Efficient large-scale simulations of a uniformly driven system

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We present results from an efficient algorithm for simulating systems of locally connected elements that are subject to uniformly increasing stresses and that discharge when these stresses reach some threshold. Previously, large-scale simulations of such systems have been hindered by the very-time-consuming search for those elements that are going to discharge next. We avoid this by using a suitable data structure, reducing computer CPU times by several orders of magnitude in typical cases. In particular, we present simulations for a simple version of the Burridge-Knopoff model introduced by Olami, Feder, and Christensen [Phys. Rev. Lett. **68**, 1244 (1992)]. Due to the substantially larger lattices and longer simulation times presently used we find that the conclusions of these authors have to be modified considerably.

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I. INTRODUCTION

Recently there has been much interest in spatially extended systems of coupled elements which can support a limited amount of some "stress" and transfer at least part of this stress to their neighbors when discharging. Such models are often characterized by scale invariant distributions of the resulting avalanches. Some applications include sandpiles [1–7], forest fires [8–13], pinned domain walls in magnets or pinned flux tubes in type-II superconductors [14], earthquakes [15–22], invasion percolation growth [23], and pinned charge-density waves [24]. In those cases where the discharges appear in avalanches with a broad and scale invariant distribution of sizes, these phenomena were subsumed as self-organized criticality (SOC) [1].

Typically, such phenomena are modeled by means of a discrete space-time lattice. In some cases also the variable at each lattice point is discrete, in which case one deals with a (deterministic or probabilistic) cellular automaton (CA). Otherwise, such systems are called coupled map lattices [25].

The most straightforward simulation of such a model consists of going through the lattice in a regular fashion, updating each variable according to the local rule. While such a strategy is appropriate in a CA where some nontrivial action is going on at most sites, it is not appropriate for SOC, since there most sites are stable and stationary. Thus in a regular passing through the lattice one spends most of the time in checking that no action has to be taken, and such algorithms tend to be very slow even if the vectorization and parallelization possible with such algorithms are fully exploited.

A much better strategy for SOC (and for a number of other systems [26,27]) consists of keeping a list of "active sites." During each time step, the entire list is examined and a new list for the next generation is established. Since the number of active sites is typically very small in the critical state, this already makes a big improvement [4].

In addition, a modest improvement is obtained by replacing the d coordinates of a site in a d-dimensional lattice by a single index. In this way the number of variables in the list of active sites is reduced and the addressing of the array elements is simplified. Also, this simplifies the implementation of the following trick, which causes a dramatic improvement in some cases.

While the above algorithm is adequate in cases where the stresses increase stochastically (as in the sandpile model of [1]), it is not adequate for large-scale simulations of models where the stresses increase uniformly and continuously [11,17,20]. In such models (which of course are coupled map lattices and not CA, in contrast to statements made, e.g., in [20]) one has to search between any two avalanches for that element which is closest to its critical state, because it will be this element that will discharge next. On a lattice of N sites, this takes a time $O(N)$. Since in some of these models the average size of the avalanches triggered by these discharges is $O(1)$, the improvement will be by a factor $O(N)$ if the search for the weakest element can be eliminated.

It is exactly the latter which I propose in this paper. In the following section I shall describe the specific model [20] dealt with in the following. A fast algorithm for it will be described in Sec. III, though it should be obvious that the idea is more general. In Sec. IV I will present results obtained by it. While I reproduce essentially all results of [20] on the small lattices studied by these authors (up to 50×50 sites), we will see that the extrapolation to larger lattices is not as straightforward as suggested in [20]. The paper finishes with a discussion in Sec. V.

II. MODEL

The specific model I will simulate is that of [20] (see also [21,22]) with either "open" [20] or periodic boundary conditions (BC's). In this model, a real variable F_i (called "stress" in the following) is attached to each point

$\mathbf{i}$ of a quadratic lattice of size $L \times L$. In the initial state, the values of F are randomly distributed in $[0,1]$ with uniform distribution. The evolution is defined by two mechanisms which seem at first sight to be very similar to those in the sandpile model of [1]:

(1) Every $F_{\mathbf{i}}$ increases with the same speed, $dF_{\mathbf{i}}/dt = v$.

