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Non-Abelian symmetries of stochastic processes: Derivation of correlation functions for random-vertex models and disordered-interacting particle systems

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We consider systems of particles hopping stochastically on d-dimensional lattices with space-dependent probabilities. We map the master equation onto an evolution equation in a Fock space where the dynamics are given by a quantum Hamiltonian (continuous time) or a transfer matrix (discrete time). Using non-Abelian symmetries of these operators we derive duality relations, expressing the time evolution of a given initial configuration in terms of correlation functions of simpler dual processes. Particularly simple results are obtained for the time evolution of the density profile. As a special case we show that for any SU(2) symmetric system the two-point and three-point density correlation functions in the N-particle steady state can be computed from the probability distribution of a single particle moving in the same environment. We apply our results to various models, among them partial exclusion, a simple diffusion-reaction system, and the two-dimensional six-vertex model with space-dependent vertex weights. For a random distribution of the vertex weights one obtains a version of the random-barrier model describing diffusion of particles in disordered media. We derive exact expressions for the averaged two-point density correlation functions in the presence of weak, correlated disorder.

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I. INTRODUCTION

Stochastically hopping particles on a lattice represent simple models for diffusion-reaction processes in various media. The quantitative description of the time evolution starting from a given initial condition is naturally a difficult problem, in particular if one deals with systems of interacting particles rather than with a single-particle problem. For special processes, however, there are duality relations allowing one to map the time evolution of a given process to that of another process in which the quantity one wishes to calculate is easier to obtain [1]. One particularly well-studied process is the symmetric exclusion process. It describes a system of particles on a lattice with exclusion interaction, i.e., such that each lattice site is either empty or occupied by at most one particle. In each (infinitesimal) time step one particle can hop from a lattice site x to another site y with certain probability $p_{x,y} = p_{y,x}$. For this model the time evolution is very well understood [2]. For instance, the time evolution of the density distribution of the N-particle system is completely determined by the probability distribution of a *single* particle. The understanding is less complete if one deals with partial exclusion, where each lattice site can be filled by up to $m > 1$ particles, or in the case of asymmetric hopping rates $p_{x,y} \neq p_{y,x}$. An even more difficult problem is posed if reactions occur, i.e., if particles can be created or annihilated or change to a different species of particles.

The main aim of this paper is to derive duality relations from symmetries of the system dynamics. We will map the state space of the stochastic model to a Fock space, where the time evolution is described by a linear operator [3–13]. Particularly in one dimension this mapping has proven to be a powerful tool as for many equilibrium and nonequilibrium models the time-evolution operator is given by (or related to) the Hamiltonian of an integrable quantum chain [6, 13]. Thus part of the vast amount of knowledge that has accumulated for these models over the past decade and some of the methods applied for solving them such as the Bethe ansatz [6, 14] are useful also for the classical systems from which the Hamiltonian was obtained and leads the way to new predictions.

While the integrability of these one-dimensional models is ensured by the existence of an infinite set of mutually commuting symmetry operators and is only valid for specific choices of the coupling constants, many of these systems have other non-Abelian global symmetries [such as SU(2) invariance] which are independent of the coupling constants and are also present in higher-dimensional analogs of these models. It is these symmetries of the d-dimensional models that we are going to exploit.

In Sec. II and in Appendixes A and B we show that the known duality relations for the symmetric exclusion process can be derived very easily from the SU(2) symmetry of the corresponding time evolution operator (which is a spin-$\frac{1}{2}$ Heisenberg ferromagnet with space-dependent coupling constants). Repeating the same calculation we derive similar relations for SU(2) symmetric higher spin models corresponding to partial exclusion where each lattice site can be occupied by up to $m > 1$ particles. In particular, we show that the time evolution of the density distribution of N particles for *any* SU(2) symmetric system is given by the one-particle probability distribu-

tion if initially each lattice site was occupied by at most one particle. Furthermore, we extend our calculations to (a) multitime correlation functions (a particularly simple expression is obtained for the three-point density correlation function) and (b) models with other symmetries. The adaption of our calculation to systems with a given symmetry is straightforward and allows for the derivation of duality relations for a very wide class of models.

In Sec. III we apply our results to some specific interacting many-particle systems, among them the random-barrier model with partial exclusion. While the single-particle probability distribution for a homogeneous system (i.e., with translationally invariant hopping rates) is easy to compute, this becomes a difficult task in the presence of disorder [15–23] or other inhomogenities. Single-particle diffusion in a random-barrier environment is an interesting field of physics, including phase transitions in systems where the moments of the inverse hopping rates do not exist [15, 16, 24]. We give a brief review of some known results which we generalize to many particle systems using our results of Sec. II. The symmetries studied Sec. II occur also in processes which are not purely diffusive. In order to illustrate this point we also discuss a simple two-species reaction-diffusion model.

One obtains quantum *Hamiltonians* for processes in continuous time, but one may also study stochastic processes in discrete time. In this case the mapping to a Fock space gives rise to a *transfer matrix* which for many interesting problems in *one* dimension is related to the transfer matrix of integrable two-dimensional systems such as vertex models [14, 25–27]. In Sec. IV we will utilize this mapping for a study of correlated disorder in a particular version of the one-dimensional many-particle random-barrier model with exclusion. While much is known about the continuous-time single-particle random-barrier model with uncorrelated disorder, where all hopping rates p_x are independently chosen random numbers with some translationally invariant distribution μ, only few results are available about systems with spatially correlated disorder [21–23]. Furthermore, these results are not automatically applicable to the vertex-model version of the process and we shall investigate the limits of their validity. This mapping is interesting in itself as the quantity we are going to study corresponds to the arrow-arrow correlation function of disordered-vertex models. We shall also point out the existence of another type of phase transition for specifically chosen hopping probabilities.

In Sec. V we summarize our results and in the Appendixes we present some details of our calculations to Sec. II (in Appendixes A and B) and we repeat and generalize the mapping of Ref. [25] of the diffusion problem to the six-vertex model (in Appendix C).

II. CORRELATIONS IN INTERACTING MANY-PARTICLE SYSTEMS

In this section we will show how correlation functions of interacting (generically disordered) many-particle systems are related to simpler quantities such as the probability distribution of a single particle moving in the same environment. Examples are the relations arising from the self-duality of exclusive diffusion [1, 2]. We rederive them from the SU(2) symmetry of the time-evolution operator of this process and thereby generalize to arbitrary SU(2)-invariant stochastic processes. In order not to complicate our discussion we restrict ourselves in this section to systems containing only one species of particles. The generalization of our approach to multispecies models is briefly discussed in Secs. II C and III B.

Let us consider a many-particle system on a d-dimensional lattice described by stochastic occupation numbers $\underline{n} = \{n_j\}$. Its dynamics are given by a master equation of the following form:

$$\partial_t F(\underline{n}, t) = -H' F(\underline{n}, t), \tag{2.1}$$

where H' is some linear operator acting on the n_j. We may also consider discrete-time dynamics where the time derivative in Eq. (2.1) is replaced by a discrete difference. Instead of H' a transfer matrix T' is then used (see Sec. IV).

According to Doi's formalism the master equation is mapped onto an evolution equation in Fock space [3,13,9]:

$$\partial_t | F(t) \rangle = -H | F(t) \rangle \tag{2.2}$$

or a similar equation in the case of discrete time. The solution of Eq. (2.2) can be written as

$$| F(t) \rangle = U_t | F(0) \rangle \quad \text{with} \quad U_t = e^{-Ht} \ . \tag{2.3}$$

(For a discrete-time dynamics U_t after t time steps is given by the tth power T^t of the transfer matrix T.)

A particular configuration $\underline{n}$ corresponds to a state $| \underline{n} \rangle = \prod_{j=1}^{L} (c_j^\dagger)^{n_j} | 0 \rangle$, where $| 0 \rangle$ is the vacuum containing no particles and L is the number of available lattice sites. A vector in Fock space is decomposed as $| F(t) \rangle = \sum_{\{n_j\}} F(\underline{n}, t) | \underline{n} \rangle$ and a scalar product is defined by $\langle \underline{n} | \underline{m} \rangle = \prod_{j=1}^{L} [n_j! \delta_{n_j, m_j}]$.

For noninteracting classical particles the creation operators $c_j^\dagger$ acting on site j and their adjoint operators c_j, which annihilate particles, obey bosonic commutation relations [3]. If we consider a system of particles with exclusion, i.e., if at most one particle is allowed to be on each lattice site, the creation and annihilation operators fulfill Pauli-type commutation rules: operators on the same lattice site have fermionic anticommutation relations whereas operators for different lattice sites commute [7, 9]. For simplicity we will refer to exclusive particles as fermions. In the more general case, where up to m particles can occupy one lattice site, the creation and annihilation operators can be represented by spin-$m/2$ ladder operators of SU(2). We call this situation partial exclusion.

Here we are interested in the time evolution of a given initial configuration and in particular in steady-state correlation functions. If the physical quantities $A(\underline{n})$, $B(\underline{n})$, and $C(\underline{n})$ are analytical functions of the occupation numbers we find [28]

$$\langle A(t) B(0) \rangle = \langle s | A U_t B | F(0) \rangle, \tag{2.4}$$

$$\langle A(t_2) B(t_1) C(0) \rangle = \langle s | A U_{t_2 - t_1} B U_{t_1} C | F(0) \rangle, \tag{2.5}$$

with $\langle\, s \mid = \langle\, 0 \mid \exp(\sum_j c_j)$ [3, 7] and A, B are the corresponding functions of the particle number operators n_j. Because of the normalization condition

$$\langle\, s \mid H = 0, \tag{2.6}$$

which has to be valid for any H describing a stochastic process, we may insert U_{-t} between the first two terms on the right-hand side (rhs) of Eq. (2.4) and obtain an expression very similar to the one in ordinary quantum mechanics. Note that averages are linear in the probability distribution here while they are bilinear in the state in quantum theory. For special problems, however, linearity and bilinearity may coincide [see (2.21)].

Of particular interest are two-point correlation functions of particle numbers in a steady state. For the moment we restrict ourselves to systems with particle conserving dynamics. As shown below (see Sec. III B) this condition can be relaxed for many-species models. The two-point density correlation function in the steady state for a N-particle system is given by

$$G_N(x,y;t) = \langle n_x(t) n_y(0) \rangle_{\rm st}, \tag{2.7}$$

where a subscript x denotes the label of the lattice site corresponding to a position x in space and n_x is the particle number in x. Averaging is performed over the stationary N-particle state.

