# Spectral $1/f$ noise derived from extremized physical information

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We consider the evolution of a temporal signal $X(t)$ that is an intrinsic random field of order zero. In the sense of a certain measurement-estimation experiment, the state of disorder of $X(t)$ should increase toward an equilibrium state. The disorder of $X(t)$ is measured by its "physical information" $\mathcal{J}$, and the equilibrium state is determined by the condition that $\mathcal{J}$ be an extremum. The equilibrium state is shown to have a power spectrum $S(\omega)$ of the form $\omega^{-\alpha}, 1 \le \alpha \le 2$, that of $1/f$ noise.

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## INTRODUCTION

A power spectrum of the form $1/\omega^{\alpha}$, $\omega$ the frequency, is commonly called "$1/f$ noise." As a physical phenomenon, $1/f$ noise describes an astonishingly diverse range of phenomena. Only a partial list includes voltage fluctuations in resistors, semiconductors, vacuum tubes and cell membranes [1–5], traffic density on a highway [6], economic time series [7], musical pitch and volume [8], sunspot activity [9], flood levels on the river Nile [9], and the rate of insulin uptake by diabatics [10]. (See succeeding references for other $1/f$ phenomena.) What single effect could exist that would cause such a disparate array of phenomena to share the same form of power spectrum?

The name "$1/f$ noise" implies that a $1/f$ power spectrum describes "noise" behavior, as if noise is the only phenomenon that all such effects could conceivably have in common. In fact, this intuitive notion agrees with the theme of this paper. This is that the related concept of *disorder,* in particular extreme physical disorder, gives rise to all $1/f$ phenomena.

Numerous mathematical models have been advanced to achieve $1/f$ noise in specific scenarios. Examples are for fractal shot noise [11], filtered white Gaussian noise [12], fractionally integrated white noise [13], fractal Brownian motion [14], superposition relaxation processes with different time constants [15], fractal renewal processes [16], quantum particles in a disordered lattice in the presence of dissipative forces [17], and a diffusion process driven by a white noise boundary condition [18].

The latter is of particular interest, since a diffusion process obeys increasing entropy [19], which implies increasing disorder. Maximum disorder will be the basis for the derivation in this paper, although disorder will be measured by Fisher information rather than by entropy.

The aim of this paper is to establish a universal model for $1/f$ noise. A recent attempt at a universal model was that of extremal dynamics [20], which predicts a sequence of $1/f$ laws of different powers $\alpha$ depending on $\omega$; however, most $1/f$ phenomena are characterized by a single $\alpha$. It will be seen that, likewise, our model has limitations, specifically in the assumption of an intrinsic random field ($IRF_0$) noise process (defined below), which leads to an overly restrictive range $1 \le \alpha \le 2$ for the $1/f$ power.

## PROBLEM DEFINITION

Let $S(\omega)$ denote the power spectrum (defined below) for a temporal signal $X(t)$. A $1/f$ power spectrum $S(\omega)=\omega^{-\alpha}$ must obey nonstationary statistics [21], since (as has been amply confirmed experimentally [22]) the spectrum generally holds down to the smallest $\omega$ that is measurable. For example, in weather data a $1/f$ noise phenomenon has been observed down to $\omega=10^{-10}$ Hz or 1 cycle in 300 years [9]. A small $\omega$ corresponds to a large time $t$, indicating a correlation time extending back to the onset of the process. Hence fluctuations $X(t)$ have an absolute dependence upon time and are therefore nonstationary. The "strength" of the nonstationarity is, on this basis, dependent upon the strength of $S(\omega)$ near the origin, i.e., the magnitude of $\alpha$. For example, in [8] typical records $X(t)$ for values $\alpha=0$, 1, and 2 are plotted, showing decreasing randomness as $\alpha$ increases. In the context of musical compositions $X(t)$, which obey a $1/f$ phenomenon, it has been observed that power $\alpha=0$ defines music that sounds too discordant or random, $\alpha=2$ defines music that is too repetitious and "boring," and $\alpha=1$ defines just the right tradeoff between randomness (novelty) and repetition [8]. Mozart's music reputedly obeys $\alpha=1$.

