Mean field model for spatially extended systems in the presence of multiplicative noise

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We present a mean field model that describes the effect of multiplicative noise in spatially extended systems. The model can be solved analytically. For the case of the ϕ^4 potential it predicts that the phase transition is shifted. This conclusion is supported by numerical simulations of this model in two dimensions.

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I. INTRODUCTION

Since the late 1970s there has been an increasing interest in the influence of external noise on phase transitions and bifurcations [1]. In particular, it was found that multiplicative noise can change the location of the critical point. This conclusion was reached through the study of zero-dimensional models, based on a phenomenological equation for the macroscopic order parameter, which is subject to perturbation by a multiplicative noise of finite amplitude. The resulting fluctuations of the order parameter are macroscopic and the location of the critical point becomes a matter of contention. Instead of noise-induced bifurcations or phase transitions, researchers have coined the terms stochastic bifurcations or noise-induced transitions to refer to a change in the shape of the probability density of the order parameter. Other criteria, yielding different results for the location of critical points, such as changes in sign of the Lyapunov exponents or the properties of the extrema of the probability density, have also been proposed [2]. In spite of these interpretation problems, the zero-dimensional models have been useful to describe systems with macroscopic fluctuations, such as lasers excited by noisy signals [3], and have been the paradigm of the so-called *noise-induced transitions* for more than a decade.

Recently, a spatially extended system—the Swift-Hohenberg equation—has been studied under the influence of a multiplicative noise affecting the evolution of microscopic variables [4]. The numerical simulations presented in this paper indicate that a transition to roll structures appears in a regime in which a deterministic analysis predicts a homogeneous solution. In other words, an ordered state has been induced by the multiplicative noise. A similar phenomenon was observed in a real experiment involving a photosensitive reaction [5]. The important difference with the above discussed zero-dimensional models is that these structures are stable, indicating that the macroscopic state of the system retains its well-defined nonfluctuating character. Furthermore, the shift in the bifurcation point has the opposite sign of that predicted by the zero-dimensional theory and is in good agreement with the prediction based on the sign of the Lyapunov exponent (linearized theory). These three facts: the absence of fluctuations in the macroscopic state of the system, the negative shift of the bifurcation, and the goodness of the linear analysis, are quite intriguing for the noise-induced-transition community since they are in contradiction with what has been found so far in zero-dimensional systems. A theoretical analysis of multiplicative noise perturbing spatially extended systems is necessary to clarify this new situation. The difficulty of such an analysis, arising from the lack of detailed balance or of an equilibrium potential, is probably the reason why attention has been focused for so long on the study of zero-dimensional systems. In this paper, we study the effect of multiplicative noise in spatially distributed systems by taking into account the spatial coupling in a mean field kind of way. The resulting mean field model can be solved exactly, even in the presence of multiplicative noise, and its predictions are in qualitative agreement with the above-mentioned observations and with detailed numerical simulations of the ϕ^4 model in two dimensions. In spite of its simplicity, this exactly solvable model is, to our knowledge, the first one to explain the properties of a system driven by multiplicative noise with infinitely many degrees of freedom.

II. MODEL

To avoid unnecessary mathematical complications, we will consider here a model defined on a lattice with a single scalar variable. The state of the system is thus characterized by the values $\{x_i\}$ of a variable x at the lattice sites i of a d-dimensional cubic lattice. The rate of change of $\{x_i\}$ is given by the following Langevin equation:

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$$\dot{x}_i=-\frac{\partial U(x_i)}{\partial x_i}+\frac{D}{2d}\sum_n (x_n-x_i)+\xi_m^{(i)}g(x_i)+\xi_a^{(i)} . \quad (1)$$

The sum over n runs over the nearest neighbors of i. The corresponding term stands for the discretized form of the diffusion operator. $\xi_a^{(i)}$ and $\xi_m^{(i)}$ represent independent Gaussian white noises with zero mean value and correlation

$$\langle \xi_{a,m}^{(i)}(t)\xi_{a,m}^{(j)}(t')\rangle=\sigma_{a,m}^2\delta_{ij}\delta(t-t') . \quad (2)$$