The connected two-point correlation function is defined by

$$C_N(x,y;t) = \langle \{n_x(t) - \langle n_x \rangle_{\rm st}\}\, \{n_y(0) - \langle n_y \rangle_{\rm st}\} \rangle_{\rm st}, \tag{2.8}$$

where the arguments t and 0 are dropped in the one point averages, because $\langle n_x(t) \rangle_{\rm st}$ does not depend on time. In a similar way we define the connected three-point function as

$$D_N(x,y,z;t_2,t_1) = \langle\, \{n_x(t_2) - \langle n_x \rangle_{\rm st}\}\, \{n_y(t_1) - \langle n_y \rangle_{\rm st}\} \times \{n_z(0) - \langle n_z \rangle_{\rm st}\} \rangle_{\rm st}. \tag{2.9}$$

First we consider a system with one particle only. It can be described using a master equation (2.1) or much easier by means of the following probability distribution:

$$P(x,t;y,0) = P(x,t|y,0) P(y,0), \tag{2.10}$$

which gives the probability of finding the particle in y at time 0 and in x at time t. The first expression on the rhs is the corresponding conditional probability. For the two-point correlation function defined above we obtain

$$G_1(x,y;t) = \sum_{\{n_j\},\{n'_j\}}^{(1)} n_x n'_y F(\underline{n},t;\underline{n}',0) = P(x,t|y,0) P_{\rm st}(y), \tag{2.11}$$

where $F(\underline{n},t;\underline{n}',0)$ is the combined probability of configuration $\underline{n}$ at time t and $\underline{n}$ at time 0. [A superscript (k) at the sum means that the sum runs over states with a total particle number k only.] Assuming, furthermore, the steady state to be homogeneous we get

$$G_1(x,y;t) = \frac{1}{L} P(x,t|y,0)\ . \tag{2.12}$$

Our aim is (a) to express the N-particle correlation functions C_N and D_N in terms of the one-particle distribution $P(x,t|y,0)$ and (b), more generally, to express the time evolution of an N-particle initial configuration in terms of correlators in sectors with less than N particles. The latter quantities are generally easier to compute. Problem (a) is just a special case of problem (b).

A. The fermionic case

First we study the fermionic case, i.e., systems of particles with total exclusion (each site can be occupied by at most one particle).

We will study systems the dynamics of which are given by an arbitrary SU(2)-symmetric Hamiltonian H, i.e.,

$$[H, S^{\pm}]_- = 0, \tag{2.13}$$

$$[H, S^{z}]_- = 0, \tag{2.14}$$

where $S^{\pm,z}$ are the generators of SU(2). (In the discrete time case H has to be replaced by the transfer matrix T.)

As demonstrated in the example of Sec. III B it is actually sufficient to demand the symmetry to hold in a subspace of H. We would like to stress that the SU(2) symmetry is not a technical assumption but the main ingredient leading to the results derived below.

It is easy to check that in the fermionic case the operators

$$S^+ = \sum_{j=1}^{L} c_j\ ,\quad S^- = \sum_{j=1}^{L} c_j^\dagger\ ,\quad S^z = \sum_{j=1}^{L} (\tfrac{1}{2} - n_j), \tag{2.15}$$

with $n_j = c_j^\dagger c_j$ satisfy the commutation relations of SU(2). Introducing Pauli matrices $\boldsymbol{\sigma}_j = \{c_j^\dagger + c_j, -i(c_j^\dagger - c_j), 1 - 2n_j\}$ we can write the general form of a SU(2)-symmetric Hamiltonian for the stochastic time evolution of exclusive particles as

$$H = -\frac{1}{2} \sum_{j,k} p_{j,k}(t)\, [\boldsymbol{\sigma}_j \boldsymbol{\sigma}_k - 1] - \frac{1}{4} \sum_{j,k,l,m} p_{j,k,l,m}(t)\, [\boldsymbol{\sigma}_j \boldsymbol{\sigma}_k - 1]\, [\boldsymbol{\sigma}_l \boldsymbol{\sigma}_m - 1] + \cdots, \tag{2.16}$$

where the generically time-dependent quantities $0 \le p_{\ldots}(t) < \infty$ are arbitrary real-valued functions of t. [For multispecies models they can be operators commuting with the generators (2.15).] The summations over n pairs of indices $k_1, \ldots, k_{2n}$ run over the set $1 \le k_1 < k_2 \le k_3 < k_4 \le \cdots \le k_{2n-1} < k_{2n} \le L$, where L is the number of sites in the d-dimensional lattice.

As a consequence of the symmetry some interesting properties result.

(i) The number of particles in the system is a conserved quantity because of Eq. (2.14).

(ii) The Hamiltonian is Hermitian.

(iii) Because of the normalization condition (2.6) and the commutators (2.13) we find

$$H|\,N\,\rangle = 0 \quad \text{with} \quad |\,N\,\rangle = \frac{1}{N!}(S^-)^N|\,0\,\rangle \qquad (2.17)$$

and

$$\langle\, N\,|H = 0 \quad \text{with} \quad \langle\, N\,| = \frac{1}{N!}\langle\, 0\,|(S^+)^N. \qquad (2.18)$$

Hence the normalized N-particle state ($N < L$)

$$|\,N\,\rangle_{\text{norm}} = \binom{L}{N}^{-1} |\,N\,\rangle = \binom{L}{N}^{-1} \sum_{\{n_j\}}^{(N)} |\,\underline{n}\,\rangle \qquad (2.19)$$

is a stationary solution of the problem assigning the same probability to any possible configuration of the n_j, which has a total particle number N. [The summation in (2.19) runs over states with $n_j = 0, 1$ and $\sum_{j=1}^{L} n_j = N$.] The averaged occupation numbers in state $|\,N\,\rangle_{\text{norm}}$ are $\langle n_x \rangle = N/L$.

From Eqs. (2.4) and (2.7) we find for the two-point correlation function

$$G_N^{\text{ferm}}(x,y;t) = \langle\, s\,|n_x U_t n_y|\,N\,\rangle_{\text{norm}}, \qquad (2.20)$$

where the steady states (2.19) are used. Because of $\langle\, s\,| = \sum_{N=0}^{L}\langle\, N\,|$, particle conservation and the orthogonality of the $|\,N\,\rangle$ we get

$$\binom{L}{N} G_N^{\text{ferm}}(x,y;t) = \langle\, N\,|n_x U_t n_y|\,N\,\rangle. \qquad (2.21)$$

Using the SU(2) symmetry the following recursion relation can be derived from Eq. (2.21) [see Appendix A, Eq. (A8)]:

$$G_{N+1}^{\text{ferm}}(x,y;t) = \frac{L-N-1}{L-N}\frac{N+1}{N} G_N^{\text{ferm}}(x,y;t) + \frac{1}{L}\frac{N+1}{L-N} \ . \qquad (2.22)$$

From this one finds an exact expression for G_N in terms of G_1:

$$G_N^{\text{ferm}}(x,y;t) = N\frac{L-N}{L-1} G_1^{\text{ferm}}(x,y;t) + \frac{N}{L}\frac{N-1}{L-1}, \qquad (2.23)$$

which reduces in the thermodynamic limit $L, N \to \infty, \rho = N/L = \text{const}$ to

$$G_N^{\text{ferm}}(x,y;t) = N(1-\rho) G_1^{\text{ferm}}(x,y;t) + \rho^2, \qquad (2.24)$$

where ρ is the mean density of the N-particle system. Inserting Eq. (2.12) into Eq. (2.24) and using the definition (2.8) of the connected two-point correlation function we find

$$C_N^{\text{ferm}}(x,y;t) = \rho(1-\rho) P(x,t|y,0) \ . \qquad (2.25)$$

Equation (2.25) is symmetric with respect to $\rho \leftrightarrow (1-\rho)$. This reflects the particle-hole symmetry in a system where occupation numbers are restricted to 0 and 1 only. The amplitude of the correlation function has its maximum at $\rho = 1/2$.

Following a similar procedure as above we find for the connected three-point function in the thermodynamic limit

$$D_N^{\text{ferm}}(x,y,z;t_2,t_1) = (\rho - 3\rho^2 + 2\rho^3) P(x,t_2|y,t_1) P(y,t_1|z,0) \ . \qquad (2.26)$$

Note that this quantity vanishes for $\rho = 1/2$.

Because the properties of a one-particle system are well-known for many environments, Eqs. (2.23), (2.25), and (2.26) provide a useful tool to study (finite or infinite) many-particle systems in any dimension provided the SU(2) symmetry holds (at least for a subspace).

A more general duality relation allowing one to map an M-particle process onto a process with $N < M$ particles was proved by Spitzer [2] for exclusion processes with constant symmetric hopping rates. We will show how this relation arises from the SU(2) symmetry of the Hamiltonian.

Instead of studying a master equation for occupation numbers $\{n_j(t)\}$ we may consider the stochastic quantity $A_N(t)$ which is the set of lattice sites occupied at time t (N is the particle number). The dynamics of the system are then completely described by the probability distribution $P_C[A_N(t) = B]$, i.e., by the probability that the stochastic process $A_N(t)$ has the value B at time t provided it started with the value C. More general quantities are $P_C[A_N(t) \subset B]$ and $P_C[A_N(t) \supset B]$ giving the probabilities for $A_N(t) \subset B$ and $A_N(t) \supset B$, respectively, under the condition $A_N(0) = C$.

We consider two processes $A_N(t)$ and $A_M(t)$ describing an N-particle and an M-particle system moving in identical environments. Using the SU(2) symmetry of the time evolution the following duality relation is proved in Appendix B:

$$P_{A_M}[A_M(t) \supset A_N] = P_{A_N}[A_N(t) \subset A_M] \ . \qquad (2.27)$$

The probability to find all points of A_N occupied at time t by the process $A_M(t)$ with $A_M(0) = A_M$ is equal to the probability that all particles lie in A_M for the process $A_N(t)$ with $A_N(0) = A_N$. This statement is the same as in [2], but the class of processes considered here is more general than those discussed by Spitzer. The SU(2) symmetry is the main property ensuring (2.27).