Correlation with the past implies memory. Keshner [22] plots the autocorrelation functions for $RC$ circuits that approximate a $1/f$ spectral law for each of $\alpha=0$, 1, and 2, and finds these to have increasingly negative slopes in the order $\alpha=1,2,0$. Thus, a system with $\alpha=1$ has a very long memory. The closer $\alpha$ is to 1, the greater is the influence of the distant past when compared with that of the recent past. For $\alpha$ near either 0 or 2 the $X(t)$ process is influenced by the recent past much more strongly than by the distant past. In summary, a $1/f$ noise process has memory, and the extent of memory is governed by the size of $\alpha$.

Nonstationary statistics, however, present a problem of definition of the power spectrum. The usual route to its definition is the Wiener-Khintchine theorem [23], according to which $S(\omega)$ is the Fourier transform of a stationary autocorrelation function. However, there is an alternative [24,25]. Consider a real-valued, temporal, stochastic signal $X(t)$ over a time interval $(0,T)$ $T$ finite. It has an associated (complex) Fourier spectrum

$$Z_T(\omega)=\int_0^T dt\, X(t)e^{-i\omega t}\Big/\sqrt{T}\,,\quad i=\sqrt{-1}\,, \tag{1}$$

and a periodogram

$$I_T(\omega)=|Z_T(\omega)|^2\,. \tag{2}$$

As an example, the signal $X(t)$ may be a randomly selected musical composition, where $X(t)$ is the instantaneous squared voltage wave form [8]. For simplicity, assume that the DC component of $X(t)$ has been subtracted out, so that $\langle X(t)\rangle=0$. (This is equivalent to subtracting out a fixed amount from the power spectrum at the origin, which has no effect on its shape elsewhere.)

Define a power spectrum

$$S(\omega)=\lim_{T\to\infty}\langle I_T(\omega)\rangle,\quad \omega\neq 0\,. \tag{3}$$

In practice, the infinite limit can be well approximated by practicable time spans of modest length, since most musical compositions (and signals) are eventually ergodic. Any of Mahler's symphonies, e.g., are certainly long enough to be ergodic. We seek to derive $S(\omega)$ as obeying a $1/\omega^\alpha$ form, $\alpha$ constant.

Equation (3) shows that we are seeking an equilibrium, or time-invariant, form for $S(\omega)$. The principle of extreme physical information (EPI) may be used to derive such equilibrium functions [26]. This principle is introduced in the next paragraph, and is described more fully in the second section following. Some previous uses of EPI have been the derivation of the stationary forms of the Schrödinger wave equation, Klein-Gordon equation, and Dirac equation [27].

The EPI procedure is briefly as follows. (a) Form a total information quantity $\mathcal{J}$, which is the difference between a Fisher information term $I$ and a "constraint" information $J$,

$$\mathcal{J}=I-J\,. \tag{4}$$

Fisher information $I$ is of a universal form [(6b) below], while $J$ defines the particular scenario. Both $I$ and $J$ are to be expressed as functionals of the unknown distribution, here $S(\omega)$.

(b) The latter is then varied so that both conditions

$$\mathcal{J}=I-J=(\text{extremum}) \tag{5a}$$

and

$$\mathcal{J}=I-J=0,\quad I=I[S(\omega)],\quad J=J[S(\omega)]\,, \tag{5b}$$

are met [28]. This procedure will be followed below [see Eqs. (20)–(22)] to form an output equilibrium law $S(\omega)$.