(2) As soon as one of the $F_{\mathbf{i}}$ becomes equal to the critical value $F^* = 1$, it relaxes to $F'_{\mathbf{i}} = 0$, and a fraction $\alpha F_{\mathbf{i}}$ ($\alpha < 1/4$) is distributed to each of its four neighbors (boundary sites will be discussed below). If any of the neighbors $\mathbf{j}$ becomes thereby supercritical ($F'_{\mathbf{j}} = F_{\mathbf{j}} + \alpha F_{\mathbf{i}} \geq F^*$), it will also relax to $F''_{\mathbf{j}} = 0$, and again a fraction $\alpha F'_{\mathbf{j}}$ is given to each of its neighbors. The sequence of discharges triggered in this way will be called an *avalanche*.

If more than one site is supercritical at any time, the discharges are assumed to happen simultaneously. Step (2) is repeated [without applying step (1) between the discharges] until all sites are subcritical. Thus, formally we consider the limit $v \to 0$.

Notice that relaxation is always to $F' = 0$. This is in contrast to the sandpile model of [1]. It renders the model non-Abelian [2], and the temporal order of the discharges has to be strictly followed (this excludes a recursive depth-first algorithm as discussed in [4]). Also, since only a fraction 4α of F is distributed to the neighbors, the model does not conserve the total stress $\sum_{\mathbf{i}} F_{\mathbf{i}}$ (again in contrast to [1]); a fraction $(1 - 4\alpha)F$ of the stress is dissipated.

The last (and in some sense most important) difference to the sandpile model is that the model is strictly deterministic. It is easily seen that the model does not show deterministic chaos either. Indeed, if all discharges happen at $F_{\mathbf{i}} = F^*$, then an infinitesimal change $\delta F_{\mathbf{i}}$ leads just to an infinitesimal change $\delta t = -\alpha\, \delta F_{\mathbf{i}}$ of the time at which this site discharges. If, on the other hand, $F_{\mathbf{i}} > F^*$ at the discharge, there is no time shift but a fraction of $\delta F_{\mathbf{i}}$ is passed to the neighbors. In the first case, the evolution is conservative in the sense of Liouville's theorem with all Lyapunov exponents equal zero, while in the second case it is dissipative (notice that the notions of "conservation" and "dissipation" here do not refer to the conservation of stress but of phase-space volume).

Let us finally discuss the boundary conditions. Since stress is not conserved, it seems most natural to use periodic BC's. This is again in sharp contrast to the sandpile model. There, periodic BC's would lead to an infinite avalanche after a finite amount of sand had been added, and to a violation of ergodicity on finite lattices. This problem would be absent in the limit of infinite lattices, but this limit is reached very slowly [4]. But no such problem is expected in models without stress conservation, and periodic BC's (when possible) usually involve the smallest finite-size corrections.

In addition, I also simulated "open" BC's, following [20]. In this case, an additional layer of sites with $F_{\mathbf{i}} = -\infty$ is added at the boundary. This layer will never discharge, but it absorbs the discharges of the neighboring sites, the rules for which thus remain formally unchanged.

III. ALGORITHM

At the first time step in an avalanche, the list of "active" sites contains all sites with $F_i = F^*$, while at later times it will contain all sites which have received some stress from neighboring discharges (here we have already replaced the vector site index $\mathbf{i}$ by a single integer i, as discussed in the Introduction).