In terms of correlation functions Eq. (2.27) reads

$$\left\langle M \left| \prod_{i\in A_N} n_i U_t \prod_{j\in A_M} n_j \right| M \right\rangle = \sum_{B\subset A_M} \left\langle N \left| \prod_{j\in B} n_j U_t \prod_{i\in A_N} n_i \right| N \right\rangle \ , \qquad (2.28)$$

where B are sets of N lattice sites. Choosing $N = 1$ in (2.28) one obtains the above relations for two-point functions. For the time-dependent density profile $\rho_{A_N}(x,t)$ of system which starts with initial configuration A_N (2.28) gives

$$\rho_{A_N}(x,t) = \sum_{B_1\subset A_N} \rho_{B_1}(x,t) \ . \qquad (2.29)$$

This rephrases the known validity of Fick's law for exclusive diffusion.

B. Partial exclusion

In Sec. II A we have studied models with total exclusion such that each lattice site can be occupied by at most one particle. Now we turn to partial exclusion, where up to m particles can sit on a given site j. Besides this being interesting in its own right, one arrives at such a model via real-space renormalization of the fermionic model.

In this case the occupation numbers n_j can take the integer values $0 \leq n_j \leq m$. Translating these properties into operator language leads to a representation in terms of spin $s = m/2$ matrices. The number operator n_j is given by the matrix $m/2 - s_j^z$ and the creation and annihilation operators are defined as $s_j^-|..n_j..\rangle = (m+1-n_j)|..n_j+1..\rangle$ and $s_j^+|..n_j..\rangle = (n_j+1)|..n_j-1..\rangle$. The latter ones satisfy the SU(2) algebra $[s_l^+, s_m^-] = 2\delta_{l,m}s_l^z$ and $[s_l^z, s_m^\pm] = \pm\delta_{l,m}s_l^\pm$. As above we assume the time-evolution operator to commute with the total spin operators $S^{\pm,z} = \sum s_j^{\pm,z}$. For spin s the most general SU(2) symmetric Hamiltonian reads

$$H = -p(t) - \frac{1}{2}\sum_{j,k}\sum_{\mu=1}^{m} p_{j,k}^{(\mu)}(t)\,[\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k]^\mu - \frac{1}{4}\sum_{j,k,l,n}\sum_{\mu,\nu=1}^{m} p_{j,k,l,n}^{(\mu,\nu)}(t)\,[\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k]^\mu\,[\boldsymbol{\sigma}_l\boldsymbol{\sigma}_n]^\nu + \cdots. \tag{2.30}$$

It describes processes where particle jumps from site j to site k have probability rates proportional to $n_j(m-n_k)$. Note that for special choices of the functions $p_{k_1,\ldots}^{(\mu_1,\ldots)}(t)$ the symmetry can be higher than SU(2) (see below). The powers of the matrices $\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k = 2(s_j^+s_k^- + s_j^-s_k^+ + 2s_k^zs_j^z)$ allow for the possibility that more than one particle jumps in each time step.

As the calculations of Sec. II A are based only on the symmetry and not on the properties of the spin-$\frac{1}{2}$ representation they can be repeated here. In particular, the analogs of Eqs. (A3) and (A4) are $\langle N|s_j^- = \langle N-1|(m-n_j)$ and $\langle N|S^- = \langle N-1|(mL-N+1)$. Equations (A1), (A2), and (A5) remain unchanged after replacing $c^\dagger$ by s^- and c by s^+. Eqs. (2.17) and (2.18) are still valid with the operators $S^\pm$ defined by Eqs. (2.15) with the above replacment. Using these relations and the normalization condition

$$|\,N\,\rangle_{\rm norm} = \binom{mL}{N}^{-1}|\,N\,\rangle \tag{2.31}$$

one can again write down and solve a recursion relation in N for the two-point function (2.7). One finds

$$G_N(x,y;t) = \frac{N}{L}\frac{mL-N}{mL-1}P(x,t|y,0) + \frac{N}{L}\frac{mN-m}{mL-1} \tag{2.32}$$

and therefore for the connected two-point function in the thermodynamic limit

$$C_N(x,y;t) = \rho\left(1-\frac{\rho}{m}\right)P(x,t|y,0), \tag{2.33}$$

where again $\rho = \langle n_x\rangle_{\rm st} = N/L$. The amplitude is symmetric under $\rho \to m-\rho$ and has its maximum at $\rho = m/2$. (m is the maximum number of particles allowed on each lattice site.) In these expressions $P(x,t|y,0)$ is again the probability distribution for a single particle moving in the environment defined by the coupling constants $p_{k_1,\ldots}^{(1)}(t)$. Note that $P(x,t|y,0)$ does not depend on m and not on the coupling constants $p_{k_1,\ldots}^{(\mu_1,\ldots)}$ with upper (greek) indices larger than one. These coupling constants give the probabilities that more than one particle jumps at the same time. For $m = 1$ one recovers the result (2.25) of Sec. II A.

Duality relations such as Eq. (2.27) exist for higher spins as well. They are established as demonstrated in Appendix B. Only their structure is not as simple as for the $s = 1/2$ case in general. As an example we give a result for $s = 1$ ($m = 2$).

As in Sec. II A the system dynamics are described by a set $A_k(t)$ of occupied lattice sites. Contrary to the $s = 1/2$ case this is not a complete description here, because it does not specify which of the sites are doubly occupied. (A complete description using two sets results in more complicated duality relations than the one derived here.) Let $P_C[A_N(t) = B]$ be the probability that the process $A_N(t)$ has the value B at time t under the initial condition $|F_C\rangle$. The state $|F_C\rangle$ assigns equal probability to all configurations contained in C. In a similar way we define $P_C[A_N(t) \subset B]$ and $P_C[A_N(t) \supset B]$. One possible duality relation reads

$$P_{A_{k+1}}[A_{N+1}(t) \supset A_k] = \frac{1}{2}\frac{2L-N-k}{N+1-k}\frac{N-k}{2k-N+2}P_{A_{k+1}}[A_N(t)\supset A_k] + \frac{1}{N+1-k}\frac{2k+1-N}{k+1}\sum_{B_k\subset A_{k+1}}P_{A_k}[A_N(t)\supset B_k], \tag{2.34}$$

where N is a particle number and the index $k \leq N$ gives the cardinality of the set A_k.

Now consider $k = N$ where each site in the initial set A_k is occupied by exactly one particle. Iterating (2.34) we obtain (2.27) and consequently (2.29), which therefore are also valid for partial exclusion. We conclude that for an initial condition where each lattice site is occupied by at most one particle the dynamics of the density profile is determined by the one-particle probability distribution, irrespective of the possible double occupancies in intermediate states. This remains true for arbitrary spin $s = m/2$. But as opposed to the case of total exclusion, the initial condition A_N is a special choice here. For an arbitrary initial condition and arbitrary spin one can find other expressions for the time dependence of the density profile. It turns out that it is given in terms of the probability distributions for l-particle systems where $1 \leq l \leq m$. This can be shown using the selection rules

for matrix elements of SU(2). The calculation of similar relations for higher spin is lengthy but remains straightforward.

C. Other symmetries: The bosonic case

From our discussion it is clear that similar recursions and duality relations can be obtained if the time-evolution operator has non-Abelian symmetries other than SU(2) (Abelian symmetries do not lead to degeneracies which are essential in our calculations). A simple example is the Hamiltonian (2.30) with $p_{j,k}^{(1)} = p_{j,k}^{(2)} = \cdots = p_{j,k}^{(m)}$ and all other coupling constants 0. In this case H is SU$(m+1)$ symmetric and describes the fully symmetric diffusion of m species of exclusive particles. Here the higher symmetry allows for additional relations which we do not want to write down. Also the q-deformed versions of these symmetry algebras can be used to obtain results of similar nature. In this case the time-evolution operator describes driven diffusion in media with reflecting boundary conditions [29].

Another special case which we want to discuss in more detail is a Hamiltonian commuting with generators $N = \sum c_j^\dagger c_j$ and $S^\pm$ (2.15) of a harmonic-oscillator algebra with $[c_j^\dagger, c_k] = \delta_{j,k}$. This corresponds to noninteracting particles.

As in the fermionic case the total particle number is conserved, H is Hermitian, and Eqs. (2.17) and (2.18) result from the symmetry (2.13). Only the normalization constants and the explicit form of the steady states are different:

$$| N \rangle_{\text{norm}} = \frac{N!}{L^N} | N \rangle = \frac{N!}{L^N} \sum_{\{n_j\}}^{(N)} \left(\prod_{j=1}^{L} \frac{1}{n_j!} \right) | \underline{n} \rangle \ . \tag{2.35}$$

These states correspond to homogeneous probability distributions as for fermions. If the system is in the state $| N \rangle_{\text{norm}}$ any configuration with particle number N has a probability proportional to $\prod_{j=1}^{L} \frac{1}{n_j!}$.

In a way analogous to that in Sec. II A we find a recursion relation

$$G_{N+1}^{\text{boson}}(x,y;t) = \frac{N+1}{N} G_N^{\text{boson}}(x,y;t) + \frac{N+1}{L^2} \tag{2.36}$$

from which follows

$$G_N^{\text{boson}}(x,y;t) = N G_1^{\text{boson}}(x,y;t) + \rho^2 \tag{2.37}$$

in the thermodynamic limit. Instead of Eq. (2.33) we obtain for the connected two-point correlation function for bosons

$$C_N^{\text{boson}}(x,y,z) = \rho P(x,t|y,0) \ . \tag{2.38}$$

The probability on the rhs is the same as in the fermionic case because there is no distinction between fermion and boson for a single-particle system. The amplitude of the correlation function can be obtained from the model with partial exclusion by taking the limit $m \to \infty$. This gives a quantitative description of the effect caused by the particle exclusion for the class of systems considered here.

III. APPLICATIONS TO CONTINOUS-TIME PROCESSES

In this section we illustrate our results in several models. First we observe that double-hopping processes, i.e., pieces in the Hamiltonian proportional to a term $\sigma_k \sigma_l \sigma_m \sigma_n$ with k,l,m,n pairwise different, can be ignored as long as one is interested only in the two- or three-point functions (2.8) or (2.9) or in the dynamics of the density $\rho_{F_0}(x,t) = \langle N | n_x U_t | F_0 \rangle$. (Here $|F_0\rangle$ is some arbitrary initial condition.) The double-hopping operators do not change the steady state and do not enter the dynamics of these quantities which are completely determined by the single-particle excitations. We shall therefore consider in this section only Hamiltonians H with single-hopping operators.

