Escape rates in bistable systems induced by quasimonochromatic noise

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Path-integral techniques are used to understand the behavior of a particle moving in a bistable potential well and acted upon by quasimonochromatic external noise. In the limit of small diffusion coefficient, a steepest-descent evaluation of the path integral to leading order enables mean first-passage times and the transition times from one well to the other to be computed. The results and general approach are compared with computer simulations of the process. It is found that the bandwidth parameter Γ has a critical value above which particle escape is by white-noise-like outbursts, but below which escape is by oscillatory-type behavior.

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I. INTRODUCTION

The study of noise-driven systems frequently focuses on one of the simplest possible cases, the motion of an overdamped particle in a bistable potential $V(x)$, in order to investigate the essential features of a particular type of noise [1]. The model is described by the Langevin equation

$$\dot{x} + V'(x) = \xi(t) , \tag{1}$$

where the noise is taken to be Gaussianly distributed with zero mean. The nature of the noise is then completely characterized by the correlator,

$$\langle \xi(t)\xi(t') \rangle = D\ C(|t-t'|) . \tag{2}$$

Apart from the case of white noise, $C(|t-t'|) = 2\delta(t-t')$, the most studied situation [2] is where the noise is exponentially correlated,

$$C(|t-t'|) = \frac{1}{\tau}\exp(-|t-t'|/\tau) .$$

The power spectrum

$$\tilde{C}(\omega) = \int_{-\infty}^{\infty} ds\ e^{i\omega s}C(|s|) \tag{3}$$

may also be used to specify the noise, and the expansion of its inverse

$$\tilde{C}^{-1}(\omega) = \frac{1}{2}(1+\kappa_1\omega^2+\kappa_2\omega^4+\cdots) \tag{4}$$

introduces coefficients κ_i which define the type of noise under consideration. White noise corresponds to $\kappa_i = 0$ ($i \geq 1$) and exponentially correlated noise $\kappa_i = 0$ ($i \geq 2$), $\kappa_1 = \tau^2$. Clearly the most straightforward generalization consists in taking $\kappa_i = 0$ ($i \geq 3$). The case of a single noise correlation time and $\kappa_1, \kappa_2 > 0$ has already been investigated in an analogous fashion to the exponentially correlated case [3]. Since $\tilde{C}^{-1}(\omega)$ must be positive for all ω (for the Gaussian functional integral over the noise to exist), κ_2 must be positive. However, κ_1 need not be, and a clearly interesting limit is where $\tilde{C}^{-1}(\omega)$ is nearly zero for some value of ω. This corresponds to what has been called quasimonochromatic noise (QMN) [4] or harmonic noise [5], and is the subject of this paper. These previous studies suggest that QMN differs in an essential way from white and exponentially correlated noise in that, for instance, the particle passes over the top of the potential barrier many times before making a well transition. One of the aims of this paper will therefore be to calculate the escape time for the system (1) with QMN. We will use path integrals, as in earlier work by Dykman and co-workers [4, 6, 7], however our approach will be quite different and, we believe, more systematic and transparent.

II. PATH-INTEGRAL APPROACH

The correlation function for QMN in frequency space is

$$\langle \xi(\omega)\xi(\omega') \rangle = \frac{2D}{(\omega^2-\omega_0^2)^2+4\Gamma^2\omega^2}2\pi\delta(\omega+\omega') . \tag{5}$$

If we define a new diffusion coefficient

$$\tilde{D} = \frac{D}{\omega_0^4} , \tag{6}$$

then $\tilde{C}^{-1}(\omega)$ is of the form (4) with $\kappa_1 = -2\omega_0^{-2}[1-2(\Gamma/\omega_0)^2]$ and $\kappa_2 = \omega_0^{-4}$. In the limit $\Gamma \ll \omega_0$, $\tilde{C}(\omega)$ is sharply peaked at the finite frequency $(\omega_0^2-2\Gamma^2)^{1/2} \approx \omega_0$, hence the name "quasimonochromatic noise." We will be working within this limit for the rest of the paper. In the same way as exponentially correlated noise may also be viewed as resulting from a stochastic process $\tau\dot{\xi} + \xi = \eta$, where η is Gaussianly distributed white noise with zero mean and strength D, QMN can be viewed as the result of passing white noise through a harmonic oscillator filter,

$$\ddot{\xi} + 2\Gamma\dot{\xi} + \omega_0^2\xi = \eta , \tag{7}$$

where the white noise η has strength D. We also require that ξ and $\dot{\xi}$ are both zero as $t \to \pm\infty$ for Eq. (7) to be

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equivalent to (5).