In order to implement step (1) of the evolution, one first has to find the site(s) with the largest stress $F_{\max}$, and then increment all sites by $F^* - F_{\max}$. When implemented naively, this involves two passes through the lattice. The second pass can be avoided by not increasing F_i, but instead decreasing F^*. Of course, the rule for discharge then has to be replaced by

$$F_i \to F'_i = F^* - 1, \qquad F_j \to F'_j = F_j + \alpha(F_i - F^* + 1). \tag{1}$$

The first pass can be eliminated by using a suitable data structure from which the most stressed sites can be found directly. The idea is that the interval $[F^* - 1, F^*]$ of possible stress values is divided into M bins of size $1/M$, so that we only have to search in the highest bin. If $M \approx N$ (where $N = L^2$ is the number of lattice sites), and if the distribution of F over the lattice is not too singular, this will reduce CPU time by a factor $\sim O(N)$.

This can be done in principle in many ways. The most naive implementation would use M arrays, one array for each bin. This would be very inefficient, as the sizes of these arrays would fluctuate significantly, leading to a waste of storage. Instead, we use a structure that has been found to be very efficient in finding nearest neighbors in time sequence analysis [28, 29]. Here a set of M linked lists is used (one list for each bin), with M pointers to the heads of these lists. Since the total length of these lists is exactly N, we can write them into one large array LIST of size N when using FORTRAN. The header pointers are written then into an array BOX of size M.

Let us now describe the insertion of a site i, either during the initialization phase or after F_i has changed. At the beginning, all elements of BOX are empty, while LIST can contain arbitrary values. If F_i falls into bin k, the previous content of BOX(k) is written into LIST(i) and the new value of BOX(k) is set to i.

After the initiation phase, all elements of BOX corresponding to empty bins will be empty, while those corresponding to nonempty bins contain the site of the last value of F in this bin. The sites corresponding to this bin are simply found by following the links until an empty element of LIST is encountered, which indicates that the end of this particular list is reached. When a site i with F_i in bin k discharges or gets a contribution from a discharging neighbor, it has to be removed from the kth list and inserted in the list corresponding to the new F'_i. Removing it from its old list is done by first checking whether BOX(k) $= i$. If yes, then we simply replace it by BOX(k) $= 0$. If not, then we follow the list until we reach a j such that LIST(j) $= i$, and replace it by LIST(j) =LIST(i).

There is one final aspect which should be mentioned. Since the allowed interval $[F^* - 1, F^*]$ is continuously

shifted during the simulation, it is unfeasible to divide exactly this interval into M bins. Instead, we define the bin index by $k = 1 + (\lfloor MF_i \rfloor \bmod M)$. In this way a discharging site remains in the same bin if its stress before the discharge is equal to F^*. When searching the next site in this bin, we then have to check whether its stress is near F^* (so that it will discharge) or near $F^* - 1$ (in which case it has just discharged).

Obviously, the algorithm becomes inefficient if the system synchronizes strongly so that most of the sites are in only a few bins. Eliminating an element from the corresponding lists will then become very time consuming. In the present model this happens for values of α close to $1/4$ ($\alpha = 1/4$ is the conservative limit). But for $\alpha \leq 0.23$ and $L \geq 50$ I found the algorithm already much faster than the naive one. For $\alpha \leq 0.1$ the speed was essentially independent of L and α, with $\approx 10^4$ discharges per second on a DEC 2100 UNIX computer workstation.

IV. RESULTS

The main result of [20] was that the model showed scaling except for very small α ($\alpha_c \approx 0.05$). This was seen mainly in a power law for the probability distribution that an avalanche involves $\geq s$ discharges, $P(s) \sim s^{2-\tau}$. The exponent τ depended on α. It was obtained with two different types of nonperiodic boundary conditions. It was surprising, as scaling was expected only in the conservative limit $\alpha = 1/4$, in analogy to what happens in the sandpile model [6]. In the following we shall see that this result has to be modified.