In any scenario the solution [here $S(\omega)$] will satisfy (5b), since (5b) is an axiom [axiom (iii) below] of the approach. However, a solution to (5b) does not necessarily satisfy (5a), since generally a root of a function (say, a polynomial) is not necessarily an extremum as well. For example, in the scenario of relativistic quantum mechanics the Klein-Gordon equation obeys both (5a) and (5b), while the Dirac equation obeys (5b) but not necessarily (5a) (depending on the form of the potential field present) [27]. *A tenet of the theory is that every solution to either (5a) or (5b) has physical significance,* i.e., occurs in nature. We call a solution that obeys both (5a) and (5b) a "principal solution" of the EPI problem.

A principal solution for $S(\omega)$ will be sought below. This is for two reasons: (a) Since a principal solution arises as the solution to *either* (5a) or (5b), it is, in a sense, a dominant solution, which complies with the ubiquitous nature of the $1/f$ law; and (b) an information $\mathcal{J}$ that satisfies both properties (5a) and (5b) is also, mathematically, an "information divergence" (see [29]). Briefly, an information divergence measures the "distance" between two functionals. Terms $I=I[S(\omega)]$ and $J=J[S(\omega)]$ in Eq. (4) are such functionals; and $I-J$ is obviously a measure of the distance between $I$ and $J$. Hence, $\mathcal{J}$ is an information divergence. This class of information quantities includes Kullback-Leibler *entropy* and Shannon information as members. [Recall that both are expressible as the difference of two entropy terms, as in (4).] Extremum principle (5a) then represents a generalized second law of thermodynamics, where the maximum is replaced by an extremum. This gives added physical significance to the solution $S(\omega)$ found below.

## TEMPORAL EVOLUTION AND DISORDER

We next describe the evolution of the time signal $X(t)$ in terms of Fisher information. It will be shown that, as $T\to\infty$, the disorder of $X(t)$ increases and consequently $I\to$a minimum value. This provides a basis for use of the EPI approach.

Consider the gedanken measurement experiment in Fig. 1. Time signal $X(t)$ is a musical composition; say, a

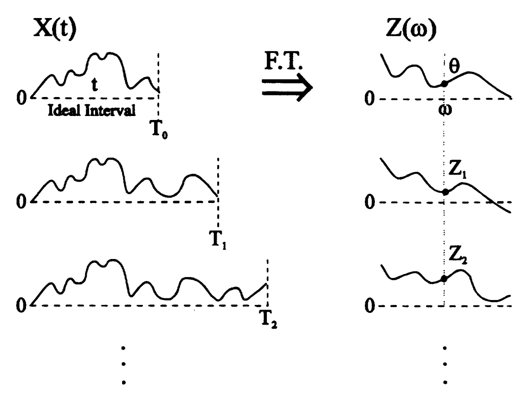


FIG. 1. Gedanken measurement-estimation experiment. The unknown tone amplitude $\theta(\omega)$ is caused by signal $X(t)$ over ideal interval $(0,T_0)$. Subsequent tone amplitudes $Z_1(\omega),Z_2(\omega),\ldots$ are due to listening over ever-longer time intervals.

randomly selected violin sonata. Signal $X(t)$ is produced over increasing time intervals, $(0,T_0)$, $(0,T_1)$, $(0,T_2),\ldots,$ where $T_0<T_1<T_2\cdots$. Suppose that a note $\omega$ occurs in the first interval $(0,T_0)$, and with complex amplitude $Z_0(\omega)\equiv\theta(\omega)$ via Eq. (1). However, we are not listening during the first interval, and so do not know either $X(t)$ over the interval or $\theta(\omega)$, $\omega$ fixed. Instead, we know spectral amplitudes $Z_n(\omega)$, $n=1,2,\ldots,$ over increasing, nested time intervals $(0,T_n)$. The observable numbers $Z_n(\omega)$ are formed through Eq. (1), but without our knowledge of the underlying $X(t)$ values. *From simple observation of any one $Z_n(\omega)$ we are to best estimate $\theta(\omega)$.* Which value $Z_n(\omega)$ ought to lead, on average, to the best estimate? How should mean-square error in the estimate vary with interval length $T_n$?