We can use (7) to transform from the probability density functional for white noise to that for $\xi(t)$ [8],

$$P[\xi] = \mathcal{N}\exp\left(-\frac{1}{4D}\int_{-\infty}^{\infty} dt\left[\ddot{\xi} + 2\Gamma\dot{\xi} + \omega_0^2\xi\right]^2\right) . \quad (8)$$

Now we impose that $\xi,\ \dot{\xi} \to 0$ as $t \to \pm\infty$ so that Eq. (8) becomes

$$P[\xi] = \mathcal{N}'\exp\Bigg(-\frac{\omega_0^4}{4D}\int_{-\infty}^{\infty} dt\Bigg[\ \xi^2 - \frac{2}{\omega_0^2}\left(1 - 2\frac{\Gamma^2}{\omega_0^2}\right)\dot{\xi}^2 + \frac{1}{\omega_0^4}\ddot{\xi}^2\Bigg]\Bigg) . \quad (9)$$

Using (1) and (6) this gives the probability density functional for $x(t)$,

$$P[x] = \mathcal{N}''J[x]\exp\left[-\frac{S[x]}{\tilde{D}}\right] , \quad (10)$$

where $J[x]$ is a Jacobian factor which is irrelevant to leading order in D and $S[x]$ is the generalized Onsager-Machlup functional given by

$$\begin{aligned} S[x] = \ & \frac{1}{4}\int_{-\infty}^{\infty} [\dot{x} + V'(x)]^2 \\ & - \frac{2}{\omega_0^2}\left(1 - \frac{2\Gamma^2}{\omega_0^2}\right)[\ddot{x} + \dot{x}V''(x)]^2 \\ & + \frac{1}{\omega_0^4}[x^{(3)} + \ddot{x}V''(x) + \dot{x}^2V'''(x)]^2\, dt . \end{aligned} \quad (11)$$

Probability distributions, correlation functions, etc. can now be obtained by integrating over functions $x(t)$ with weight given by (10). For $D \to 0$ these path integrals may be evaluated by steepest descents; the paths that dominate the integrals are the ones for which $\delta S[x]/\delta x = 0$. Using (11) we obtain a sixth-order nonlinear differential equation for the paths. If this is multiplied by $\dot{x}$ and integrated with respect to time we can derive a fifth-order nonlinear equation where $x^{(n)}$ is the nth derivative of x with respect to time, given by

$$\begin{aligned} & \omega_0^4(\dot{x}^2 - V'^2) + 2\omega_0^2\left(1 - \frac{2\Gamma^2}{\omega_0^2}\right)(2x^{(3)}\dot{x} - \ddot{x}^2 + 2\dot{x}^3V''' - \dot{x}^2V''^2) \\ & +2x^{(5)}\dot{x} - 2x^{(4)}\ddot{x} + (x^{(3)})^2 + 10x^{(3)}\dot{x}^2V''' + 10\ddot{x}\dot{x}^3V'''' + 2\dot{x}^5V''''' \\ & - 2x^{(3)}\dot{x}V''^2 + \ddot{x}^2V''^2 - 4\ddot{x}\dot{x}^2V''V''' + \dot{x}^4V'''^2 - 2\dot{x}^4V''V'''' = 0 . \end{aligned} \quad (12)$$

Even in the limit $\Gamma/\omega_0 \to 0$, this is a formidable looking equation. It seems natural to try to make a perturbative analysis in the interesting large ω_0 limit, which would be somewhat analogous to an expansion in τ^2 for the exponentially correlated noise problem. But the QMN case is more subtle; the near vanishing of $\tilde{C}^{-1}(\omega)$ at $\omega \approx \omega_0$ gives rise to rapidly oscillating solutions which have to be carefully extracted from (12). In order to make further progress it is useful to have a specific potential in mind; we choose the much used bistable potential $V(x) = -x^2/2 + x^4/4$. In the dimensionless units we are using, we are working in the limits $\Gamma \ll \omega_0$ and $\omega_0 \gg 1$. The nature of the extremal solution can be glimpsed by studying what happens near to the turning points $x = -1, 0, +1$ where this potential can be approximated by a parabola. Let $x = x_i + \rho$, where x_i is the turning point and ρ is a small quantity. Taking only the lowest order of powers of ρ we get the approximate potential to be $V(\rho) = a_i + \frac{1}{2}b_i\rho^2$ where $a_i = -1/4$ and $b_i = 2$ if $x_i = \pm 1$ and $a_i = 0$ and $b_i = -1$ if $x_i = 0$. Equation (12), when expressed in terms of ρ, will have solutions of the form $\rho = \sum_{n=1}^{6} \mathcal{C}_n e^{\alpha_n t}$, where $\mathcal{C}_n$ and α_n are constants. Substitution into the equation gives

$$\alpha_n = \pm b_i \text{ or } \alpha_n = \pm(\Gamma \pm i\Omega) , \quad (13)$$

where $\Omega = \sqrt{\omega_0^2 - \Gamma^2}$. Clearly we can only have $\mathrm{Re}(\alpha_n) > 0$ and $\mathrm{Re}(\alpha_n) < 0$ solutions separately corresponding to the correct boundary condition $\rho \to 0$ as $t \to \pm\infty$. Note that the oscillatory solutions are independent of the shape of the potential. In the large ω_0 limit, where the form of the potential is unimportant, the oscillatory solutions of interest to us have precisely this form. In order to make this more concrete, first look at the case of the nonlinear potential gradient split into N segments between $x = -1$ and $x = 1$, where the potential gradient is approximated by N linear segments:

$$V'(x)|_{x_i} = (3x_i^2 - 1)\left(x - \frac{2x_i^3}{3x_i^2 - 1}\right) . \quad (14)$$