A. Many-particle diffusion in a random barrier environment

A well-known example is the diffusion of exclusive particles in a random environment which is constant in time [1, 2, 22]. Here we consider a system of partially exclusive particles. Our discussion is to a large extent a review and rederivation of known results which we shall use for comparison in Sec. IV.

In one dimension the time evolution is defined by the Hamiltonian (see also [7, 12])

$$H = -\sum_j \{ p_j [c_{j+1}^\dagger c_j - c_{j+1} c_{j+1}^\dagger c_j^\dagger c_j] + p_{j-1} [c_{j-1}^\dagger c_j - c_{j-1} c_{j-1}^\dagger c_j^\dagger c_j] \} \tag{3.1}$$

$$= -\frac{1}{2} \sum_j p_j [\sigma_j \sigma_{j+1} - 1] \ . \tag{3.2}$$

This is the Hamiltonian of a generalized Heisenberg spin-s ferromagnet with space-dependent spin-spin coupling. The local hopping probabilities $0 < p_i < \infty$ are random numbers with distribution μ. We shall use the notation $\overline{Q}$ for averages of quantities Q with respect to μ. In particular, we denote the mean value of p_x by $\overline{p}$ and its variance $\overline{(p_x - \overline{p_x})^2}$ by σ^2. (We assume μ to be translationally invariant.)

The SU(2) symmetry of H is obvious for each realization of the environment and $\rho = N/L$ is the same for each realization. Hence we can average the linear equations (2.33) over the environment. Without any additional calculation one immediately obtains the result

$$\overline{C_N(x,y;t)} = \rho \left(1 - \frac{\rho}{m} \right) \overline{P(x,t|y,0)} \ . \tag{3.3}$$

Equation (3.3) gives the averaged two-point correlation functions for the many-particle system provided $\overline{P(x,t|y,0)}$ is known. The latter quantity is the averaged solution of the one-particle master equation

$$\partial_t P(x,t|y,0) = p_x[P(x+1,t|y,0) - P(x,t|y,0)] + p_{x-1}[P(x-1,t|y,0) - P(x,t|y,0)] \quad . \tag{3.4}$$

The exclusion interaction enters only in the amplitude of the correlation function.

We briefly review some known results for the one-particle process. The simplest case is that of classical diffusion, i.e., with deterministic $p_x = D \ \forall x$, for which $\overline{P(x,t|y,0)} = P(x,t|y,0)$ is well known from textbooks [30]. For $x,t \gg 1$ it is given by

$$P(x,t|y,0) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}} \tag{3.5}$$

and its Fourier-Laplace transform $\tilde{S}(k,\omega)$ is

$$\tilde{S}(k,\omega) = \frac{1}{\omega + Dk^2} \quad . \tag{3.6}$$

Next we consider a random-barrier environment where the hopping rates $0 < p_x < \infty$ are distributed with a measure μ which is translationally invariant and ergodic. These assumptions are sufficient to show that the stochastic process defined by H (3.1) converges to Brownian motion in the limit $x,t \to \infty$ with x^2/t =const [22]. Here we are interested in the correction to Brownian motion of the averaged process for x and t large, but finite.

If we also assume that μ is a product measure, i.e., if the hopping rates p_x are *uncorrelated* random variables, then the short-wavelength low-frequency behavior of the Fourier-Laplace transform $\overline{\tilde{S}(k,\omega)}$ of $\overline{P(x,t|y,0)}$ is known [15–20]. One introduces a generalized diffusion constant $D(k,\omega)$ by writing

$$\overline{\tilde{S}(k,\omega)} = \frac{1}{\omega + D(k,\omega)k^2} \tag{3.7}$$

and finds [18, 19]

$$D(k,\omega) = D_0 + D_1\sqrt{\omega} + D_2\omega + \cdots - k^2(E_0 + E_1\sqrt{\omega} + \cdots) + \cdots . \tag{3.8}$$

Here D_j and E_j are functions of the moments $\mu_l = \overline{(p^{-1} - \overline{p^{-1}})^l}$ of the inverse hopping rates. Expanding them around $D = \overline{p}$ and neglecting higher powers of σ^2 as well as higher moments $\sigma_l = \overline{(p-\overline{p})^l}$ one obtains

$$D_0 = \overline{p^{-1}} \approx D - \sigma^2 D^{-1}, \tag{3.9}$$

$$D_1 = \frac{1}{2} D_0^{5/2} \mu_2 \approx \frac{1}{2}\sigma^2 D^{-3/2} \quad . \tag{3.10}$$

Within the framework of our approximation D_2 is 0 and $E_0 = D_0/12$, $E_1 = D_1/12$, and we can write

$$D(k,\omega) = D\left(1 - \frac{k^2}{12}\right)\left(1 - \frac{\sigma^2}{D^2}(1 - \tfrac{1}{2}\sqrt{D\omega})\right) \quad . \tag{3.11}$$

The prefactor $1 - k^2/12$ has its origin in the Fourier transform of the lattice Laplacian and is present also in the ordered system. Up to order σ^2 the uncorrelated disorder affects only the time dependence of the correlation function. For $\omega, k \to 0$ and ω/k^2 fixed one finds from (3.5) the diffusion constant $D_0^{-1} = \overline{p^{-1}}$.

The problem of corrections to Brownian motion in the case of *correlated* disorder has been addressed in [21] and [23]. In Sec. IV we will study this problem for a different exclusion process, described by the transfer matrix of the six-vertex model and show under which conditions disorder correlations are negligible.

B. A simple diffusion-reaction model

The two-species model we are going to study next is an example for a system where the SU(2) symmetry holds only in a subspace. In this subspace, which contains the steady states, we will find simple expressions for correlation functions.

Consider two species A and B (the corresponding operators are denoted with an upper index) diffusing and reacting on a lattice. Each site may contain one A particle or one B particle or it is empty. A certain configuration corresponds to a Fock space vector of the type $|\, A\, A\, 0\, B\, 0\, \ldots\rangle$ where an A at the jth position means the site j is occupied by an A particle and similar for B and 0. The dynamics is assumed to be based on next-neighbor interactions only. It is defined by the infinitesimal time transitions

$$\begin{aligned}
|\, 0\, A\,\rangle &\longrightarrow \frac{1}{1+\beta}|\, A\, 0\,\rangle + \frac{\beta}{1+\beta}|\, B\, 0\,\rangle, \\
|\, 0\, B\,\rangle &\longrightarrow \frac{\alpha}{1+\beta}|\, A\, 0\,\rangle + \frac{\alpha\beta}{1+\beta}|\, B\, 0\,\rangle, \\
|\, A\, 0\,\rangle &\longrightarrow \frac{1}{1+\beta}|\, 0\, A\,\rangle + \frac{\beta}{1+\beta}|\, 0\, B\,\rangle, \\
|\, B\, 0\,\rangle &\longrightarrow \frac{\alpha}{1+\beta}|\, 0\, A\,\rangle + \frac{\alpha\beta}{1+\beta}|\, 0\, B\,\rangle,
\end{aligned} \tag{3.12}$$

where the indicated states are supposed to represent the occupation of nearest neighbor in a d-dimensional lattice. All other possible pairs of configurations remain unchanged. Equations (3.12) describe diffusion with constant, space-independent hopping rates as well as reactions $A \longrightarrow B$ and $B \longrightarrow A$, which are coupled to the jump events. We will assume $\alpha = 1$ in which case H has an SU(2) invariant subspace.

Writing the probabilities of occupations 0, A, and B for a single site j as a column vector $\tilde{v}$ we may introduce matrices

$$\tilde{\tau}_j^x = \frac{1}{\sqrt{1-\beta}}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix}, \quad \tilde{\tau}_j^y = \frac{1}{\sqrt{1-\beta}}\begin{pmatrix} 0 & -i & -i \\ i & 0 & 0 \\ i\beta & 0 & 0 \end{pmatrix}, \quad \tilde{\tau}_j^z = \frac{1}{\sqrt{1-\beta}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad . \tag{3.13}$$

The system dynamics is then given by the Hamiltonian

$$H = -\frac{1}{2}\sum_j (\tilde{\tau}_j \tilde{\tau}_{j+1} - 1) \ . \tag{3.14}$$

The symmetry properties become obvious after a basis transformation given by the matrix

$$U = \begin{pmatrix} \sqrt{1+\beta} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -\beta & 1 \end{pmatrix} \ . \tag{3.15}$$

The $\tilde{\tau}$ matrices transform to

$$\tau_j = U \tilde{\tau}_j U^{-1}, \tag{3.16}$$

which read explicitly

$$\tau_j^x = \frac{1}{\sqrt{1-\beta}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \tau_j^y = \frac{1}{\sqrt{1-\beta}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \tau_j^z = \frac{1}{\sqrt{1-\beta}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ . \tag{3.17}$$

The components of the transformed column vectors $v = U\tilde{v}$ can be understood as the probabilities of occupations 0, a, or b at the lattice site j, where quasiparticles a and b are introduced. Their dynamics are described by

$$H = -\frac{1}{2}\sum_j (\tau_j \tau_{j+1} - 1) \ . \tag{3.18}$$

From Eqs. (3.17) and (3.18) we see that no b particles are produced and that the weight of b occupation at any lattice site decays exponentially in time. Hence the steady state contains no b particles. Consequently, consideration of the dynamics in the b-particle free subspace gives all the information necessary for the analysis of steady-state correlations. In this subspace, which is invariant of course, the σ matrices reduce to Pauli matrices, i.e., the dynamics is SU(2) symmetric there. All the results from Secs. II A and III A apply to the dynamics of a-occupation numbers then. Translating this in expressions for A- and B-occupation numbers we find

$$\langle n_x^A \rangle_{\rm st} = \frac{1}{1+\beta}\rho, \tag{3.19}$$

$$\langle n_x^B \rangle_{\rm st} = \frac{\beta}{1+\beta}\rho, \tag{3.20}$$

where

$$\rho = N/L = \sum_{x=1}^{L} \langle n_x^a \rangle_{st}/L \quad \text{with} \quad n_x^a = n_x^A + n_x^B \tag{3.21}$$

is the particle density.

Because of Eqs. (2.25) and (3.5) we find for $x, t, L, N \gg 1$ the following connected correlation function:

$$\langle n_x^a(t) n_y^a(0) \rangle_{\rm st} - \rho^2 = \rho(1-\rho)\frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \ . \tag{3.22}$$

Hence the number of particles at a lattice site shows diffusive behavior and the averaged number of B particles at a given lattice site x and the averaged number of A particles there have a constant ratio β. Choosing $\alpha \neq 1$ breaks the SU(2) symmetry and the correlations become more complicated.