If the time sequence $X(t)$ over interval $(0,T_0)$ were known, by Eq. (1) $\theta(\omega)$ would be known with zero error. Therefore we call interval $(0,T_0)$ the "ideal data interval." Suppose that the next interval $(0,T_1)$ includes the ideal interval plus a small amount. Then, by Eq. (1), its Fourier transform $Z_1(\omega)$ should depart from $\theta(\omega)$ by a small amount. Likewise, an optimum *estimate* of $\theta(\omega)$ made on the basis of observation $Z_1(\omega)$ should incur small mean-square error. The trend continues. Interval $(0,T_2)$ includes the ideal plus more "tail" of $X(t)$ than its predecessor. Therefore, the resulting $Z_2(\omega)$ will incur more error from $\theta(\omega)$ than did $Z_1(\omega)$, and so will any estimate of $\theta(\omega)$ based upon $Z_2(\omega)$. *Hence, as time $T$ increases, the optimized mean-square error $e^2$ in knowledge of $\theta(\omega)$, $\omega$ fixed, should increase.*

The Cramer-Rao inequality [30] states that optimum error $e^2$ varies inversely with available information. This is for a one-dimensional unknown $\theta$. However, our unknown $\theta(\omega)$ is complex and therefore two dimensional. An outgrowth of the Cramer-Rao inequality is that [31] for a 2-D unknown the error relates to a Fisher information quantity $I$ as

$$\frac{1}{e_r^2}+\frac{1}{e_i^2}=I,\quad e^2\equiv e_r^2+e_i^2\ , \tag{6a}$$

$$I=\int\int dZ_r dZ_i\frac{(\partial p/\partial Z_r)^2+(\partial p/\partial Z_i)^2}{p}\ , \tag{6b}$$

where $p=p(Z_r,Z_i)$ is the probability law defining the joint fluctuations of the real and imaginary parts $Z_r,Z_i$, respectively, of $Z_T$. The integration limits for all integrals $dZ_r dZ_i$ are infinite. Note that by Eq. (6b) quantity $I$ is the trace (here a two-term trace) of the usual Fisher information matrix [30]. It is this scalar, trace quantity that derives physical laws [26]. The trace $I$ also relates, via a Poisson equation, to many other forms of information including Kullback-Leibler entropy, and forms due to Jeffreys, Rao, and Wootters [32]. In this sense, $I$ is a kind of "mother information."

From the form of (6b), $I$ also measures the roughness of $p$, through the strength of its gradient; see also [33]. The broader and smoother a function $p(Z_r,Z_i)$ is, the more randomness it exhibits in its variables $Z_r$, $Z_i$. Hence, $I$ is also a measure of the amount of disorder that is present.

Mean-square error $e^2$ in estimation of $\theta$ was found to increase with $T$. Also, by (6a) with $e_r^2=e_i^2=e^2/2$ (since $Z_r$ and $Z_i$ are identically distributed; see below),

$$I=4/e^2\ . \tag{7a}$$

Then $I$ must decrease with $T$. It follows that as $T\to\infty$, $I$ tends toward a minimum value,

$$I(p)=(\text{minimum})\ . \tag{7b}$$

The solution $p$ that attains the minimum, and the value of the minimum, depends upon one or more inputs of constraint information regarding $p$. Achieving (7b) subject to these constraints was the basis for a derivation procedure that predated EPI called minimum Fisher information (MFI); see [34].

## EXTREME PHYSICAL INFORMATION (EPI) PRINCIPLE

The EPI principle is an offshoot of relation (7b). As discussed [35], it is a stronger approach than MFI since it has a wider scope of derivation, and follows from a physically meaningful, axiomatic definition. A quantity $\mathcal{J}$ called *physical information* is to obey the following axioms [36].