The additive noise term $\xi_a^{(i)}$ models the presence of thermodynamic fluctuations, while the multiplicative noise $\xi_m^{(i)}$ represents the effect of an external noise. Note that both types of noises are assumed to be independent of each other, white in time and uncorrelated in space. The multiplicative noise term will be interpreted according to the Stratonovich calculus. Furthermore, we will concentrate on the case of the so-called ϕ^4 model, although more complicated models can be treated along similar lines. The potential U is thus given by

$$U(x)=-\frac{\alpha}{2}x^2+\frac{\beta}{4}x^4 . \quad (3)$$

To keep matters simple, we also assume that the external noise acts directly on the control parameter α, i.e., $g(x)=x$. Finally we note that we can set $\beta=1$ and $\sigma_a=1$ by an appropriate choice of the units for the x_i variable and the time variable. We choose this parametrization given by D and α instead of some of those used in the literature (see Ref. [6] for a review of such parametrizations) because the limit $D\to\infty$ clarifies some points of the analytical treatment of our model. A Langevin equation with this type of potential, but without the multiplicative noise term, has been used to describe a wide variety of both equilibrium [7,8] and nonequilibrium phenomena [9,10] and has been studied in great detail. In particular it is known that the system undergoes a phase transition in dimensions larger or equal to two (see, e.g., [6] for a recent study of the critical properties in dimension 2; for a discussion about universality in 1 dimension see [11]; for the effect of colored noise see [12]).

An analytic study of Eq. (1) is very difficult. In particular, the explicit form of the steady-state solution for the multivariate probability $P(\{x_i\})$ is only known in the absence of multiplicative noise, when it is nothing but the thermal equilibrium state. In order to make progress, we decouple the behavior of one cell from the others by the following mean field assumption: the spatial coupling of cell i to its neighbors is replaced by a coupling to the average value or mean field $\mu(t)$

$$\dot{x}_i=\alpha x_i-x_i^3+D(\mu-x_i)+\xi_a^{(i)}+\xi_m^{(i)}x_i \quad (4)$$

and the value of $\mu(t)$ has to be calculated self-consistently imposing $\mu(t)=\langle x_i(t)\rangle$. The advantage of the mean field ansatz is that Eq. (4) is now closed in the variable x_i. Moreover, the self-consistent equation for the stationary value of μ is nonlinear, opening the possibility for multiple solutions which are to be expected in the case of the breaking of ergodicity associated to a phase transition. Similar mean field models, but in the absence of multiplicative noise, have been studied in detail under the name of the *independent-site approximation* in the context of structural phase transitions [7,13] and in other related problems [14–20].

The Langevin equation (4) is equivalent to the following Fokker-Planck equation [1] for the probability distribution $P(x;t)$ of the process $x(t)$ [we will drop the subscript i in the following since Eq. (4) is similar for every site i]:

$$\frac{\partial}{\partial t}P(x;t)=\left[\frac{\partial}{\partial x}[-\alpha x+x^3-D(\mu-x)]+\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{\sigma_m^2}{2}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}x\right]P(x;t) . \quad (5)$$

The stationary solution of this equation is found to be

$$P_{\rm st}(x)=N\exp\left[-2\int^x dy\frac{-\alpha y+y^3-D(\mu-y)+\sigma_m^2 y/2}{\sigma_m^2y^2+1}\right] , \quad (6)$$

N being a normalization constant, and the following self-consistent equation for $\mu=\lim_{t\to\infty}\mu(t)$ results:

$$\mu=\phi(\mu) \quad (7)$$

with

$$\phi(\mu)=\frac{\int_{-\infty}^{\infty}dz\, z\exp\left[-2\int_0^z dy\frac{-\alpha y+y^3-D(\mu-y)+\sigma_m^2y/2}{\sigma_m^2y^2+1}\right]}{\int_{-\infty}^{\infty}dz\exp\left[-2\int_0^z dy\frac{-\alpha y+y^3-D(\mu-y)+\sigma_m^2y/2}{\sigma_m^2y^2+1}\right]} . \quad (8)$$

Before proceeding to an analysis of this equation, it is revealing to give a simple argument valid in the limit of a very large spatial coupling $D\to\infty$. From the Fokker Planck equation (5), one easily obtains the following exact evolution equation for $\langle x(t)\rangle=\mu(t)=\int_{\mathbb{R}}P(x;t)x\,dx$:

$$\langle\dot{x}\rangle=\left[\alpha+\frac{\sigma_m^2}{2}\right]\langle x\rangle-\langle x^3\rangle . \quad (9)$$

In the limit $D\to\infty$ the fluctuations of the variable x around its mean value $\langle x\rangle$ are expected to vanish since the coupling term $D(\mu - x)$ in (4) will prevent such fluctuations. Consequently the steady-state equation (9) for $\mu = \langle x \rangle$ becomes

$$\left[\alpha + \frac{\sigma_m^2}{2} \right] \mu - \mu^3 = 0 . \tag{10}$$

This result predicts that a branch of two new solutions will arise at the value $\alpha = -\sigma_m^2/2 < 0$ whereas no such transition is possible in the absence of multiplicative noise for $\alpha < 0$. In other words, the transition to bistability is advanced. Note that the location of the critical point coincides in this limit with the criterion of the breakdown of linear stability: the phase transition is predicted at the point at which the coefficient of the linear term in the equation for $\langle x(t) \rangle$ vanishes. This criterion was used in the previously cited study of the Swift-Hohenberg equation [4] to locate the transition to a roll structure, and good agreement with the simulations was found. In these simulations, the internal noise amplitude was taken to be small, which is tantamount to having a strong spatial coupling. The observed agreement with the linear instability criterion is thus in accordance with our mean field analysis.