Making the substitution

$$x = \frac{2x_i^3}{3x_i^2 - 1} + \rho$$

the approximate form of the potential gradient is then given by $V' = b_i\rho$, where b_i denotes the gradient $3x_i^2 - 1$ at the point x_i. Once again ρ will be the sum of exponentials and we can say that locally at the point x_i the motion of x has the form

$$\begin{aligned} x = \ & \left(\frac{2x_i^3}{3x_i^2 - 1} + \mathcal{C}_i^1 e^{\pm b_i t}\right) \\ & + \mathcal{C}_i^2 e^{\pm\Gamma t}e^{i\Omega t} + \mathcal{C}_i^3 e^{\pm\Gamma t}e^{-i\Omega t} , \end{aligned} \quad (15)$$

where here C_i^n are constants at the particular point x_i and are such that x will be continuous.

Substituting this into Eq. (12), (15) can be shown to be a solution to lowest order in inverse powers of ω_0 which suggests that $x(t)$ has the form

$$x(t) = z_0 e^{-\Gamma|t-t_0|} e^{i\Omega|t-t_0|} + z_0^* e^{-\Gamma|t-t_0|} e^{-i\Omega|t-t_0|} + x_{\rm sm}(t) \tag{16}$$

to lowest order in inverse powers of ω_0, where z_0 is a complex number so that $x(t)$ is real. The parameter t_0 is such that $-\infty < t_0 < \infty$ and appears because of the time-translational invariance involved and can be set to zero without loss of generality. It is the time at which the amplitude of the oscillations has its greatest value and as a consequence the likelihood of a transition being made at this time is enhanced. The smooth function $x_{\rm sm}(t)$ represents the motion of the center of oscillation. We have ignored terms oscillating at higher frequencies since the magnitude of these components is related to the ratio of the height of the power spectrum (5) at the resonant frequency to that at the higher frequencies. For terms oscillating at frequencies $\pm n\Omega$ this ratio is approximately $\Gamma^2/n^2\omega_0^2$. Since we are considering the regime $\Gamma/\omega_0 \ll 1$ this ratio is very small. Hence the assumption that terms oscillating at higher frequencies have very little effect on the motion and can be ignored.

III. CALCULATION OF THE ACTION

Returning to the case of a general bistable potential $V(x)$, we substitute the expected form of the solution (16) into the action given by (11). For (11) to be finite we require that $\dot{x}$ and $\ddot{x}$ be continuous at $t = 0$, which gives $z_0 = \frac{1}{2}x_0(1 - i\frac{\Gamma}{\Omega})$, where x_0 is the amplitude of the oscillations at $t = 0$. The action consists of three contributions: one involving the oscillatory terms in (16) only, one involving cross terms between the oscillatory terms and time derivatives of $x_{\rm sm}$ and $V'(x)$, and one involving time derivatives of $x_{\rm sm}$ and $V'(x)$ only. Taking these in turn, the first is evaluated by simply substituting in the explicit expression given in (16). This may be facilitated by integrations by parts which are in effect reversing the steps taken to get from (8) to (9). The result is Γx_0^2. The second contribution may be evaluated by integrating by parts to obtain an integrand which has the form of $[\dot{x}_{\rm sm} + V'(x)]$ multiplied by a function of the oscillating terms in x which vanishes identically. However the discontinuity in $x^{(3)}$ at $t = 0$ gives rise to an extra term from this contribution which is essentially this discontinuity multiplied by the time derivative of $[\dot{x}_{\rm sm} + V'(x)]$ at $t = 0$. Using these results and writing out the third contribution in full gives for Eq. (11)

$$S[x_{\rm sm}] = \Gamma x_0^2 - \frac{2\Gamma}{\omega_0^2} x_0 \frac{d}{dt}[\dot{x}_{\rm sm} + V'(x_{\rm sm},t)]|_0 + \frac{1}{4}\int_{-\infty}^{\infty} [\dot{x}_{\rm sm} + V'(x_{\rm sm},t)]^2 - \left(\frac{2}{\omega_0^2} - \frac{4\Gamma^2}{\omega_0^4}\right)\left(\ddot{x}_{\rm sm} + \frac{dV'(x_{\rm sm},t)}{dt}\right)^2 + \frac{1}{\omega_0^4}\left(x_{\rm sm}^{(3)} + \frac{d^2V'(x_{\rm sm},t)}{dt^2}\right)^2 dt\,, \tag{17}$$

where $V'(x_{\rm sm},t)$ is defined by

$$V'(x_{\rm sm},t) = V'\left[z_0 e^{-\Gamma|t|} e^{i\Omega|t|} + z_0^* e^{-\Gamma|t|} e^{-i\Omega|t|} + x_{\rm sm}(t)\right]\,. \tag{18}$$