(i) *Disorder aspect.* $\mathcal{J}$ is a measure of the disorder, or smoothness, in a paradigm $p$ for a physical scenario. $\mathcal{J}$ measures smoothness through a direct, linear dependence upon the Fisher information $I$ of (6b),

$$\mathcal{J}\propto I(p)\ . \tag{8}$$

(ii) *Second law of thermodynamics aspect.* $\mathcal{J}$ is minimized, or more generally, extremized, by formation of $p$,

$$\mathcal{J}=(\text{extremum})\ . \tag{9}$$

(iii) *Equivalence of all paradigms.* The value of the extremum should be a universal constant over all phenomena. All phenomena are equivalent in their "information content." The information $\mathcal{J}$ value is zero. As a corollary, $\mathcal{J}$ is zero as well when the same phenomenon is viewed under different choices of coordinate system. (This is used abundantly below.)

(iv) *Invariance to coordinate space.* For a given physical scenario, the same solution $p$ to principle (9) should occur whether information $\mathcal{J}$ about unknown amplitude $\theta$ is initially expressed in the direct measurement space $\omega$ or in the Fourier conjugate space $t$. (This property is not used below.)

As shown [37], the solution to axioms (i)–(iv) is a principle (as specialized to our one-component $p$, two-dimensional $Z_r,Z_i$ problem)

$$\mathcal{J}=\int\int dZ_r dZ_i\frac{(\partial p/\partial Z_r)^2+(\partial p/\partial Z_i)^2}{p}-\int\int dZ_r dZ_i F[Z_r,Z_i,p]=(\text{extremum})=0 \tag{10}$$

that derives physical paradigms. The first right-hand term is Fisher information $I$; see Eq. (6b). This is of a fixed form independent of scenario. Its effect on the solution is to produce a smooth output $p$ [by principle (7b)] regardless of scenario. The second term is $J$ in Eqs. (4)–(5a) and (5b). Functional $F$ identifies the particular physical scenario. This gives the principle its scope of application.

## APPLICATION TO 1/f SCENARIO

EPI is a principle of wide applicability in statistical physics [26,31,34], and it would be surprising if it did not have something to say about the widespread occurrence of the $1/f$ power spectrum. Use of the principle requires identification of the two contributions $I$ and $J$.

As was discussed, a time signal $X(t)$ that exhibits $1/f$ behavior is intrinsically nonstationary, essentially because of its long memory. The latter is indicated by the blowup of $1/f$ near the origin (the so-called "infrared catastrophe" [38]). A wide class of nonstationary signals $X(t)$ was recently defined and analyzed by Solo [25]. This is the class of intrinsic random fields ($IRF_0$) of order zero [39]. An $IRF_0$ is a second-order, mean square continuous process $X(t)$ obeying $X(0)=0$, whose values are nonstationary but whose increments are stationary. A particle exhibiting ordinary Brownian motion, e.g., has these properties [40]. The $IRF_0$ class of signals achieves nonstationarity as, effectively, a time-dependent sequence of stationary processes of short duration (as anticipated by Keshner [22]). We shall regard $X(t)$ as an $IRF_0$.

It was shown [25] that such a process obeys a central limit theorem. Thus, both the real and imaginary parts of $Z_T(\omega)$ are independent Gaussian, with the same variance, at each $\omega$, and over all $\omega$. This allows us to compute $I$. If a density $p(x)$ is Gaussian, with variance $\sigma^2$, a simple calculation [using one component of Eq. (6b)] shows that

$$I=1/\sigma^2 . \tag{11}$$

Here we have $p(Z_r,Z_i)$ separable Gaussian, with $\sigma^2=S(\omega)/2$ [25]. Then Eq. (6b) gives $1/\sigma^2$ for each term, or a total of $2/\sigma^2=4/S(\omega)$. Hence,