At this point, we would like to stress the difference of this situation with what is happening in the well-known zero-dimensional case. Indeed consider Eq. (4) with $D = 0$. The stationary probability density can easily be obtained and is found to be an even function of x. Consequently, the corresponding stationary state value of the average of x is identically zero and this happens whether or not the system is linearly unstable. In fact, in the regime of linear instability $\alpha + \sigma_m^2/2 > 0$, small initial deviations from zero $\langle x(t=0) \rangle \neq 0$ will on the average grow for small times if the initial probability distribution is sharply peaked, but relax back to zero at larger times. This difference between the zero-dimensional and spatially distributed case persists even in the absence of any multiplicative noise. Indeed consider a thermally activated particle in a bistable potential, symmetric around $x = 0$ and with a metastable maximum located at this point. For small intensities of the noise, a particle that starts for example at an initial location $x \geq 0$ will, on a short time scale and following essentially the deterministic dynamics, move to the stable well located at the right-hand side of the origin. Consequently $\langle x(t) \rangle$ will increase. On a much longer time scale, however, the thermal noise will induce transitions between the two potential wells. Therefore, the value of $\langle x(t) \rangle$ will decrease until eventually the stationary state is reacted with $\mu = \langle x \rangle = 0$. In the presence of a spatial coupling between several particles each in their own potential, which is exactly the situation encountered in the ϕ^4 model, the relaxation to this symmetric situation is compromised since it is known that the system can undergo a phase transition (in two or more dimensions) with a breaking of the ergodicity between the $\mu > 0$ and $\mu < 0$ states.

III. PHASE DIAGRAM

We now turn to a more detailed analysis of the self-consistent equation (7). Since $\phi(0) = 0$, this equation possesses the trivial solution $\mu = 0$. Furthermore, the function $\phi(\mu)$ is a continuous even function of μ with $\phi'(\mu) > 0$, $\forall \mu > 0$ and $\phi(\mu) < \mu$ for large values of μ. A pair of new solutions symmetric around $\mu = 0$ will therefore appear at the values of the parameters for which $\phi'(\mu = 0) = 1$. The explicit form of $\phi'(\mu = 0)$ can be easily found:

$$\phi'(\mu=0) = \frac{2D}{\sigma_m} \langle y \arctan(\sigma_m y) \rangle^0 , \tag{11}$$

where the average $\langle \; \rangle^0$ is calculated with respect to the following probability density:

$$P^0(y) = N e^{-y^2/\sigma_m^2} (1 + \sigma_m^2 y^2)^{-1/2 + (\alpha - D)/\sigma_m^2 + 1/\sigma_m^4} , \tag{12}$$

N being a normalization constant. In the limit $D \to \infty$ one finds by steepest descent that $\phi'(\mu=0) = 1 + (2\alpha + \sigma_m^2)/2D + O(1/D^2)$, which confirms the validity of the simple analysis given above with the critical point located at $\alpha_c = -\sigma_m^2/2$. In Fig. 1, we have plotted the phase transition line $\phi'(\mu=0) = 1$ in the D versus α plane for several values of the external noise intensity σ_m. We have included the transition line obtained in the absence of multiplicative noise $\sigma_m = 0$, the location of some points at which a phase transition occurs in the two-dimensional ϕ^4 model as obtained numerically in Ref. [6] and transition points resulting from our simulations (see below). From this figure, it is clear that the multiplicative noise always advances the transition for large values of the spatial coupling, i.e., it induces a shift to lower values of α, whereas the opposite result is true for small values of this coupling.

To compare the results of the mean field theory with the two-dimensional model, we have performed numerical simulations of Eq. (1). We have also performed simu-

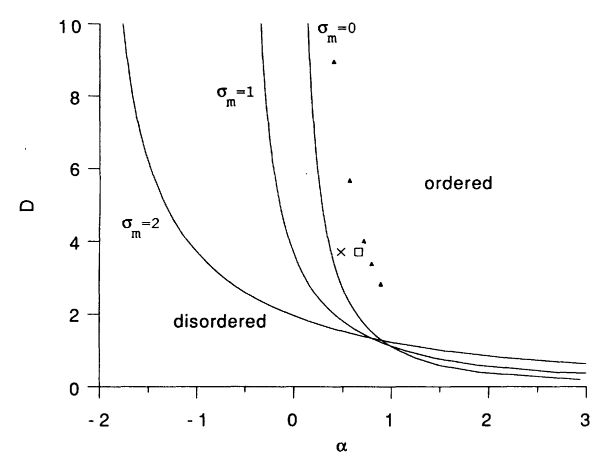

FIG. 1. Phase transition lines predicted by the mean field theory in the D versus α plane for several values of the multiplicative noise amplitude. The triangles correspond to the location of the critical point for the two-dimensional lattice in the absence of the multiplicative noise given by Ref. [6] while the cross and the square are the results of our simulations for the two-dimensional model with multiplicative noise ($\sigma_m = 1$) and without multiplicative noise ($\sigma_m = 0$), respectively. These two points result from fitting curves to the corresponding points in Fig. 2.

