \psi_i}\right) \times P_{\mathrm{st}} = 0 . \qquad (3.9)$$

And by comparing coefficients one finds the following differential equations for F_1 and F_2:

$$\frac{\partial F_1}{\partial \psi_i} = \varepsilon \frac{\partial^2 f_k}{\partial \psi_k \partial \psi_i} + f_k \frac{\partial f_k}{\partial \psi_i} , \qquad (3.10)$$

$$\frac{\partial F_2}{\partial \psi_i} = \nabla_{ik}^2 f_k . \qquad (3.11)$$

These equations can be easily solved by considering the definition of f_k given in (2.3) and taking into account relations (3.3). The solution is

$$F_1 = \frac{1}{4} \sum_i \left[\frac{1}{2}(1 - 12\varepsilon)\psi_i^2 - \psi_i^4 + \frac{1}{2}\psi_i^6 - \left(\nabla^+\psi\right)_i^2 - \psi_i^3 \left(\nabla^2\psi\right)_i + \frac{1}{2}\left(\nabla^2\psi\right)_i^2\right] , \qquad (3.12)$$

$$F_2 = \frac{1}{2} \sum_i \left[-\frac{1}{2}\left(\nabla^+\psi\right)_i^2 + \frac{3}{4}\psi_i^2 \left(\nabla^2\psi^2\right)_i - \psi_i^3 \left(\nabla^2\psi\right)_i + \frac{1}{2}\left(\nabla^2\psi\right)_i^2\right] . \qquad (3.13)$$

The discrete steady probability density is then found by introducing (3.8), (3.12), and (3.13) into (3.5):

$$P_{\mathrm{st}}(\psi) \sim \exp\left\{-\frac{1}{4\varepsilon} \sum_j \left[\left(-1 + \frac{\tau}{2} - 6\tau\varepsilon\right)\psi_j^2 + \left(\frac{1}{2} - \tau\right)\psi_j^4 + \left(1 - \tau - \lambda^2\right)\left(\nabla^+\psi\right)_j^2 + \frac{\tau}{2}\psi_j^6 + \left(\lambda^2 + \frac{\tau}{2}\right)\left(\nabla^2\psi\right)_j^2 - \left(\tau + 2\lambda^2\right)\psi_j^3 \left(\nabla^2\psi\right)_j + \frac{3}{2}\lambda^2\psi_j^2\left(\nabla^2\psi^2\right)_j\right]\right\}. \qquad (3.14)$$

The last four terms of this expression can be shown to be irrelevant by means of a renormalization-group analysis [23]. Hence the expression for the steady probability density in continuum space takes the form

$$P_{\mathrm{st}} \sim \exp\left\{-\frac{1}{4\varepsilon} \int d\mathbf{x} \left[\left(-1 + \frac{\tau}{2} - 6\tau\varepsilon\right)\psi^2 + \left(\frac{1}{2} - \tau\right)\psi^4 + \left(1 - \tau - \lambda^2\right)\left(\nabla\psi\right)^2\right]\right\}. \qquad (3.15)$$

This is the expression up to first order in τ and λ^2 for the steady-state probability density of the field variable ψ. It can be checked that in a mean-field approach the result for a zero-dimensional non-Markovian stochastic process (i.e., $\lambda = 0$ and $\tau \neq 0$) is recovered [10]. Since the effects of τ and λ are not clear from their appearance in expression (3.15), we present in Appendix B an exactly solvable model in which one can see the softening role of these two parameters.

IV. COMMENTS AND CONCLUSIONS

We have presented numerical and analytical results of the effects of a nonwhite external noise in the time-dependent Ginzburg-Landau model. Since the system is close to the equilibrium case when the noise is white, this model permits the comparison between a well-known phenomenology of equilibrium and the phenomena we expect in nonequilibrium situations. In this sense the noise parameters λ and τ, which control the departure from equilibrium, are assumed to be small in the theoretical approximate approach. A numerical analysis of the model shows that the role of the correlation time and length of the noise is to decrease its effective intensity. This can be understood in a linear model which is solved exactly in Appendix B.

ACKNOWLEDGMENTS

This research was supported in part by the Dirección General de Investigación Científica y Técnica (Spain) under Project No. PB90-0030. We acknowledge CESCA (Centre de Supercomputació de Catalunya) and Cray Research, Inc. for CPU time in their Cray's YMP, where these calculations have been performed. One of us (J.G.O.) thanks Fundació Catalana per a la Recerca for financial support and the people of the Chemistry

Research and Engineering group at Cray Research Park (Eagan, Minnesota) for their hospitality and help while part of this work was carried out. We also thank R. Toral for fruitful comments.