$$I(\omega)=4/S(\omega) . \tag{12}$$

This is the behavior at one frequency $\omega$. Since $Z_T(\omega)$ is independent over frequencies, the information quantities (12) add [34], and the total information is

$$I=4\int_\Omega d\omega \Big/ S(\omega) . \tag{13}$$

This is the amount of Fisher information present about many (now) unknown tone amplitudes $\theta(\omega)$, $\omega\in\Omega$, $\Omega=(\omega_1,\omega_2)$, in independent, Gaussian data values $Z_T(\omega)$, $\omega\in\Omega$. The dc "tone" $\omega=0$ is excluded from $\Omega$; it has no physical reality. All subsequent integrals are over range $\Omega$.

The other contributor to $\mathcal{J}$ is $J$. At first, allow $J$ to have a general form

$$J=\lambda\int d\omega\, F[S(\omega),\omega] , \tag{14}$$

where $F$ is a general function of $S$ and $\omega$. Obviously $F$ must be known if solution $S(\omega)$ is to be found.

Subtracting (14) from (13) [see (4)] results in a physical information,

$$\mathcal{J}=4\int d\omega \Big/ S(\omega)-\lambda\int d\omega\, F[S(\omega),\omega] . \tag{15}$$

We next find function $F$ by demanding $S(\omega)$ to be a principal solution of EPI.

## FINDING $F[S(\omega),\omega]$

A principal solution $S$ satisfies both axioms (ii) and (iii). Then the solution obtained by extremizing (15) is the same as by equating (15) to zero. The Lagrangian for problem (15) is

$$L=4/S-\lambda F(S,\omega) . \tag{16a}$$

The Euler-Lagrange extremum solution is

$$\frac{\partial L}{\partial S}=0=-\frac{4}{S^2}-\lambda_1\left[\frac{\partial F}{\partial S}\right] . \tag{16b}$$

The condition that (15) be zero is satisfied by equating $L$ of (16a) to zero,

$$0=+4/S-\lambda_2 F(S,\omega) . \tag{16c}$$

We allow for different Lagrange parameters $\lambda_1,\lambda_2$ in (16b) and (16c) since they are independent solutions. Placing Eq. (16c) in the same form as (16b) by multiplying through (16c) by $-1/S$ gives

$$0=-4/S^2+\lambda_2 F(S,\omega)/S . \tag{16d}$$

Since both (16b) and (16d) must have one solution, we equate the two. The result is a simple differential equation, with solution

$$F(S,\omega)=G(\omega)S^b, \quad b=-\lambda_2/\lambda_1 . \tag{17}$$

The new function $G(\omega)$ arises out of the *partial* derivative $\partial/\partial S$ operation in (16b), causing an integration constant $G$ to become an integration function $G(\omega)$. The information (15) now becomes

$$\mathcal{J}=4\int d\omega \Big/ S(\omega)-\lambda\int d\omega\, S(\omega)^b G(\omega) . \tag{18}$$

The form of $G(\omega)$ is found next.

## FINDING $G(\omega)$

By axiom (iii), $\mathcal{J}$ should remain invariant at value zero to different choices of the underlying coordinate system (here $\omega$). In past uses of axiom (iii), invoking invariance to moving frame of reference gave rise to the Lorentz transformation group of special relativity, and invoking invariance to arbitrary geometrical distortion of coordinate space gave rise to the kinetic equations of general relativity [26].