The potential can be written as a sum of a smoothly varying part and oscillating parts:

$$V'(x_{\rm sm},t) = V'_{\rm sm}(x_{\rm sm},t) + \sum_{n(\neq 0)} V'_n(x_{\rm sm},t) e^{in\Omega t}\,, \tag{19}$$

where $V'_{\rm sm}$ and V'_n are smooth functions. We will denote the second term by $V'_{\rm osc}(x_{\rm sm},t)$. The first term equals, by integrating over a period,

$$V'_{\rm sm}(x_{\rm sm},t) = \frac{1}{2\pi}\int_0^{2\pi} V'\left[z_0 e^{-\Gamma|t|} e^{is} + z_0^* e^{-\Gamma|t|} e^{-is} + x_{\rm sm}(t)\right] ds\,. \tag{20}$$

We can now integrate Eq. (17) by parts to give

$$S[x_{\rm sm}] = \Gamma x_0^2 + \frac{1}{4}\int_{-\infty}^{\infty} [\dot{x}_{\rm sm} + V'_{\rm sm}(x_{\rm sm},t)]^2 - \left(\frac{2}{\omega_0^2} - \frac{4\Gamma^2}{\omega_0^4}\right)\left(\ddot{x}_{\rm sm} + \frac{dV'_{\rm sm}(x_{\rm sm},t)}{dt}\right)^2 + \frac{1}{\omega_0^4}\left(x_{\rm sm}^{(3)} + \frac{d^2V'_{\rm sm}(x_{\rm sm},t)}{dt^2}\right)^2 dt + \frac{1}{4}\int_{-\infty}^{\infty} \{2[\dot{x}_{\rm sm} + V'_{\rm sm}(x_{\rm sm},t)] + V'_{\rm osc}\}\left\{V'_{\rm osc} + \left(\frac{2}{\omega_0^2} - \frac{4\Gamma^2}{\omega_0^4}\right)\frac{d^2V'_{\rm osc}(x_{\rm sm},t)}{dt^2} + \frac{1}{\omega_0^4}\frac{d^4V'_{\rm osc}(x_{\rm sm},t)}{dt^4}\right\}dt\,, \tag{21}$$

where contributions due to discontinuities at $t = 0$ have been omitted since they are down by powers of $1/\omega_0$ on the leading term.

We now show that the third term of (21) is of order $1/\omega_0$. First consider the oscillating term with $n = \pm 1$. The curly brackets in the third term of (21) can be approximated by

$$\left\{1 - \left(\frac{2}{\omega_0^2} - \frac{4\Gamma^2}{\omega_0^4}\right)\Omega^2 + \frac{\Omega^4}{\omega_0^4}\right\}\left(V'_{+1}e^{i\Omega t} + V'_{-1}e^{-i\Omega t}\right) \approx O\left(\frac{1}{\omega_0^2}\right) .$$

The terms in the potential oscillating at higher frequencies are also of order $1/\omega_0$ by the same argument used at the end of Sec. II.

The action is therefore given by

$$S[x_{\rm sm}, x_0) = \Gamma x_0^2 + \frac{1}{4}\int_{-\infty}^{\infty}(\dot{x}_{\rm sm} + V'_{\rm sm})^2 dt \quad + \quad O\left(\frac{1}{\omega_0}\right) . \tag{22}$$

The extremal paths are found from $\delta S/\delta x = 0$, or in terms of the new variables $\{x_{\rm sm}(t), x_0\}$, $\delta S/\delta x_{\rm sm} = 0$, where we are not yet varying x_0. This yields to lowest order in ω_0^{-1} the equation

$$\frac{d}{dt}[\dot{x}_{\rm sm}(t) + V'_{\rm sm}(x_{\rm sm}, t)] - [\dot{x}_{\rm sm}(t) + V'_{\rm sm}(x_{\rm sm}, t)]\frac{\partial V'_{\rm sm}(x_{\rm sm}, t)}{\partial x_{\rm sm}} = 0 . \tag{23}$$

For convenience this can be separated into two first-order coupled differential equations which are given by

$$\dot{x}_{\rm sm}(t) + V'_{\rm sm}(x_{\rm sm}, t) = f(t) ,$$

$$\dot{f}(t) = \frac{\partial V'_{\rm sm}(x_{\rm sm}, t)}{\partial x_{\rm sm}} f(t) . \tag{24}$$

So now the action given by (22) can be written as

$$S = \Gamma x_0^2 + \frac{1}{4}\int_{-\infty}^{\infty} f^2(t)dt \quad + \quad O\left(\frac{1}{\omega_0}\right) . \tag{25}$$

We now introduce a specific potential, the quartic potential $V(x) = -x^2/2 + x^4/4$, and attempt to calculate the action for a well transition and the action for a mean first passage. First we have to find $V'_{\rm sm}$. It is given by

$$V'_{\rm sm}(x_{\rm sm}, t) = x_{\rm sm}^3 - x_{\rm sm}\left(1 - \frac{3x_0^2}{2}\left\{1 + \frac{\Gamma^2}{\Omega^2}\right\}e^{-2\Gamma|t|}\right) . \tag{26}$$

We can now use (26) to integrate Eqs. (24). In this case there is no easy transformation of variables $\dot{x}(t) = y(x)$ as used to solve similar equations in [9]. Notice that in the limit $x_0 \to 0$ the solution of Eqs. (24) is that of the extremal path of the white-noise case, that is, $\dot{x}_{\rm sm}(t) = -V'_{\rm sm}$ and $f = 0$ or $\dot{x}_{\rm sm}(t) = +V'_{\rm sm}$ and $f = 2V'_{\rm sm}$. The first case is the *downhill* solution and leads to a zero contribution to the action. This case also remains a solution even when $x_0 \neq 0$. The second case is the *uphill* solution but this no longer remains a solution if $x_0 \neq 0$ because of the explicit time dependence of the potential.