A. Periodic boundary conditions

When simulating small lattices with periodic BC's, I found a rather surprising result: on small large lattices and for relatively small α, nearly all initial configurations led quickly [after $O(L^2)$ avalanches] to a strictly periodic state with period exactly equal to L^2. In this state there are strong correlations such that *none* of the discharges leads to an overcritical neighbor, and thus all "avalanches" consist of single isolated events. It does happen occasionally that several sites become degenerate and thus discharge simultaneously (hence simply counting the simultaneously discharging sites is not a good measure for the size of avalanche), but these discharges are not causally related. Causally induced chains of topplings arise very rarely, only for fairly large α (see Fig. 1) and not at all in the periodic states mentioned above.

It is easy to see that a strictly periodic state with period L^2 and with only trivial avalanches (called the "ordered state" in the following) is a valid solution. During one period of such a state, each site discharges exactly once, loosing thereby one unit of stress, and receives 4α units from its four neighbors. Thus all F_i decrease by the same amount $1 - 4\alpha$ during one period. Indeed, one easily verifies that a continuum of such states exists, since a small change in the initial conditions will generally lead to the same small change at large times. What is less clear is why the set of these states has such a large basin of attraction that nearly all initial configurations tend towards it at least locally.

Figure 1 suggests that there is a sharp threshold $\alpha_c \approx 0.18$ below which no avalanches with more than one discharge appear in the final state, so that the lattice orders into one of the above periodic states. Actually, the situation is not so simple. Since the evolution is not chaotic, every state finally becomes periodic (eventually with period $\neq L^2$) for all α. More extensive simulations showed that the approach towards disordered states is also rather fast and roughly exponential. Thus after a rather short time any configuration effectively becomes frozen in a periodic state which may or may not be ordered. Even for rather small α (much smaller than 0.18) there is a low but nonvanishing probability that some defects get frozen in an otherwise ordered state, thus smearing out the ordering phase transition. On the other hand, the fast freezing happens also for large α, suggesting that there is no sharp freezing transition either.

Thus there is actually no sharp threshold, but in the following we shall use $\alpha_c \approx 0.18$ as an effective threshold between approximately ordered and strongly disordered asymptotic states.

The asymptotic state on a periodic lattice is not ergodic since it is neither unique nor chaotic. Thus differ-

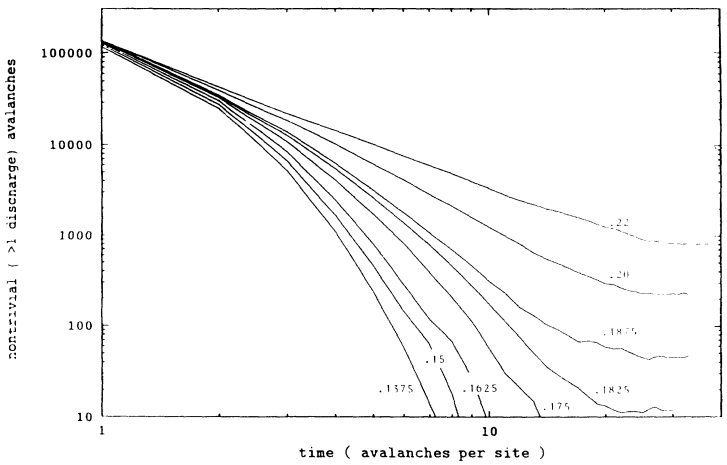

FIG. 1. Number of nontrivial avalanches (>1 discharges) during successive bins of L^2 avalanches on a lattice with periodic BC's. Only a single realization of a lattice with $L \geq 600$ is shown for each of the eight values of α: 0.1375, 0.15, 0.1625, 0.175, 0.1825, 0.1875, 0.2, and 0.22.

ent initial conditions would lead to different final states. In all simulations presented in this paper, I used uncorrelated initial conditions with F_i uniformly distributed in $[0,1]$. I also made runs with $F_i \in [0,a]$, $a < 1$. The results for these runs were indeed different, but not qualitatively.