Imagine that a solution $S(\omega)$ to (15) has achieved $\mathcal{J}=0$. Axiom (iii) requires that $\mathcal{J}$ remain zero under, in particular, an arbitrary *change of units* in $\omega$. Define a new unit $\omega_1=a\omega$, $a$ constant. Then the new power spectrum $S_1$ obeys

$$S_1(\omega_1)=\frac{1}{a}S\left[\frac{\omega_1}{a}\right] . \tag{19a}$$

The new information $\mathcal{J}_1$ is of the form (18),

$$\mathcal{J}_1=4\int d\omega_1 \Big/ S_1(\omega_1)-\lambda(a)\int d\omega_1 S_1(\omega_1)^b G(\omega_1) \ . \quad (19b)$$

We used the fact that parameter $\lambda \equiv \lambda(a)$ will generally vary with unit $a$. Substituting (19a) into (19b), and changing integration variables back to

$$\omega=\frac{\omega_1}{a} , \quad (19c)$$

gives

$$\mathcal{J}_1=4a^2\int d\omega \Big/ S(\omega)-\lambda(a)a^{1-b}\int d\omega\, S(\omega)^b G(a\omega) \ . \quad (19d)$$

Compare Eqs. (18) and (19d). The extremum solution $S(\omega)$ to (18) attained $\mathcal{J}=0$. In order for the extremum solution to (19d) to retain $\mathcal{J}_1=0$, the Lagrangians in (18) and (19d) must be proportional. We see that they are (with proportionality constant $a^2$) if and only if $G(\omega)$ satisfies

$$\lambda(a)a^{1-b}G(a\omega)=\lambda(1)a^2G(\omega), \quad \lambda(1)\equiv\lambda \ . \quad (19e)$$

If $\lambda$ depends upon unit $a$ as a power law,

$$\lambda(a)=\lambda(1)a^c, \quad c=\text{const} , \quad (19f)$$

then the solution to (19e) is

$$G(\omega)=\omega^k, \quad k=1+b-c \ . \quad (19g)$$

Interestingly, this is independent of unit $a$. If, on the other hand, $\lambda(a)$ does not have the special form (19f) the answer for $G(\omega)$ will still be a power-law solution as in (19g), but the power will now depend on unit $a$.

## SOLUTION

With $F$ and $G$ now known by Eqs. (17) and (19g), the physical information (15) becomes

$$\mathcal{J}=4\int d\omega \Big/ S(\omega)-\lambda\int d\omega\, S(\omega)^b\omega^k \ . \quad (20)$$

Parameters $b$ and $k$ are undetermined numbers. The information quantity $J$ (far-right term) that fixes the scenario is a generalized Mellin transform of $S(\omega)$. In the particular case $b=1$, $J$ becomes the ordinary Mellin transform. The Mellin transform has been shown [41] to be a solution to classes of fractional differential equations. Fractional and fractal effects (as previously noted) of many types dominate the analyses of $1/f$ noise.

We may now find the equilibrium solution $S(\omega)$. The Lagrangian in (20) is

$$L=L[\omega,S(\omega)]=\frac{4}{S}-\lambda S^b\omega^k \ . \quad (21)$$

The solution by either Euler-Lagrangian equation $\partial L/\partial S=0$ or $L=0$ is the same (as required above),

$$S(\omega)=C\omega^{-\alpha}, \quad C,\alpha=\text{const}, \quad \alpha=1-c/(b+1)\geq 0 \ . \quad (22)$$

Equation (19g) was also used. The exponent is negative because, physically, $S(\omega)$ should attenuate with $\omega$, not grow. The case $\alpha=0$ represents white noise. The case $c=0$ is of interest. By Eq. (22) it causes pure $1/\omega$ noise, and this is independent of $b$. Also, by Eq. (19f), $\lambda$ does not then depend upon the choice of unit $a$.

Solo [25] has shown that a solution (22) is consistent with the $\text{IRF}_0$ assumption if $1\leq\alpha\leq 2$. Empirically, this includes the majority of $1/f$ phenomena [22]. However, there are physical cases for which $\alpha<1$ [3] or $\alpha>2$ [42]. These are beyond the scope of this derivation. It appears that the $\text{IRF}_0$ assumption is slightly too restrictive in this regard. Indeed, the only property of an $\text{IRF}_0$ process that was used is that its spectrum $Z_T(\omega)$ obeys a central limit theorem [see Eq. (11) *et vecin.*]. It may be that a less restrictive process exists that likewise obeys a central limit theorem.