In order to keep the white-noise solution as the limiting case of $x_0 \to 0$ we must use the boundary conditions, along with Eqs. (24), given by

$$x_{\rm sm}(-\infty) = -1 ,$$

$$x_{\rm sm}(\infty) = 0 . \tag{27}$$

We can now use a substitution $s = e^{-|t|}$ in Eqs. (24) and integrate them numerically using the COLSYS package [10]. We have to ensure that $x_{\rm sm}$ and f are continuous at $t = 0$ in order for the original integral (25) to make sense. Once we have calculated f we can obtain the action (25) which is a function of the parameters x_0, ω_0, and Γ. In Fig. 1 S is plotted as a function of x_0 for several values of the bandwidth parameter Γ and a fixed value of ω_0. We find that S varies very little with

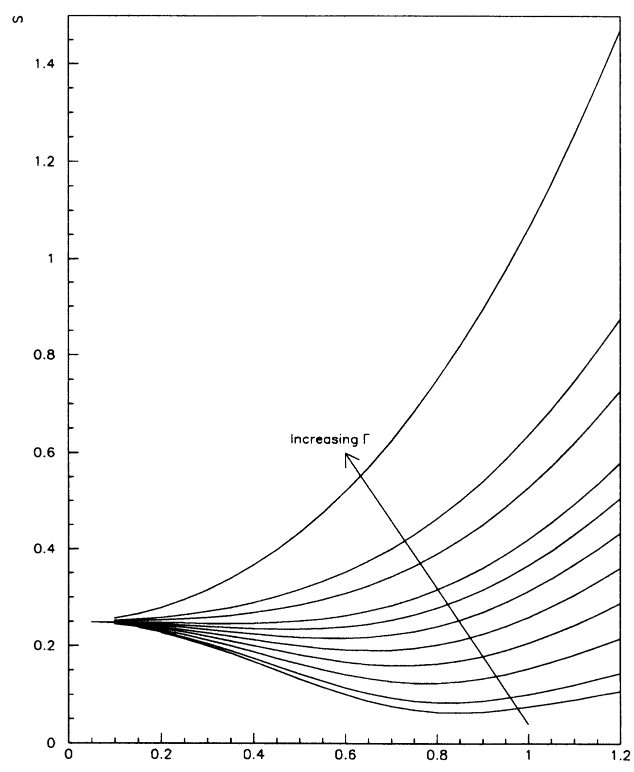

FIG. 1. The action calculated from (25) for varying values of x_0. Notice how as Γ decreases a minimum begins to form for values of $x_0 \neq 0$.

ω_0 as long as this parameter is large enough and in what follows we fix it to have the value 10.0. It still remains to minimize S with respect to x_0 which means we have to determine the value of x_0 for which $\partial S/\partial x_0 = 0$ and $\partial^2 S/\partial x_0^2$ is positive. The value of the action where these criteria hold is denoted by S^* and is calculated numerically. It is then plotted as a function of the bandwidth parameter Γ in Fig. 2. The ratio S^*/Γ is plotted as a function of Γ in Fig. 3. In the limit $\Gamma \to 0$ the value S^*/Γ will be shown to tend to a finite quantity. In the work of Dykman and co-workers [4, 6, 7] the white-noise strength has a linear dependence on Γ and so this value is studied to make a comparison between our work and theirs.

There are a number of interesting points in Figs. 1–3 which are worthy of further discussion. In Fig. 1 there are two important limits which can be picked out. The first is the $x_0 \to 0$ limit. One sees that the action tends to 0.25, which is the value found in the white-noise case—the barrier height being 0.25 in dimensionless units. The second limit is when $x_0 \to \infty$. The action now tends to Γx_0^2 and we see that the major contribution of the action arises due to the energy associated with the oscillatory motion. In Fig. 2 we see that for $\Gamma \gtrsim 0.46$, the minimal action S^* is just that obtained when the noise is white. It follows that the particle escapes from one well to another by white-noise-type outbursts. This means that the action associated with mean first passages and the action associated with well transitions are equal. On the other hand, for $\Gamma \lesssim 0.46$ the minimal action occurs when $x_0 \neq 0$. As a consequence we deduce that the particle escapes from one well to another by an oscillatory type of motion. This leads to differences between the action associated with a mean first passage and the action associated with a well transition. If one plots the envelope of x against t subject to the boundary condition (27), then one finds that the particle actually crosses the potential barrier top many times before $t = +\infty$. Finally, in Fig. 3 one sees that for $\Gamma \gtrsim 0.46$, the result is as for white noise (having the constant value $1/4\Gamma$), whereas for $\Gamma \lesssim 0.46$, S^*/Γ peaks around $\Gamma \approx 0.05$. The value at $\Gamma = 0$ can be calculated analytically and is found to be $\frac{2}{3}$, as we now show.