Besides the ordering aspect seen in the periodicity with period L^2 ("temporal order") the final states are also ordered spatially. Indeed, for *all* initial configurations the final states are highly ordered (even if not in the above temporal sense). The degree of temporal ordering can be measured by the number of nontrivial avalanches (i.e., the deviation from the temporally periodic state). A measure of spatial ordering is suggested by looking at the distribution of stresses. In Fig. 2 is shown the probability distribution (not normalized) of $F^* - F_i$ for $\alpha = 0.1825$ (which is close to the transition point) and for $\alpha = 0.22$, which is in the aperiodic regime. Surprisingly, this indicates that the temporally disordered state (large α) is spatially more ordered and vice versa. For $\alpha \to 1/4$, the distribution tends towards a sum of five delta peaks at $F = m/4$, $m = 0,1,\ldots,4$. The same is indicated by the distribution of the stress difference between neighbors (Fig. 3). The latter is essentially a measure of local ordering, while the distribution of $F^* - F_i$ measures global ordering.

Figures 2 and 3 indicate that for large α, i.e., close to the limit of stress conservation, it is very difficult for the system to find the periodic state, since it has a much smaller basin of attraction. Thus the system remains longer in the dissipative aperiodic state, which then leads to a higher degree of spatial ordering.

This dichotomy of spatial order together with temporal disorder (and vice versa) has not been observed previously in critical phenomena. On the other hand, it is rather common in cellular automata [34] and coupled map lattices [25]. It demonstrates that the concept of "ordering" is less trivial than often assumed.

The effective discretization of stress seen for large α was observed in a slightly different model by Zhang [30]. It seems also related to the well known synchronization in coupled relaxation oscillators [31–33].

B. Open boundary conditions

If we use nonperiodic BC's (as in [20]), the situation is very different. I agree with the findings of [20] that one finds large avalanches for nearly all values of α, and that these avalanches are roughly described by scaling laws. Indeed, in contrast to [20] I found such scaling also for