APPENDIX A: FOKKER-PLANCK APPROXIMATION FOR NON-MARKOVIAN LANGEVIN EQUATIONS IN EXTENDED SYSTEMS

Let us consider the discrete Langevin equation (2.1) with a noise colored in space and time. We are looking for an evolution equation for the probability density

$$P(\boldsymbol{\psi},t) = \langle \delta(\boldsymbol{\psi}(t) - \boldsymbol{\psi}) \rangle \ , \tag{A1}$$

where the average is taken over the initial conditions and different realizations of the noise (this is the so-called *Van Kampen lemma* [24]). On the other hand, a continuity equation for the evolution of $\langle \delta(\boldsymbol{\psi}(t) - \boldsymbol{\psi}) \rangle_{\rm IC}$ (average taken over initial conditions only) must hold:

$$\frac{\partial}{\partial t} \langle \delta(\boldsymbol{\psi}(t) - \boldsymbol{\psi}) \rangle_{\rm IC} = -\frac{\partial}{\partial \psi_i} \dot{\psi}_i \langle \delta(\boldsymbol{\psi}(t) - \boldsymbol{\psi}) \rangle_{\rm IC} \, . \tag{A2}$$

This equation is the stochastic Liouville equation. Its average over the noise $\xi_i(t)$ leads to an expression for the evolution of the probability density defined above:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \psi_i} f_i P - \frac{\partial}{\partial \psi_i} \langle \xi_i(t) \delta(\boldsymbol{\psi}(t) - \boldsymbol{\psi}) \rangle \ . \tag{A3}$$

The remaining average in (A3) can be calculated by means of Novikov's theorem [25]:

$$\langle \xi_i(t) \delta(\boldsymbol{\psi}(t) - \boldsymbol{\psi}) \rangle = \int_0^t dt' \, \varepsilon \, h_{ij}(t,t') \times \left\langle \frac{\delta(\delta(\boldsymbol{\psi}(t) - \boldsymbol{\psi}))}{\delta \xi_j(t')} \right\rangle \ . \tag{A4}$$

It can easily be seen that the following equality holds:

$$\frac{\delta\left(\delta(\boldsymbol{\psi}(t) - \boldsymbol{\psi})\right)}{\delta \xi_j(t')} = -\frac{\partial}{\partial \psi_k} \frac{\delta \psi_k(t)}{\delta \xi_j(t')} \, \delta(\boldsymbol{\psi}(t) - \boldsymbol{\psi}) \ . \tag{A5}$$

This result, together with Novikov's theorem, leads to a new expression for the equation we are trying to derive:

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial \psi_i} f_i P + \frac{\partial^2}{\partial \psi_i \partial \psi_k} \int_0^t dt' \, \varepsilon \, h_{ij}(t,t') \times \left\langle \frac{\delta \psi_k(t)}{\delta \xi_j(t')} \, \delta(\boldsymbol{\psi}(t) - \boldsymbol{\psi}) \right\rangle \ . \tag{A6}$$

The problem of the exact evaluation of this last average and its relation to the probability density remains unsolved. Thus an approximation has to be done at this point. Let us assume that τ is small. Then the correlation function $h_{ij}(t,t')$ of the noise appearing in the integral will be a sharply peaked function of $t-t'$. This fact allows us to use a Taylor-series expansion of the response function appearing in the average of Eq. (A6),

$$\frac{\delta \psi_k(t)}{\delta \xi_j(t')} \simeq \left. \frac{\delta \psi_k(t)}{\delta \xi_j(t')} \right|_{t=t'} + \frac{d}{dt'} \left. \frac{\delta \psi_k(t)}{\delta \xi_j(t')} \right|_{t=t'} (t'-t) \ . \tag{A7}$$

We need to calculate now the response function at equal times and its first time derivative. To do so we formally integrate Eq. (2.1) to obtain

$$\psi_k(t) = \psi_k(0) + \int_0^t ds \, \left[f_k(\boldsymbol{\psi}(s)) + \xi_k(s) \right] \ . \tag{A8}$$

Functional differentiation of this expression leads through (A7) to [10,26]

$$\frac{\delta \psi_k(t)}{\delta \xi_j(t')} \simeq \delta_{kj} - \frac{\partial f_k(\boldsymbol{\psi}(t))}{\partial \psi_j(t)} (t'-t) \ . \tag{A9}$$

By introducing this result into Eq. (A6), making use of relation (A1), and performing the integral in time we find

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \psi_i} f_i P + \varepsilon \frac{\partial^2}{\partial \psi_i \partial \psi_k} \left(h^0_{ik} + \tau h^1_{ij} \frac{\partial f_k}{\partial \psi_j} \right) P \ . \tag{A10}$$

This is the final expression for the approximate Fokker-Planck equation corresponding to a stochastic discretized field process driven by an additive colored noise. Transient terms have been neglected by extending the time integrals from 0 to ∞, and also the following definitions have been used:

$$h^0_{ij} \equiv \int_0^\infty ds \, h_{ij}(s) \ , \quad \tau h^1_{ij} \equiv \int_0^\infty ds \, s \, h_{ij}(s) \ . \tag{A11}$$