IV. CORRELATION FUNCTIONS IN A RANDOM-SIX-VERTEX MODEL

In this section we restrict ourselves to one space dimension only and consider discrete-time dynamics. It was shown by Kandel *et al.* that the diagonal-to-diagonal transfer matrix of the six-vertex model for a certain one-parameter family of vertex weights describes diffusion of particles with exclusion in one dimension [25]. In this mapping the vertex weights are hopping probabilities p and the time evolution proceeds along one diagonal of a square lattice while the space extends along the diagonal perpendicular to the time diagonal (see Fig. 1). They consider the standard case where the vertex weights do not depend on the position of the vertex in the two-dimensional lattice. As in Sec. III we want to study space-dependent hopping rates. In the framework of the mapping to the vertex model on a square lattice this is achieved by introducing vertex weights which are constant on the time diagonal of the lattice but vary along

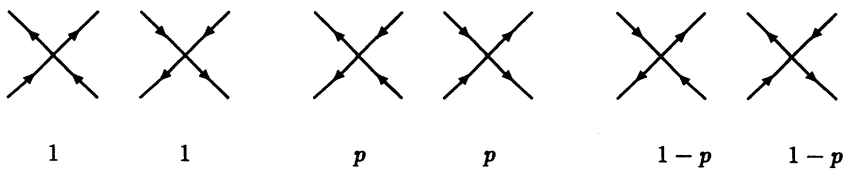

FIG. 1. Allowed vertex configurations in the six-vertex model and their Boltzmann weights. Up-pointing arrows correspond to particles, down-pointing arrows represent vacant sites. In the dynamical interpretation of the model the Boltzmann weights give the transition probability of the state represented by the pair of arrows below the vertex to that above the vertex.

the space diagonal. For the convenience of the reader we repeat this mapping in Appendix B and generalize it to arbitrary space-dependent vertex weights.

A study of this model is interesting for several reasons. First of all, the calculation of correlation functions for the vertex model is interesting in its own right. Kandel *et al.* derived an expression for the connected two-point function (2.8) only for the special value $p = 1/2$. Here we derive the exact expression for arbitrary values of p. Second, the vertex model describes an exclusion model with simultaneous many-particle hopping and it is not *a priori* clear whether the known results for the continous-time process with single-particle hopping quoted and partially rederived in Secs. II and III A hold in this model. Third, for hopping rates $p \simeq 1$, the model indeed does *not* have diffusive behavior, instead particles move relativistically close to their light cone. The vertex model is in so far more general than the continuous-time model and we shall study the crossover from relativistic motion to classical diffusion.

The dynamics of the exclusion process which leads to the generalized transfer matrix of the six-vertex model are defined as follows: We present the state of the system with L sites (L even) at time t by the quantity $\underline{n}(t) = \{n_1(t), n_2(t), \ldots, n_L(t)\}$ where $n_x(t)$ counts the number of particles on site x and can take the values 0 or 1. The time evolution consists of two steps. Suppose the system is in the state $\underline{n}(t)$ with t an integer. In the first half-time step $t \to t + 1/2$ we divide the chain of L sites into pairs of sites $(1,2)$, $(3,4)$, ..., $(L-1, L)$. If both sites in a pair $(2x-1, 2x)$ are occupied or empty then they remain so with probability 1. (We exclude the possibility of particle creation or annihilation.) If there is one particle and one hole then the particle hops to the unoccupied site in the pair with probability p_{2x-1} and remains where it was with probability $1 - p_{2x-1}$. Note that the hopping probability in such a pair is the same for both directions, i.e., it does not depend on whether the particle is on site $2x-1$ or on site $2x$. These hopping rules are applied in parallel to all pairs in the chain. In the second half-time step $t+1/2 \to t+1$ we shift the pairing of the chain by one lattice unit such that the pairs are now $(2,3)$, ... $(L,1)$ (we assume periodic boundary conditions). We apply the same rules as above, but the hopping probabilities in a pair $(2x, 2x+1)$ are now p_{2x}. From these rules one can derive a master equation for the probability distribution $F(\underline{n}, t)$ (2.1).

Instead of working with a master equation, we directly study the transfer matrix T, which encodes these hopping rules as discussed in Sec. II. We choose as a basis of the Fock space, the same basis as in Sec. III where the presence of a particle corresponds to spin down and the absence of a particle corresponds to spin up. Then the transfer matrix is given by

$$T = T^{\text{even}} T^{\text{odd}} = \prod_{j=1}^{L/2} T_{2j}(p_{2j}) \prod_{j=1}^{L/2} T_{2j-1}(p_{2j-1}), \qquad (4.1)$$

where

$$T_j(p_j) = 1 - \frac{p_j}{2}(\boldsymbol{\sigma}_j \boldsymbol{\sigma}_{j+1}) \qquad (4.2)$$

with the Pauli matrices $\sigma^{x,y,z}$. The time-evolution operator T^t for t time steps is defined as the kth power of T if $t = k$ is integer and as $T^{\text{odd}} T^k$ if $t = k + 1/2$. We use periodic boundary condition and occasionally label the spatial indices from $-L/2$ to $L/2 - 1$. (Because of the periodic boundary condition they are defined mod L.) The local transfer matrices $T_j(p_j)$ act as unit operator on all sites except on the pair $(i, i+1)$. The SU(2) symmetry of T is obvious and therefore the time-dependent connected two-point correlation function in the steady state

$$C_N(x, y; t) = \langle\, s \,|n_x T^t n_y|\, N \,\rangle - \rho^2 \qquad (4.3)$$

in the sector with $N = \rho L$ particles is given by the correlation function $G_1(x, y; t)$ in the same environment in the one-particle sector as in (2.25). This is true for any choice of the p_j and we can restrict our discussion to the one-particle sector as in the continuous-time case. We shall omit the index 1 in the correlator and simply write $G(x, y; t)$. In order to avoid boundary effects we shall furthermore work in the thermodynamic limit $L \to \infty$.

In the one-particle sector one finds (see Appendix C)

$$G(x, y; t + 1/2) = \begin{cases} (1 - p_{x-1}) G(x, y; t) + p_{x-1} G(x-1, y; t), & x \text{ even} \\ (1 - p_x) G(x, y; t) + p_x G(x+1, y; t), & x \text{ odd.} \end{cases} \qquad (4.4)$$

In the same way one obtains

$$G(x, y; t + 1) = \begin{cases} (1 - p_x) G(x, y; t + 1/2) + p_x G(x+1, y; t + 1/2), & x \text{ even} \\ (1 - p_{x-1}) G(x, y; t + 1/2) + p_{x-1} G(x-1, y; t + 1/2), & x \text{ odd} \,. \end{cases} \qquad (4.5)$$

From these recursion relations, which are the analog of the master equation (3.4), together with the initial condition

$$G(x, y; 0) = \delta_{x,y} \qquad (4.6)$$

one can compute the exact correlation function for arbitrary values of the local hopping probabilities p_x. Note that these hopping rules are not left-right symmetric.

The simplest nontrivial case is the homogeneous model $p_x = \text{const} = p$. In this case the system is invariant under translations by two lattice units and $G(x, y; t) = G(x + 2z, y + 2z; t)$ depends only on whether y is even or odd and on the distance $r = x - y$. For the special choice $p = 1/2$ the correlation function was computed (in a different way) by Kandel *et al.* [25] (see Appendix B).

We find that for arbitrary values of p the solution to the recursion relations (4.4) and (4.5) with initial condition (4.6) for x even and t integer is given by

$$G(x,y;t)=p^{2t}\delta_{-r,2t}+\sum_{k=1}^{t-|r|/2}\binom{t-r/2}{k}\binom{t-1+r/2}{k-1}p^{2t-2k}(1-p)^{2k},\quad y\text{ even} \tag{4.7}$$

$$G(x,y;t)=\sum_{k=0}^{t-(|r|-1)/2}\binom{t-(r-1)/2}{k}\binom{t+(r-1)/2}{k}p^{2t-2k-1}(1-p)^{2k+1},\quad y\text{ odd}\ . \tag{4.8}$$

For x odd and t integer one finds

$$G(x,y;t)=G(y,x;t),\qquad y\text{ even} \tag{4.9}$$

$$G(x,y;t)=G(y+1,x+1;t),\qquad y\text{ odd}\ . \tag{4.10}$$

The correlator for half-odd integer values of t is given by relations (4.4). For finite densities $\rho=N/L$ one obtains the exact connected correlation function (in the thermodynamic limit) by multiplying the expressions (4.7) by $\rho(1-\rho)$. Note that for $p=1/2$ these expressions simplify considerably and we recover the result of Ref. [25] [see Eqs. (B7)–(B10) in Appendix B].

Now we study some limiting cases. In the limit $r,t\to\infty$ with r^2/t fixed the distinction between even and odd distances vanishes and the correlator has the form

$$C_N(r,t)=\rho(1-\rho)(4\pi Dt)^{-1/2}e^{-r^2/4Dt} \tag{4.11}$$

with the diffusion constant $D=p/(1-p)$ which establishes the diffusive behavior of the process for $p\neq 0,1$.

For $p=1$ the correlation function reduces to the δ function $C_N(r,t)=\rho(1-\rho)\delta_{\pm r,2t}$ for even distances and $C_N(r,t)=0$ for odd distances. In this case the process is *not* diffusive. The correlation function is invariant under the scale transformations $r\to\lambda r$, $t\to\lambda t$ corresponding to a dynamical exponent $z=1$ as opposed to $z=2$ for normal diffusion. This result can be understood as follows: If $p=1$, particles which are on odd lattice sites at integer time steps move to right at a constant rate of two lattice units in space direction per full time step while particles on the even sublattice move to the left with same velocity which is the velocity of light in the system. Thus we have a system of noninteracting massless relativistic right and left movers. More generally, the distinction between even and odd space coordinates becomes physically meaningful in the limiting case when all p_j are close to or equal to one. If p_x is not equal but close to 1, one expects the particles to remain relativistic but massive. For times smaller than a crossover time scale $\xi_t\sim 1/(1-p)$ the correlator is nonvanishing only near the light cone and for even distances. On larger time scales the system converges to Brownian motion. This feature of the vertex model makes it more interesting than the continuous-time formulation by the Hamiltonian (3.1), which allows only for nonrelativistic diffusion. The homogeneous massive system is studied in detail in [31].