As $\Gamma \to 0$, the paths for f found numerically around the point where $\partial S/\partial x_0 = 0$ and $\partial^2 S/\partial x_0^2$ is positive are found to tend to zero for all values of t. Setting $f = 0$ in (24) enables us to obtain the uphill solution analytically. For $t < 0$ one finds that

$$x_{\rm sm}^{-2}(t) = 2e^{-2\left[t-\frac{3x_0^2}{4\Gamma}\left(1+\frac{\Gamma^2}{\Omega^2}\right)e^{2\Gamma t}\right]} \times \int_{-\infty}^{t} e^{2\left[t'-\frac{3x_0^2}{4\Gamma}\left(1+\frac{\Gamma^2}{\Omega^2}\right)e^{2\Gamma t'}\right]} dt' \qquad (28)$$

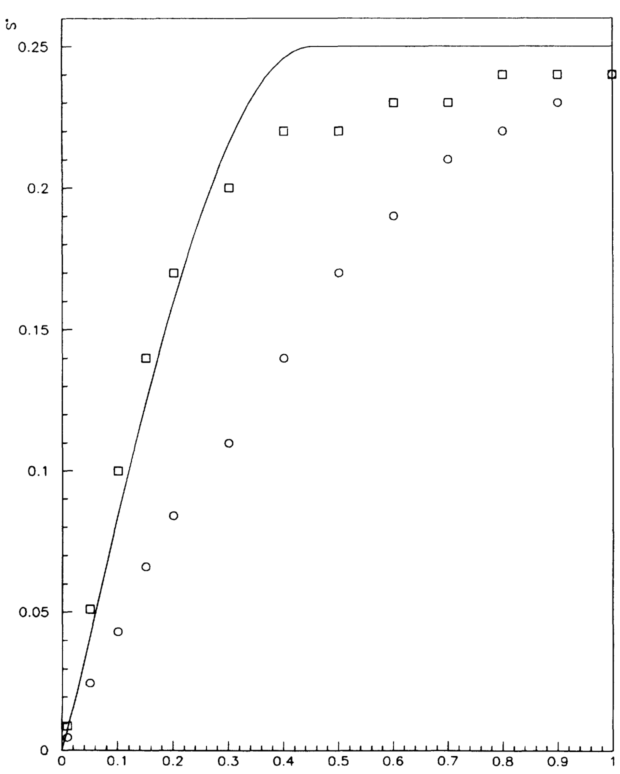

FIG. 2. The action S^* versus Γ together with results from simulations. Squares represent mean well transitions and circles mean first passages. Both show a gradual decrease in the value of the action.

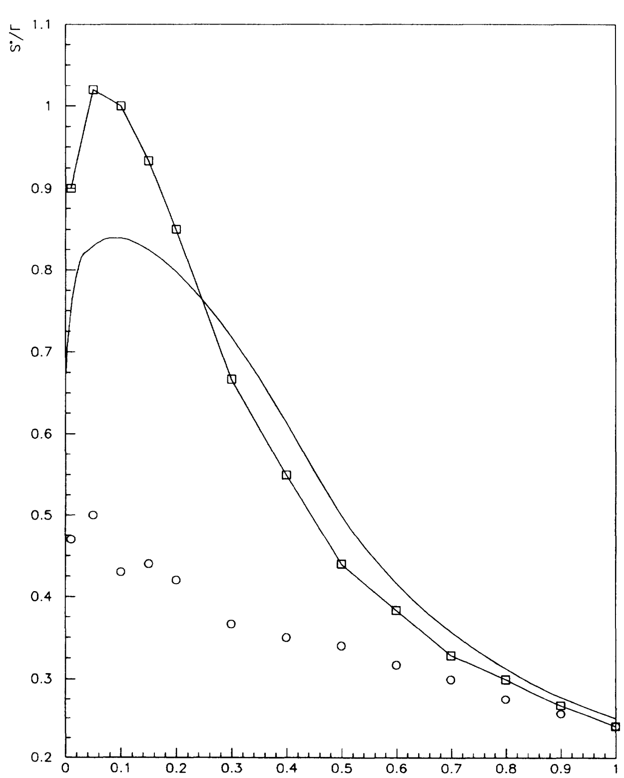

FIG. 3. The ratio S^*/Γ versus Γ together with results from simulations. Squares represent mean well transitions and circles mean first passages. Notice that the value of S^*/Γ tends to a finite value as Γ tends to zero and that its value peaks at $\Gamma \approx 0.05$.