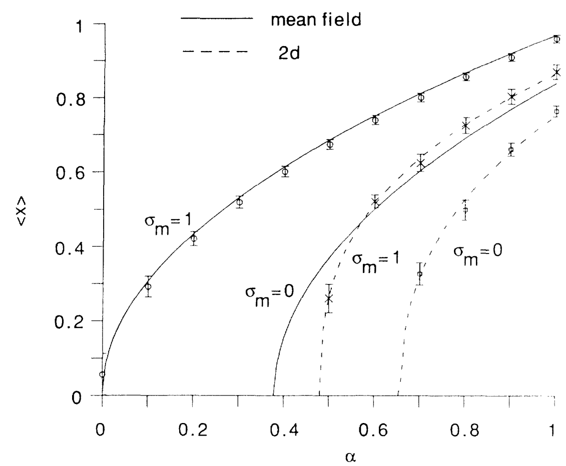

FIG. 2. The order parameter $\langle x \rangle$ as a function of α with a fixed value of $D=3.7$ for the mean field model (full lines) and the two-dimensional lattice (dashed lines), in both cases without multiplicative noise ($\sigma_m=0$) and with multiplicative noise ($\sigma_m=1$). The full lines correspond to the analytic result, Eq. (7), and the dashed lines are the result of a power-law fit to the numerical data (crosses for $\sigma_m=1$ and squares for $\sigma_m=0$). The circles correspond to simulations of the mean field model and are in perfect agreement with the curve given by Eq. (7).

lations of the mean field model itself, with each site coupled to all the others. The location of the transition is estimated by plotting the stationary value of $\langle x \rangle$ as a function of α (see Fig. 2). For the two-dimensional system, we have used square lattices of different sizes $N \times N$, $N=50,100,150$ and the integration of the Langevin Eq. (1) has been done by means of a standard Euler algorithm with time step between $\Delta t=[10^{-2},10^{-3}]$. Additive and multiplicative noises were implemented into the algorithm by means of a standard procedure [21]. In Fig. 2 we have plotted the results of our simulations. The points corresponding to the mean field model (circles) are in very good agreement with the theoretical curve. The simulation of the two-dimensional model gives a higher critical value of α than the mean field model. This is a usual feature of mean field approximations and it also occurs in absence of the multiplicative noise. One can notice that the discrepancy between the two-dimensional and the mean field model is increased by the noise. In any case, the simulations confirm the negative shift induced by the multiplicative noise.

IV. CONCLUSIONS

To conclude, our mean field model predicts that external noise acting on the control variable α in the ϕ^4 model will advance the location of the critical point for a sufficiently strong spatial coupling and delay it for a weak coupling. Moreover, for a strong spatial coupling, the location of the critical point coincides with the onset of linear instability. The same qualitative features are observed in the numerical simulation of a two-dimensional system. The fact that multiplicative noise can *induce order* may at first seem surprising, but this may be intuitively understood as follows. Consider the case of a negative value of α. No transition is possible in the presence of additive noise only. However, with multiplicative noise, the value of the control parameter plus the noise contribution at a given site will be positive from time to time and the system will then locally behave as if it is in a bistable state. If the noise intensity is sufficiently strong, this situation will be encountered at about half of all the sites and a symmetry breaking under the influence of a sufficiently strong spatial coupling cannot be ruled out. This is precisely what is predicted by our mean field model. It remains to be seen of course what is the range of validity of this model. We expect that it is correct in a system with a sufficiently high dimensionality. If we assume that the multiplicative noise does not change the universal properties of the system, the critical dimension above which the mean field theory becomes exact for the ϕ^4 model will be $d=4$. Furthermore, as is made plain by the intuitive discussion presented above, we believe that the shift of the phase transition under influence of the multiplicative noise is a very general phenomenon, also to be observed in lower dimensions and not restricted to the ϕ^4 model. Apart from the evidence discussed in the introduction [4,5] and the one gathered from the simulations presented in this paper, preliminary simulation results indicate that the introduction of a fluctuating temperature field on the boundary plates of a Bénard cell leads to the appearance of the convection rolls below criticality [22]. This again is in agreement with our general assertion that spatially uncorrelated or weakly correlated external noise, which is coupled in a non-additive way to the state variables of the problem, can shift the location of the transition point.

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