Now we study the average behavior of the correlation function in a random environment. We assume all p_x to be distributed in the interval $0<p_x\leq 1$ with a translationally invariant distribution μ. From the results obtained in [22] one might guess that if μ is ergodic, the process will converge to Brownian motion as it does in the continuous-time case for hopping rates $0<p_x<\infty$.

From our discussion in the preceding paragraph we see that one has to distinguish two cases. (a) $1-\overline{p}$ is of order $1/t$: Expanding the averaged correlation function $\overline{C}_N(r,t)$ to lowest order in the moments of the distribution μ gives the correlator of a homogeneous system with fixed hopping rate $\overline{p}$. As shown above in this case the process is not diffusive and therefore the guess is wrong. (b) $1-\overline{p}$ is larger than of order $1/t$: Here the lowest-order term is of the form (4.11) with $D=\overline{p}/(1-\overline{p})$. In what follows we compute the next nonvanishing contribution in the expansion of $\overline{C}_N(r,t)$ and show that to this order the guess is correct. Furthermore we shall compute explicitly the contribution to the correlation function arising from disorder correlations and make a comparison with the result reviewed in Sec. III for uncorrelated disorder.

We introduce the quantity

$$\Delta_x=p_x-\overline{p}\ . \tag{4.12}$$

For the sake of technical simplicity we choose as mean value $\overline{p}=1/2$ corresponding the diffusion constant $D=1$. As long as $\overline{p}$ is not close to 1, such a choice has no qualitative influence on the averaged correlation function. Furthermore we shall assume that the hopping probabilities of even and odd lattice sites are uncorrelated,

$$\overline{\Delta_{2x-1}\Delta_{2y}}=0 \tag{4.13}$$

while the correlations

$$\overline{\Delta_{2x}\Delta_{2y}}=\overline{\Delta_{2x-1}\Delta_{2y-1}}=\sum_\nu h(2\nu)\delta_{r,2\nu} \tag{4.14}$$

depend only the absolute value of the distance $r=|2y-2x|$. The quantity $h(0)=\sigma^2$ appearing in the sum on the rhs of Eq. (4.14) is the variance of Δ_x. Finally, we consider only distributions which are sharply centered around their mean value such that higher moments such as $\overline{\Delta_{2x}\Delta_{2y}\Delta_{2z}}$, etc. can be neglected in a perturbative expansion of the averaged correlation function in the moments of p_x.

With the definition (4.12) of the quantities Δ_x we write $A^{\rm odd}=A_0^{\rm odd}+\Delta^{\rm odd}$ and analogously $A^{\rm even}=A_0^{\rm even}+\Delta^{\rm even}$. $A_0^{\rm odd(even)}$ are the one-particle transfer matrices with all $p_x=1/2$ (see Appendix C). The averaged correlation function is averaged matrix element

$$\overline{G(x,y;t)}=\overline{(A^t)_{x,y}}=\langle\, x\,|\overline{\left(A_0+A_0^{\rm even}\Delta^{\rm odd}+\Delta^{\rm even}A_0^{\rm odd}+\Delta^{\rm even}\Delta^{\rm odd}\right)^t}|\,y\,\rangle\ . \tag{4.15}$$

Neglecting all pieces with more than two Δ matrices in this expression and using $\overline{\Delta^{\rm even}\Delta^{\rm odd}}=0$ we obtain the lowest-order correction to the correlation function in the presence of disorder

$$\Gamma(x,y;t) = \overline{G(x,y;t)} - G^{(0)}(x,y;t)$$
$$= \sum_{l=0}^{n}\sum_{m=0}^{n-l} \langle\, x\,|A_0^l \overline{(\Delta^{\rm even}A_0^{\rm odd})A_0^m(\Delta^{\rm even}A_0^{\rm odd})}A_0^{n-l-m}|\, y\,\rangle \qquad (4.16)$$
$$+ \sum_{l=0}^{n}\sum_{m=0}^{n-l} \langle\, x\,|A_0^l \overline{(A_0^{\rm even}\Delta^{\rm odd})A_0^m(A_0^{\rm even}\Delta^{\rm odd})}A_0^{n-l-m}|\, y\,\rangle \ .$$

Here $G^{(0)}(x,y;t)$ denotes the correlator of the ordered system with $p = 1/2$ and $n = t-2$. Some calculation shows that $\Gamma(x,y;t)$ can be expressed in terms of a sum of three-point correlation functions of the ordered system:

$$\Gamma(x,y;t) = 16\sum_{\nu} h(2\nu) \sum_{m=0}^{n}[(n-m)D_\nu(x,y;m) + \hat{D}_\nu(x,y;m)]D_\nu(0,0;m+\tfrac{1}{2}) \qquad (4.17)$$

with

$$D_\nu(x,y;k) = \langle\, x\,|\,(A_0^k - A_0^{k+1})\,|\,y+2\nu\,\rangle, \qquad (4.18)$$
$$\hat{D}_\nu(x,y;k) = \langle\, x\,|(A_0^{k+1/2} - A_0^{k+1})|\,y+2\nu\,\rangle \ . \qquad (4.19)$$

Multiplying Eq. (4.17) by $\rho(1-\rho)$ yields the lowest-order correction to the averaged two-point correlation function.

It is easy to compute the exact autocorrelation function $\overline{G(0,0;t)}$ if $h(2\nu) = \sigma^2\delta_{\nu,0}$, i.e., in the absence of disorder correlations. From Eqs. (4.17), (4.18), and (C7) one obtains

$$\overline{G(0,0;t)} = \left(1 + 4\sigma^2\frac{2t}{2t-1}\right)G^{(0)}(0,0;t) \ . \qquad (4.20)$$

For large times this has the expected form (3.8) and (3.9) for $D = 1$ and variance $4\sigma^2$: $\overline{G(0,0;t)} \sim (4\pi D_0 t)^{-1/2}$ with $D_0 = 1 - 4\sigma^2$.

The behavior of the correlation function for $r \neq 0$ and in the presence of disorder correlations becomes more transparent after a Fourier-Laplace transformation. The discrete Fourier-Laplace transform (see Appendix B) of $G^{(0)}(x,y;t)$ is given by

$$\tilde{S}(k,\omega) = \frac{1}{1 - e^{-\omega}\cos^2 k} \qquad (4.21)$$

while Fourier-Laplace transformation of $\Gamma(x,y;t)$ yields

$$\tilde{\Sigma}(k,\omega) = \frac{4\sin^2 k(1-e^{-\omega})}{1-e^{-\omega}\cos^2 k}[(\sigma^2(1-\tfrac{1}{2}\sqrt{1-e^{-\omega}}) - L(k,\omega)]\tilde{S}(k,\omega) \ . \qquad (4.22)$$

The function

$$L(k,\omega) = \sqrt{1-e^{-\omega}}\sum_{\nu=1}^{\infty} h(2\nu)\cos(2ik\nu) \times \left(\frac{e^{-\omega/2}}{1+\sqrt{1-e^{-\omega}}}\right)^{|2\nu|} \qquad (4.23)$$

on the rhs of Eq. (4.22) is the contribution of the disorder correlation function $h(2\nu)$ to the generalized diffusion constant. We would like to stress that up to this point all results are exact first-order contributions, i.e., valid for arbitrary integer values of r and $t \geq 0$ and arbitrary values of k and $\omega \geq 0$.

First we study the low-frequency behavior. In the limit $\omega, k \to 0$, keeping ω/k^2 fixed, $\tilde{S}(k,\omega)$ becomes the well-known quantity

$$\tilde{S}(k,\omega) = \frac{1}{\omega + k^2} \qquad (4.24)$$

while for the first-order correction (4.22) we obtain

$$\tilde{\Sigma}(k,\omega) = \frac{4k^2}{\omega+k^2}[\sigma^2 - \sqrt{\omega}\lim_{a\to 0} aL(ka,\omega a^2)]\tilde{S}(k,\omega) \ . \qquad (4.25)$$

From this expression we realize that the contribution of disorder correlations vanishes if their correlation length is finite. Furthermore, if they decay with a power law $h(2\nu) \sim |2\nu+1|^{-\alpha}$, then its contribution still vanishes for $\alpha \geq 0$. $\alpha = 0$ means that all fluctuating hopping rates p_{2x} would be equal to some quantity $p^{\rm even}$ and all p_{2x-1} would be equal to $p^{\rm odd}$ and the averaged correlation function would be an average over semihomogeneous models with different hopping rates at even and odd time steps. On the other hand, $\alpha = \infty$ corresponds to completely uncorrelated choices of the hopping rates. Only if the disorder correlations are such that they are finite (nonvanishing) on an infinite number of points do they give a finite contribution to the averaged correlation function in the infinite time limit. Here the term "inifinite" is understood as proportional to the length L in the thermodynamic limit $L \to \infty$. This result is equivalent with the ergodicity of μ. Hence the result of De Masi *et al.* is valid for the vertex model at least up to lowest nontrivial order in perturbation theory.

Now we focus on the behavior for ω, k small, but finite. We expand $\tilde{\Sigma}(k,\omega)$ up to order $\sqrt{\omega}$. For uncorrelated disorder, $L(k,\omega) = 0$, we recover the result (3.11) of Refs. [18]–[20] by expanding $\cos k$ and $\exp(-\omega)$ to first order in their respective arguments. In this case the generalized diffusion constant does not depend on the frequency. This changes for correlated disorder. Assuming a decay of the form $h(2\nu) = \sigma^2\exp(-|2\nu|/\xi)$ one obtains for the correction

$$L(k,\omega) \approx \sigma^2\sqrt{\omega}\frac{e^{-2(\sqrt{\omega}+\xi^{-1})} - \cos 2k}{\cosh(2\sqrt{\omega}+2\xi^{-1}) - \cos 2k} \ . \qquad (4.26)$$

For $1 \ll \xi \ll \omega^{-1/2}$ this expression becomes

$$L(k,\omega) \approx \sigma^2\sqrt{\omega}\frac{\xi^{-1}}{\xi^{-2}+k^2} \to \sigma^2\sqrt{\omega}\xi \ . \qquad (4.27)$$

This piece is independent of k for low frequencies $k \propto \sqrt{\omega}$.