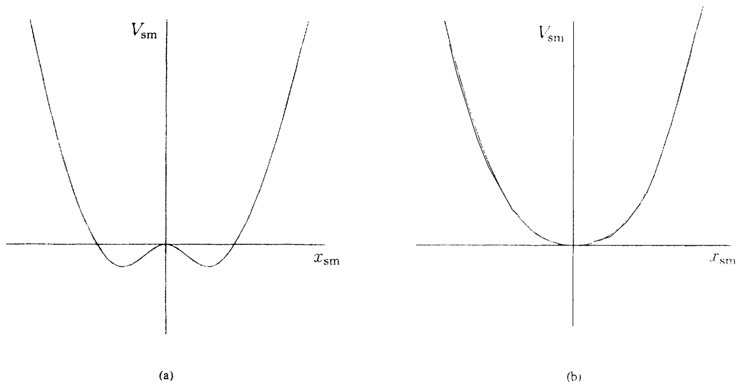

FIG. 4. The potential V_{sm} [given by (32)] versus x_{sm} in the limit $\Gamma = 0$, when (a) $x_0^2 < 2/3$ and (b) $x_0^2 > 2/3$.

and the action given by (25) becomes

$$S = \Gamma x_0^2 \,. \tag{29}$$

We can now expand $e^{2\Gamma t} = 1 + 2\Gamma t + O(\Gamma^2)$ and Eq. (28) becomes

$$x_{sm}^{-2}(t) = 2e^{\frac{3x_0^2}{2\Gamma}} e^{-2\left(1-\frac{3x_0^2}{2}\right)t+O(\Gamma)} \times \int_{-\infty}^{t} e^{-\frac{3x_0^2}{2\Gamma}} e^{2\left(1-\frac{3x_0^2}{2}\right)t'+O(\Gamma)} dt' \,. \tag{30}$$

This can now be integrated easily to give

$$x_{sm}^2(t) = \begin{cases} 1 - \frac{3x_0^2}{2}, & x_0^2 < \frac{2}{3} \\ 0, & x_0^2 \geq \frac{2}{3} \,. \end{cases} \tag{31}$$

It is easy to check that (31) holds also for positive t.

The $\Gamma \to 0$ limit can be made more transparent by proceeding in a slightly different way. If we begin from the expression

$$V_{sm}(x_{sm}, t) = -\frac{1}{2}\left(1 - \frac{3x_0^2}{2}\left\{1 + \frac{\Gamma^2}{\Omega^2}\right\} e^{-2\Gamma|t|}\right) x_{sm}^2 + \frac{1}{4} x_{sm}^4 \tag{32}$$

we see that if $x_0^2 < \frac{2}{3}$ and $\Gamma \to 0$, the potential has two minima given by $x_{sm}^2(t) = 1 - \frac{3}{2}x_0^2 e^{-2\Gamma|t|}$. However, if $x_0^2 \geq \frac{2}{3}$ the potential has a single minimum at $x_{sm} = 0$. These two possibilities are illustrated in Fig. 4. Since x_{sm} is independent of time when $\Gamma = 0$, the system is found in minima of the potential at all times. This can be seen directly from (24), where a rescaling of the time by Γ shows that when $\Gamma = 0$, $f = 0$ and $V'_{sm} = 0$ for all finite times. In spite of these simplifications it is easier to think of the time evolution of $x_{sm}(t)$ if we imagine Γ very small, but nonzero. Suppose, first of all, that $x_0^2 < \frac{2}{3}$. In the infinitely distant past V_{sm} has a double-well form with $x_{sm} = -1$. As t increases, then x_{sm} takes the value $-(1 - \frac{3}{2}x_0^2)^{\frac{1}{2}}$ and remains at this value until large positive times when it moves back to -1. This clearly violates the boundary condition $x_{sm}(+\infty) = 0$. Now suppose $x_0^2 \geq \frac{2}{3}$. Once again $x_{sm}(-\infty) = -1$, but at some large negative time the double wells merge into a single well at $x_{sm} = 0$. At large positive times double wells develop again, but in the limit $\Gamma = 0$ the system remains at $x_{sm} = 0$, which is the correct value as $t \to \infty$. We therefore see that we require $x_0^2 \geq \frac{2}{3}$ to obtain a correct picture when $\Gamma = 0$.

To determine S^*/Γ in the limit $\Gamma \to 0$ we have to minimize $S/\Gamma = x_0^2$ with respect to x_0 subject to the condition $x_0^2 \geq \frac{2}{3}$. This leads to the value $\frac{2}{3}$ for S^*/Γ in this limit, as claimed.

IV. COMPUTER SIMULATION OF QMN

Since QMN can be viewed as arising from the second-order differential equation (7), we cannot easily implement the Fox *integral* algorithm [11] for noise simulation. There are two possible approaches. The first is to use an extension of the Sancho algorithm [12] which makes use of the Box-Muller algorithm [13] for the generation of Gaussianly distributed random numbers. Then QMN is generated using a finite-difference form of (7) given by

$$\begin{aligned} f(t+\Delta t) &= 2f(t) - f(t-\Delta t) \\ &\quad -2\Gamma\Delta t[f(t) - f(t-\Delta t)] \\ &\quad -\Delta t^2 \omega_0^2 f(t) + N(t) \,, \end{aligned} \tag{33}$$