These h^k are functions in space with the same characteristic length λ. We assume that they are sharply peaked around $\mathbf{x} - \mathbf{x}' = \mathbf{0}$, so that they can be worked out as distributions having an expansion of the form

$$h^0_{ik} = \delta_{ik} + a_0 \lambda^2 \nabla^2_{im} \delta_{mk} \ , \quad h^1_{ik} = a_1 \delta_{ik} + \theta(\lambda^2) \ . \tag{A12}$$

As the parameters a_0 and a_1 can be included in the definition of τ, λ or ε, they can be assumed to be equal to one. And as we want to keep only the first corrections in τ and λ^2, we can discard the dependence of h^1 on λ^2. These expansions (A12) can be understood in the following way: the order 0 is the white-noise limit $\lambda \longrightarrow 0$ (i.e., a Kronecker δ) and the first nonzero order corresponds to λ^2 (due to space inversion symmetry), which is supposed to be a Laplacian operator (the simplest spatial correlation beyond the Kronecker δ). Now, introduction of (A12) into Eq. (A10) leads finally to Eq. (3.1), which is the final approximate Fokker-Planck equation up to the first nonzero orders in both the correlation length and correlation time of the noise.

APPENDIX B: COLORED NOISE IN A LINEAR MODEL

In order to acquire a better understanding of the effect of a colored noise in the steady-state behavior of our system let us consider a linear stable version of Eq. (1.7):

$$\frac{\partial\psi(\mathbf{x},t)}{\partial t} = \frac{1}{2}\left(-\psi + \nabla^2\psi\right) + \xi(\mathbf{x},t)\,, \tag{B1}$$

which in Fourier space takes the form

$$\frac{\partial\psi(\mathbf{q},t)}{\partial t} = -\frac{1}{2}\left(1+q^2\right)\psi(\mathbf{q},t) + \xi(\mathbf{q},t)\,. \tag{B2}$$

The noise ξ was defined in Eq. (2.4) and has a correlation in Fourier space [13]

$$\langle\, \xi(\mathbf{q},t)\, \xi(\mathbf{q}',t')\,\rangle = \frac{\varepsilon}{\tau(1+\lambda^2q^2)}\,\delta(\mathbf{q}+\mathbf{q}') \times \exp\left(-(1+\lambda^2q^2)\frac{\mid t-t'\mid}{\tau}\right)\,. \tag{B3}$$

The Fokker-Planck equation corresponding to the Langevin equation (B2) can be found by means of the procedure described in Appendix A. In this case, however, as the Langevin equation is linear, the response function can be evaluated *exactly* from Eq. (A8). The result is

$$\frac{\delta\psi_q(t)}{\delta\xi_{q'}(t')} = \delta_{qq'}\, e^{(t'-t)(1+q^2)/2} \tag{B4}$$

so that the (exact) Fokker-Planck equation is

$$\frac{\partial P}{\partial t} = \int d\mathbf{q}\, \frac{\delta}{\delta\psi(\mathbf{q},t)}\,\frac{1}{2}\left(1+q^2\right)\psi(\mathbf{q},t)\, P + \int d\mathbf{q}\, \varepsilon_{\text{eff}}(q)\, \frac{\delta}{\delta\psi(\mathbf{q},t)}\,\frac{\delta P}{\delta\psi(-\mathbf{q},t)}\,, \tag{B5}$$

where transient terms have been neglected and the effective noise intensity is given by

$$\varepsilon_{\text{eff}}(q) = \frac{\varepsilon}{(1+\lambda^2q^2)[1+\lambda^2q^2+\frac{\tau}{2}(1+q^2)]}\,. \tag{B6}$$

The steady-state probability distribution obeys (assuming zero probability flux)

$$\left(\frac{1}{2}\left(1+q^2\right)\psi(\mathbf{q}) + \varepsilon_{\text{eff}}(q)\,\frac{\delta}{\delta\psi(-\mathbf{q})}\right)P_{\text{st}} = 0\,, \tag{B7}$$

which has the following solution

$$P_{\text{st}} \sim \exp\left\{-\frac{1}{4}\int \frac{(1+q^2)\,\psi(\mathbf{q})\psi(-\mathbf{q})}{\varepsilon_{\text{eff}}(q)}\,d\mathbf{q}\right\}, \tag{B8}$$

as can be tested in a straightforward way by direct substitution in (B7). On the other hand, it can be easily seen from Eqs. (B6) and (B8) that the steady probability density in real space is

$$P_{\text{st}} \sim \exp\left\{-\frac{1}{4\varepsilon}\int\left[\left(1+\frac{\tau}{2}\right)\psi^2 + \left(1+\tau+\lambda^2\right)(\nabla\psi)^2 + \left(\lambda^2+\frac{\tau}{2}\right)\left(\nabla^2\psi\right)^2\right]d\mathbf{x}\right\}. \tag{B9}$$

Thus we have found here that, in this exact model, the role of the correlation time and length of the noise is to decrease its effective intensity, as found in our numerical simulation of the more general nonlinear model.

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