The scope of the approach can be somewhat broadened. The same solution (22) results from extremizing the information at a single frequency $\omega$,

$$\mathcal{J}=\mathcal{J}(\omega)=4/S(\omega)-\lambda F[S(\omega),\omega] \ . \quad (23)$$

Arguments (16a)–(17), and (19a)–(19g) follow for this $\mathcal{J}$ as well. Therefore, the condition for integral form (13) to hold may now be lifted, this is that $Z_T(\omega)$ be independent over frequencies $\omega$.

The EPI approach allows further generalization. Instead of Eq. (15), which has one input of scenario information, postulate the simultaneous presence of many such inputs, as in

$$I=4\int d\omega \Big/ S(\omega)-\sum_{n=1}^{N}\lambda_n\int d\omega\, F_n(S,\omega) \ . \quad (24)$$

This physically represents the presence of $N$ competing processes. Interestingly, as in the previous case ($N=1$) the functions $F_n(S,\omega)$ may again be fixed by the arguments (16a)–(17) and (19a)–(19g) that the solution $S(\omega)$ should be a principal solution and that $\mathcal{J}$ should remain zero under a linear change of coordinate $\omega$. The result is an information

$$I=4\int d\omega \Big/ S(\omega)-\sum_{n=1}^{N}\lambda_n\int d\omega\, S(\omega)^{b_n}\omega^{k_n} \quad (25)$$

[compare with Eq. (20)]. The principal solution must then obey a transcendental equation,

$$4/S(\omega)-\sum_{n=1}^{N}\lambda_n S(\omega)^{b_n}\omega^{k_n}=0 \ . \quad (26)$$

This is a polynomial equation of power $\beta=\max_n(b_n+1)$ in $S$, and so does not have a closed-form solution unless $\beta$ is 4 or less. The solution simplifies if all $b_n=b$, a constant, to

$$S(\omega)=\left[\frac{4}{\sum_{n=1}^{N}\lambda_n\omega^{k_n}}\right]^{1/(b+1)} . \quad (27)$$

In the case $b=-\frac{1}{2}, k_n=0,1,2,\ldots$, this becomes Burg's [43] maximum entropy spectral estimate. Hence, the two estimation principles of extreme physical information and maximum entropy are convergent in this case.

## DISCUSSION

The principle of extreme physical information derives paradigms of physics, i.e., phenomena that are unexplainable by other, known phenomena. The Schrödinger wave equation is a good example. The EPI approach requires one physical fact, defining the particular scenario, which is insufficient in itself to derive the paradigm. [Here, it is that $X(t)$ is an $\mathrm{IRF}_0$.] This fact combined with a condition of maximum disorder, in the Fisher sense, derives the paradigm. The answer that EPI provides to the ultimate question of why a paradigm arises, is that the paradigm is an expression by nature of extreme disorder. No other mechanism need be invoked.

Turning to the problem at hand, we note that attempts at unifying $1/f$ power spectra from a phenomenological viewpoint have been only partially successful; see surveys [3] and [44] covering decades of past work, and the more recent approaches described in the Introduction. To us, this suggests that the phenomenon is a distinct paradigm, unexplainable by other phenomena, and hence of the type derivable by EPI.

The EPI derivation rests upon the validity of the $\mathrm{IRF}_0$ assumption, and upon internal consistency of the EPI approach including its axioms. No other physical mechanism has been used. The random fields considered are, roughly speaking, filtered versions {Eq. (32) in [25]} of "nicely behaved" white noise. To the extent that such a field is present, the $1/f$ result (22) follows as an expression of extreme disorder. It would be useful to determine the extent to which the numerous physical examples of $1/f$ phenomena follow the $\mathrm{IRF}_0$ model, and if the model can be broadened to permit a slightly wider range of $\alpha$ values. We leave these questions to future research.

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