$\omega^{-1/2}$ plays the role of a crossover length scale where the correlation function changes its behavior. For $1 \ll \omega^{-1/2} \ll \xi$ one obtains a k-dependent contribution in order $\sqrt{\omega}$:

$$L(k,\omega) \approx \sigma^2 \sqrt{\omega} \frac{\sqrt{\omega}}{\omega + k^2} = \sigma^2 \frac{\omega}{\omega + k^2} \quad . \tag{4.28}$$

For a decay of the disorder correlations of the form $h(2\nu) \sim \sigma^2 |2\nu|^{-\alpha} \exp(-|2\nu|/\xi)$ the contribution of $L(k,\omega)$ to $D(k,\omega)$ is small for $\alpha > 1$, i.e., smaller than of order $\sqrt{\omega}$. In this situation the contribution of the disorder correlations can be neglected and the system behaves as it was uncorrelated.

For $0 \leq \alpha < 1$ and $\xi \gtrsim \sqrt{\omega}$ the disorder contribution is larger than of order $\sqrt{\omega}$, giving rise to a qualitative change in the frequency dependence of the diffusion constant and leading also to a k dependence.

For $\alpha = 1$ and $\xi = \infty$ one finds

$$L(k,\omega) = \frac{1}{2} \sqrt{\omega} \ln(1 + e^{-2(\omega + \xi^{-1})} - 2e^{-(\omega + \xi^{-1})} \cos 2k) \tag{4.29}$$

from which the various limiting cases can be easily derived. Our results show that with decreasing disorder correlations $\tilde{\Sigma}(k,\omega)$ increases until one reaches $\alpha = 1$. For a stronger decay only the variance σ^2 of the distribution function is relevant.

V. CONCLUSION

We have studied systems of particles hopping stochastically on lattices of arbitrary dimension with space-dependent hopping probabilities. We demonstrated how to exploit non-Abelian symmetries of such models for the derivation of duality relations which express complicated correlation functions in terms of other correlators which are easier to compute. This was done in detail for SU(2)-invariant dynamics [see (2.27] and (2.34)] generalizing earlier work on the symmetric simple exclusion process [1, 2]. Examples of such models are discussed in Secs. II–IV and include partial exclusion processes and vertex models.

We have shown that in all SU(2)-symmetric models the two-point and three-point density correlation functions in the steady state with N particles are given by the probability distribution in space of a single particle moving in the same, generically disordered and time-dependent, environment [see Eq. (2.33)]. The exclusion interaction determines only the amplitude of the correlation function. Furthermore, these correlation functions do not change if one allows for simultaneous hopping of more than particle. The time evolution of the density profile depends only on correlation functions of l-particle systems where l is not larger than the maximal number of particles allowed on each lattice site. For special initial conditions we obtain more specific results (2.29). For bosonic systems one finds analogous results (2.38) if the time-evolution operator commutes with the generators of the harmonic-oscillator algebra.

The Fock space formalism we have applied reveals relations to quantum systems. In particular, a classical diffusive system of exclusive particles is shown to be equivalent to a generalized Heisenberg ferromagnet with space- and time-dependent spin-spin coupling. Here the density correlation function corresponds to a spin correlation function. For discrete time, the time evolution operator of one-dimensional systems is the transfer matrix of some higher disordered-vertex models. Total exclusion is described by the transfer matrix of a six-vertex model with space-dependent vertex weights. In this mapping, density correlations correspond to arrow correlation functions.

Focusing on one-dimensional systems we have studied a version of the random-barrier model with spatially correlated disorder. Its time evolution is given by the diagonal-to-diagonal transfer matrix of the six-vertex model with a certain random choice of vertex weights. We study the steady state and derive expressions (4.17) and (4.22) for the averaged time-dependent two-point density correlation function in the N-particle sector in the presence of weak disorder. These expression are exact in the lowest order of the expansion of the averaged function in the moments of distribution of the hopping rates.

We study the thermodynamic limit (infinite length) and compare our results with known results for systems defined in continuous time. In the infinite-time limit $t \to \infty$ the one-particle probability distribution (which up to a constant is the density correlation function) turns out to converge to Brownian motion if disorder correlations decay to zero in space (algebraically or exponentially) and if the mean value $\bar{p}$ of the hopping rates does not approach 1 as $1/t$ or faster. The latter case describes relativistic stochastic motion in a random medium. If $\bar{p}$ approaches 1 a phase transition occurs where the dynamical exponent changes from $z = 2$ to $z = 1$.

Considering fixed (time-independent) hopping rates and large but finite times, we compare our results for correlated disorder with other known results for uncorrelated disorder. If the correlation length ξ of the disorder correlations is finite or if the correlations $h(r)$ have infinite range but decay faster than r^{-1}, the contribution of the disorder correlations becomes negligibly small for large times and the system behaves like the system with uncorrelated disorder. For infinite-ranged disorder with a slower decay, $h(r) \sim r^{-\alpha}$ where $0 \leq \alpha \leq 1$, the correlation function changes its behavior even after long times and the generalized diffusion constant $D(k,\omega)$ becomes k dependent even in the lowest order of the expansion. For uncorrelated disorder the diffusion constant depends only on ω in this approximation.

The most important problem left open in our work is a systematic approach to the study of duality relations arising from non-Abelian symmetries.

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APPENDIX A: RECURSION RELATION FOR THE TWO-POINT FUNCTION

Starting from Eq. (2.21) for the N-particle correlation function and assuming the symmetry (2.13) we will prove the recursion relation (2.22). For this we will use the following equations which can be proved easily:

$$S^-|\,N\,\rangle = (N+1)|\,N+1\,\rangle, \tag{A1}$$

$$c_j^\dagger|\,N\,\rangle = n_j|\,N+1\rangle\,, \tag{A2}$$

$$c_j|\,N\,\rangle = (1-n_j)|\,N-1\rangle\,, \tag{A3}$$

$$S^+|\,N\,\rangle = (L-N+1)|\,N-1\rangle\,, \tag{A4}$$

$$[n_j, S^-]_- = c_j^\dagger\ . \tag{A5}$$

Writing down an expression similar to (2.21) for G_{N+1}^{ferm} and using Eq. (A1) it results

$$(N+1)\binom{L}{N+1} G_{N+1}^{\mathrm{ferm}}(x,y;t) = \langle\, N+1\,|n_x U_t n_y S^-|\,N\,\rangle\ . \tag{A6}$$

By means of Eq. (A5) we obtain

$$\begin{aligned}(N+1)\binom{L}{N+1} G_{N+1}^{\mathrm{ferm}}(x,y;t) &= \langle\, N+1\,|S^- n_x U_t n_y|\,N\,\rangle \\ &\quad +\langle\, N+1\,|c_x^\dagger U_t n_y|\,N\,\rangle \\ &\quad +\langle\, N+1\,|n_x U_t c_y^\dagger|\,N\,\rangle\ .\end{aligned}$$

Because of (A2)–(A4) this results in

$$N\binom{L}{N+1} G_{N+1}^{\mathrm{ferm}}(x,y;t) = (L-N-1)\binom{L}{N} G_N^{\mathrm{ferm}}(x,y;t) + \langle\, N\,|U_t n_y|\,N\,\rangle\ . \tag{A7}$$

Because of the normalization condition (2.6) the second term on the rhs of Eq. (A7) is equal to $\frac{N}{L}\,(\frac{L}{N})$. Thus (A7) results in the following recursion relation:

$$G_{N+1}^{\mathrm{ferm}}(x,y;t) = \frac{L-N-1}{L-N}\frac{N+1}{N} G_N(x,y;t) + \frac{1}{L}\frac{N+1}{L-N}\ . \tag{A8}$$

APPENDIX B: DUALITY RELATION

We will prove the duality relation (2.27). For $M < N$ Eq. (2.27) is a triviality since both sides vanish. In case $M = N$ Eq. (2.27) follows directly from the form (2.27). The nontrivial case $M > N$ is to be proved.

Starting from

$$P_{A_M}[A_M(t) \supset A_N] = \left\langle\, M \,\middle|\, \prod_{i\in A_N} n_i U_t \prod_{j\in A_M} n_j \,\middle|\, M\,\right\rangle \tag{B1}$$

and using the definition (2.17) of the states $|\,M\,\rangle$ as well as the Hermiticity of the operators occurring in (B1) we find

$$P_{A_M}[A_M(t) \supset A_N] = \frac{1}{M}\left\langle\, M \,\middle|\, \prod_{j\in A_M} n_j U_t \prod_{i\in A_N} n_i S^- \,\middle|\, M-1\,\right\rangle\ . \tag{B2}$$

The following commutator is easily computed:

$$\left[\prod_{i\in A_N} n_i\ ,\ S^-\right]_- = \sum_{k\in A_N} c_k^\dagger \prod_{i\in A_N\setminus k} n_i\ . \tag{B3}$$

From (A2) and (B3) we get

$$\left[\prod_{i\in A_N} n_i\ ,\ S^-\right]_- |\,M-1\,\rangle = N\prod_{i\in A_N} n_i|\,M\,\rangle\ . \tag{B4}$$

Using this we find from (B2)

$$P_{A_M}[A_M(t) \supset A_N] = \frac{1}{M}\left\langle\, M \,\middle|\, \prod_{j\in A_M} n_j U_t S^- \prod_{i\in A_N} n_i \,\middle|\, M-1\,\right\rangle + \frac{N}{M} P_{A_M}[A_M(t) \supset A_N]\ , \tag{B5}$$

i.e., since S^- commutes with H

$$(M-N)P_{A_M}[A_M(t) \supset A_N] = \left\langle\, M \,\middle|\, \prod_{j\in A_M} n_j S^- U_t \prod_{i\in A_N} n_i \,\middle|\, M-1\,\right\rangle\ . \tag{B6}$$

Using (B3) and (A3) we derive the following expression similar to (B4):

$$\langle\, M\,|\left[\prod_{j\in A_M} n_j\ ,\ S^-\right]_- = \sum_{k\in A_M} \langle\, M-1\,|(1-n_k)\prod_{j\in A_M\setminus k} n_j = \sum_{k\in A_M} \langle\, M-1\,|\prod_{j\in A_M\setminus k} n_j\ , \tag{B7}$$

where we have used $\langle\, M-1\,|\prod_{j\in A_M} n_j = 0$, which is obvious, because $\langle\, M-1\,|$ is a state with $M-1$ particles only. Inserting (B7) into (B6) results in

$$P_{A_M}[A_M(t) \supset A_N] = \frac{1}{M-N} \sum_{k \in A_M} \left\langle M-1 \left| \prod_{j \in A_M \setminus k} n_j U_t \prod_{i \in A_N} n_i \right| M-1 \right\rangle$$
$$= \frac{1}{M-N} \sum_{B_{M-1} \subset A_M} P_{B_{M-1}}[B_{M-1}(t) \supset A_N], \tag{B8}$$

where the B_{M-1} are sets with $M-1$ sites. Repeating the above procedure leads to

$$P_{A_M}[A_M(t) \supset A_N] = \frac{1}{(M-N)!} \sum_{B_N \subset B_{N+1}} \sum_{B_{N+1} \subset B_{N+2}} \cdots \sum_{B_{M-1} \subset A_M} P_{B_N}[B_N(t) \supset A_N]$$
$$= \sum_{B_N \subset A_M} P_{B_N}[B_N(t) = A_N] \quad . \tag{B9}$$

Here we used the fact that $\subset$ and $=$ mean the same for sets of equal cardinality. The prefactor cancels with the number of multiple countings of the set B_N in the sums. Since for sets of equal cardinality the duality relation (2.27) is known to be true [because of Eq. (2.28)], i.e., $P_{B_N}[B_N(t) = A_N] = P_{A_N}[A_N(t) = B_N]$, we find

$$P_{A_M}[A_M(t) \supset A_N] = P_{A_N}[A_N(t) \subset A_M], \tag{B10}$$

which is the relation to be proved.