where $N(t)$ is

$$N(t) = [-4D\Delta t^3 \ln(R_1)]^{\frac{1}{2}} \cos(2\pi R_2).$$

Here R_1 and R_2 are random numbers, uniformly distributed in the range [0,1], and Δt is the step length used in the simulation. The second approach is to consider the system as three coupled differential equations and to use the work by Mannella and Palleschi [14]. In the notation of that paper the equations governing the system are written in the form of three first-order differential equations,

$$\dot{x}_1 = -V'(x_1) + x_2 \,,$$

$$\dot{x}_2 = x_3 \,,$$

$$\dot{x}_3 = -2\Gamma x_3 - \omega_0^2 x_2 + \sqrt{2D}\xi(t) \,, \tag{34}$$

where $x_1 = x$, $x_2 = f$, $x_3 = \dot{f}$, and $\xi(t)$ is Gaussian white noise of zero mean and mean square value unity.

This system can be simulated by

$$x_1(t+\Delta t) = x_1(t) + \Delta t\,[-V'(x_1(t)) + x_2(t)] + O(\Delta t^{3/2}) ,$$

$$x_2(t+\Delta t) = x_2(t) + \Delta t x_3(t) + O(\Delta t^{3/2}) , \qquad (35)$$

$$x_3(t+\Delta t) = x_3(t) + \Delta t^{1/2}\sqrt{2D}M + \Delta t\left[-2\Gamma x_3(t) - \omega_0^2 x_2(t)\right] + O(\Delta t^{3/2}) ,$$

where M is a Gaussianly distributed random number of zero mean and mean square value unity. This random number can again be generated by the Box-Muller algorithm and is given by

$$M = [-2\ln(R_1)]^{\frac{1}{2}} \cos(2\pi R_2) , \qquad (36)$$

where R_1 and R_2 are as before.

With the simulation step length Δt small enough (in our case equal to 0.001) these two approaches are consistent to fractions of a percent. Using small Δt also eliminates the need to use any corrector steps as in [14]. However a full discussion of the merits of using different algorithms or different step lengths is not included in this paper.

During the simulations care must be taken to measure both mean first-passage times and the mean well transition times since the theory predicts different values of the action for these two quantities. Since we are dealing with small values of $\tilde{D} = D/\omega_0^4$, we expect that these characteristic times will have a simple activation energy-type dependence on the associated action. It is then natural to extend this to the limit of $\tilde{D} \to 0$ by plotting the quantity (where τ is either the mean first-passage time or the mean well transition time)

$$S(\Gamma, \omega_0, \tilde{D}) = \tilde{D}\ln(\tau) \qquad (37)$$

against $\tilde{D}$. This will coincide with S^* (the quantity shown as a function of Γ in Fig. 2) when $\tilde{D}$ goes to zero. As in [15], the action has a near linear dependence on $\tilde{D}$ and the value of the actions measured for $\omega_0 = 10$ are compared with the theoretical results in Figs. 2 and 3. The squares are the actions for mean well transitions and the circles are for mean first passages. The line connecting the squares in Fig. 3 is to guide the eye and to help show the qualitative features of the simulation data.

This method is a useful technique since it allows measurement of the action from simulation data without the need of having knowledge of any prefactors.

V. CONCLUSIONS

The theoretical results are only correct to order of inverse powers of ω_0 and so for $\omega_0 = 10.0$ these could result in corrections of order 10%. The errors resulting from the simulations can occur from the following.

(i) Using a limited number of ensembles of simulation runs to predict the ensemble average of typical well transition and first-passage times.

(ii) The fact that finite $\tilde{D}$ was used, although these errors are limited using the analysis used in [15].

(iii) Using a finite but small simulation step length. A full discussion of how this affects measurements is given in [15].

(iv) Using a basic finite-step size expansion for the stochastic equations.

However, considering all possible errors the theoretical approach and the computer simulations agree well in the qualitative description of QMN. Both show that the type of transition between wells in a bistable potential is dependent on the bandwidth of the noise Γ. Both theory and simulations predict that the value of S^*/Γ has a finite value of ≈ 0.67 at $\Gamma = 0$ and that this quantity peaks at $\Gamma \approx 0.05$.

This is in agreement with digital simulations in Ref. 5 where a peak is found in the frequency of escapes at $\Gamma \approx 0.1$ (the factor of 2 difference between our result and theirs arises from a different definition of Γ).

For large values of Γ the oscillatory behavior of the motion is suppressed and the system behaves as if it is under the influence of white noise. As Γ is reduced, however, the particle begins to oscillate and this type of motion causes a well transition, which results in differences between the action for mean first-passage actions and the action for a mean well transition. Also there is a decrease in size of these two (if $\tilde{D}$ can be considered Γ independent). The suppression of oscillations with increasing Γ, but not the existence of a critical value, was noticed by Schimansky-Geier and Zülicke [5].

This means the bandwidth of the noise can be used a switching device by the system to decrease the action and hence escape times between wells. If a well transition needs to be inhibited, then the bandwidth may be broadened; if it needs boosting, then the bandwidth may be narrowed. This interesting property and the different method of motion as compared to white and exponential noise means that many natural systems may have more control over their behavior than at first expected.

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