APPENDIX C: THE SIX-VERTEX MODEL AS A DISORDERED DIFFUSIVE SYSTEM

Here we repeat the mapping of Ref. [25] of a one-dimensional diffusion problem to a six-vertex model and generalize it to a version of the random-barrier model. Consider the six-vertex model on a diagonal square lattice defined as follows: Place an up- or down-pointing arrow on each link of the lattice and assign a nonzero Boltzmann weight to each of the vertices shown in Fig. 1. (All other configurations of arrows around an intersection of two lines, i.e., all other vertices, are forbidden.) The partition function is the sum of the products of Boltzmann weights of a lattice configuration taken over all allowed configurations. In the transfer-matrix formalism up- and down-pointing arrows represent the state of the system at some given time t (Fig. 1). Each row of a diagonal square lattice is built by M of these vertices. Corresponding to the M vertices there are $L = 2M$ sites in each row. The configuration of arrows in the next row above (represented by the upper arrows of the same vertices) then corresponds to the state of the system at an intermediate time $t' = t+1/2$, and the configuration after a full time step $t'' = t+1$ corresponds to the arrangement of arrows two rows above. Therefore each vertex represents a local transition from the state given by the lower two arrows of a vertex representing the configuration on sites j and $j+1$ at time t to the state defined by the upper two arrows representing the configuration at sites j and $j+1$ at time $t+1/2$. The correspondence of the vertex language to the particle picture used in Sec. IV can be understood by considering up-pointing arrows as particles occupying the respective sites of the chain while down-pointing arrows represent vacant sites, i.e., holes.

The diagonal-to-diagonal transfer matrix T acting on a chain of L sites (L even) of the six-vertex model with space-dependent vertex weights as shown in Fig. 1 is then defined by [32]

$$T = \prod_{j=1}^{L/2} T_{2j} \prod_{j=1}^{L/2} T_{2j-1} = T^{\text{even}} T^{\text{odd}} \quad . \tag{C1}$$

The matrices T_j act nontrivially on sites j and $j+1$ in the chain, on all other sites they act as unit operator. All matrices T_j and $T_{j'}$ with $|j-j'| \neq 1$ commute. For an explicit representation of the transfer matrix we choose a spin-$\frac{1}{2}$ tensor basis where the Pauli matrix σ_j^z acting on site j of the chain is diagonal and spin down at site j represents a particle (up-pointing arrow) and spin up a hole (down-pointing arrow). In this basis $n_j = \frac{1}{2}(1-\sigma_j^z)$ is the projection operator on particles on site j and $s_j^{\pm} = \frac{1}{2}(\sigma_j^x \pm i\sigma_j^y)$ ($\sigma^{x,y,z}$ being the Pauli matrices) create (s_j^-) and annihilate (s_j^+) particles, respectively. The matrices T_j in this basis are defined by

$$T_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-p_j & p_j & 0 \\ 0 & p_j & 1-p_j & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{j,j+1} \quad . \tag{C2}$$

In the particle language the matrices T_j describe the local transition probabilities of particles moving from site j to site $j+1$ represented by the weights of the corresponding vertices. With these identifications the vertex model with a random choice of the numbers p_j in the interval $0 \leq p_j \leq 1$ becomes a discrete-time version of the random-barrier model.

The transfer matrix acts in parallel first on all odd-even pairs of sites $(2j-1, 2j)$, then on all even-odd pairs. In a model with transfer matrix $\tilde{T} = T^{\text{odd}} T^{\text{even}}$ one would start the time evolution at an intermediate half odd integer time step and there will be no difference in the physical properties of these two systems. We assume periodic boundary conditions, i.e., we identify site $L+1$ with site 1.

In the one-particle sector the transfer matrix has a simple form. We denote by $| x \rangle$ the state with the particle being on site x and $\langle x |$ is its transposed. The scalar product on this space is given by $\langle x | y \rangle = \delta_{x,y}$ ($\delta_{x,y}$ is the Kronecker symbol) and the unit operator is $1 = \sum_x | x \rangle \langle x |$. The transfer matrix A restricted to the one-particle sector reads

$$A = A^{\text{even}} A^{\text{odd}} = \sum_{j=1}^{L/2} A_{2j}(p_{2j}) \sum_{j=1}^{L/2} A_{2j-1}(p_{2j-1}) \tag{C3}$$

with local transfer matrices

$$A_j(p_j) = |\, j \,\rangle\langle\, j \,| + |\, j+1 \,\rangle\langle\, j+1 \,| - p_j(|\, j \,\rangle - |\, j+1 \,\rangle)(\langle\, j \,| - \langle\, j+1 \,|) \ . \tag{C4}$$

The steady state with eigenvalue 1 of A is the vector $\langle\, a \,| = L^{-1/2} \sum_x |\, x \,\rangle$ and $|\, a \,\rangle$, which is $\langle\, s \,|$ restricted to the one-particle sector, is its transpose. The particle number operator n_x is simply given by $n_x = |\, x \,\rangle\langle\, x \,|$ and the correlation function (4.3) is the matrix element $\langle\, x \,|A^t|\, y \,\rangle = (A^t)_{x,y}$ of A^t. This power is defined in the same way as T^t:

$$A^t = \begin{cases} A^k & \text{if } t = k \\ A^{\text{odd}} A^k & \text{if } t = k + 1/2 \ . \end{cases} \tag{C5}$$

With these definitions one can immediately derive a recursion relation for $G(x, y; t)$ with respect to x and t. We have for integer values of $t = k$

$$G(x, y; t + 1/2) = \langle\, x \,|A^{\text{odd}} A^t|\, y \,\rangle \ . \tag{C6}$$

Inserting Eq. (C4) into this expression gives the recursion relations (4.4) and (4.5) of Sec. IV.

The time-dependent connected two-point correlation function for the homogeneous model with $p = 1/2$ was computed by Kandel *et al.* [25] for full time steps in the continuum limit $L \to \infty$. We quote their result and the corresponding expression for half-odd integer time intervals which are used in Sec. IV. One obtains the following with $r = x - y$ and $\rho = N/L$.

For t an integer

$$\text{(a)} \quad C_N(x, y; t) = \rho(1-\rho) \left(\frac{1}{2}\right)^{2t} \binom{2t-1}{t + r/2}, \quad x, y \text{ even} \tag{C7}$$

$$\text{(b)} \quad C_N(x, y; t) = \rho(1-\rho) \left(\frac{1}{2}\right)^{2t} \binom{2t-1}{t + (r-1)/2}, \quad x \text{ odd}, \ y \text{ even} \tag{C8}$$

$$\text{(c)} \quad C_N(x, y; t) = \rho(1-\rho) \left(\frac{1}{2}\right)^{2t} \binom{2t-1}{t + (r-1)/2}, \quad x \text{ even}, \ y \text{ odd} \tag{C9}$$

$$\text{(d)} \quad C_N(x, y; t) = \rho(1-\rho) \left(\frac{1}{2}\right)^{2t} \binom{2t-1}{t - r/2}, \quad x, y \text{ odd}. \tag{C10}$$

For t a half odd integer

$$\text{(a)} \quad C_N(x, y; t) = \rho(1-\rho) \left(\frac{1}{2}\right)^{2t} \binom{2t-1}{t + (r-1)/2}, \quad x, y \text{ even} \tag{C11}$$

$$\text{(b)} \quad C_N(x, y; t) = \rho(1-\rho) \left(\frac{1}{2}\right)^{2t} \binom{2t-1}{t + r/2}, \quad x \text{ odd}, \ y \text{ even} \tag{C12}$$

$$\text{(c)} \quad C_N(x, y; t) = \rho(1-\rho) \left(\frac{1}{2}\right)^{2t} \binom{2t-1}{t - r/2}, \quad x \text{ even}, \ y \text{ odd} \tag{C13}$$

$$\text{(d)} \quad C_N(x, y; t) = \rho(1-\rho) \left(\frac{1}{2}\right)^{2t} \binom{2t-1}{t + (r-1)/2}, \quad x, y \text{ odd}. \tag{C14}$$

Because of the distinction of right and left movers and full and half time steps we have to define carefully Fourier and Laplace transforms of space and time-dependent correlation functions. We define the dynamic structure function $S(k, t)$ as the sum of Fourier transforms of the two-point correlation function between right movers and left movers:

$$S(k, t) = \sum_x e^{ikx} [C(2x, 0; t) + C(2x+1, 1; t)]. \tag{C15}$$

From (C7) one obtains for the correlation function $C_N(x, y; t)$ of the ordered system with $p = 1/2$ at integer times

$$S_0(k, t) = \rho(1-\rho) (\cos k)^{2t} \ . \tag{C16}$$

The discrete Laplace transform of a time-dependent quantity $f(t)$ is defined as the sum

$$\tilde{f}(\omega) = \sum_{t=0}^{\infty} e^{-\omega t} f(t) \tag{C17}$$

over full time steps only. For the dynamic structure function (C16) we obtain

$$\tilde{S}_0(k, \omega) = \frac{1}{1 - e^{-\omega} (\cos k)^2}. \tag{C18}$$

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