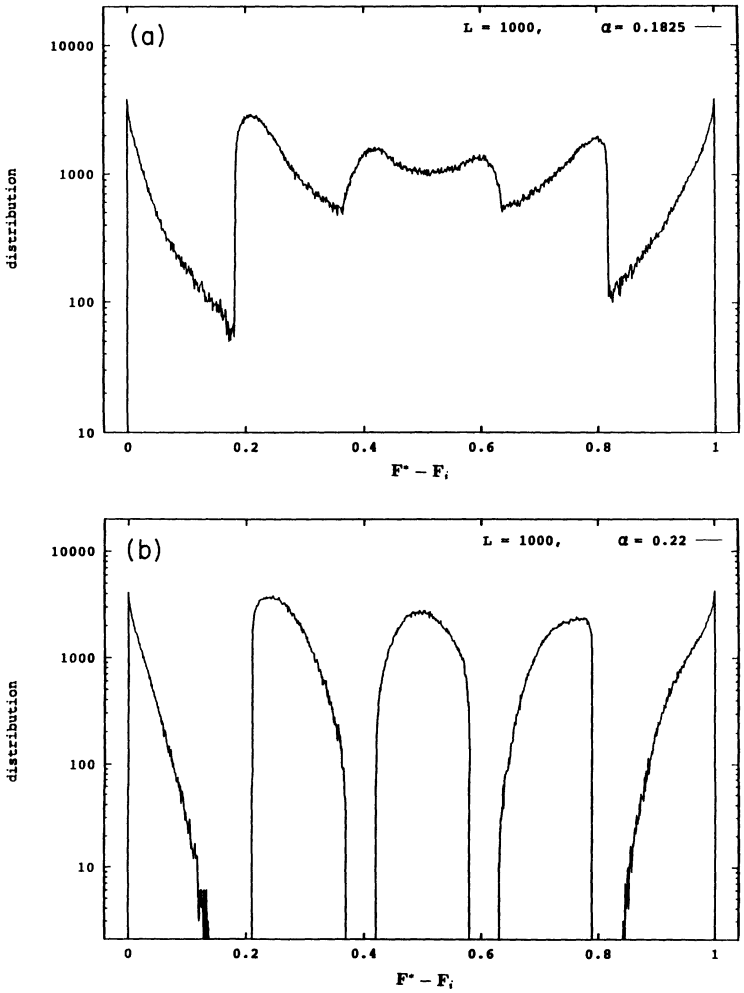

FIG. 2. Distribution of $F^* - F_i$ on a lattice with $L = 1000$ and periodic BC's. Panel (a) is with $\alpha = 0.1825$, panel (b) with $\alpha = 0.22$.

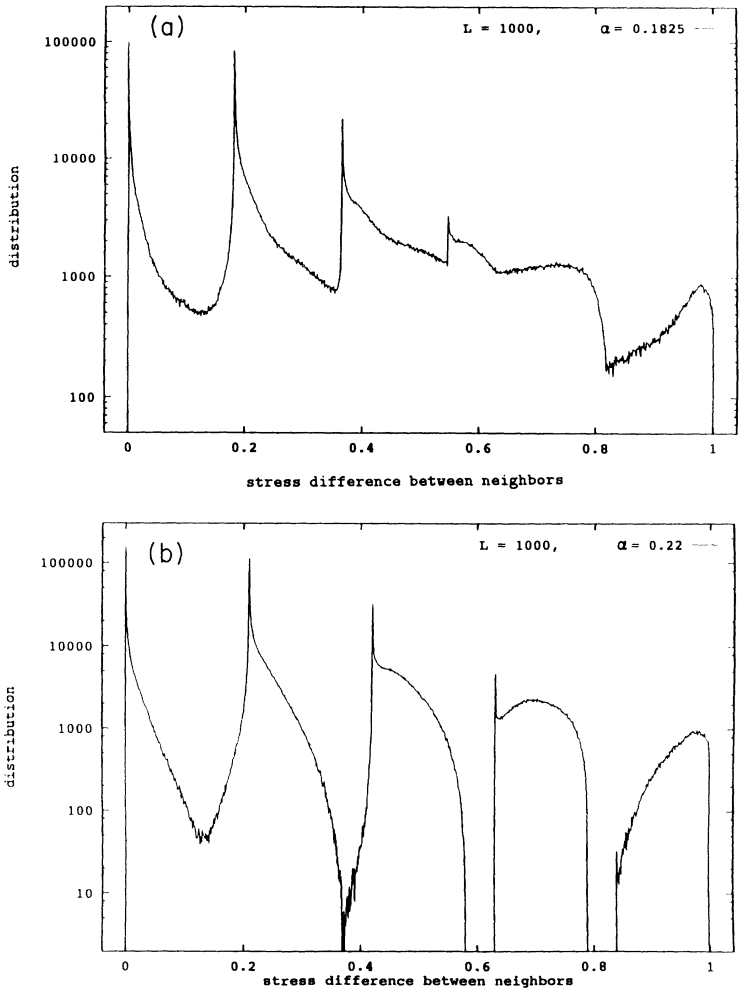

FIG. 3. Distribution of $|F_i - F_j|$ where i and j are neighboring sites, for the same parameters as in Fig. 2.

very small α ($\alpha < 0.05$) where these authors claim to see none. But only for small lattices and for fairly large α are *all* avalanches described by this scaling. In Fig. 4 I show the integrated distribution of avalanche sizes,

$$P(s) = \text{prob}\{\text{number of discharges} > s\}, \tag{2}$$

for $\alpha = 0.05$ and for three different lattice sizes. For all three curves we see roughly a power law [35] $P(s) \sim s^{2-\tau}$, $\tau \approx 2.8$, except for a very sharp and pronounced peak at $s = 1$. Furthermore, the height of this peak increases with L. More precisely, the total contribution of the power law tail decreases roughly as $1/L$ for large L. In view of the results of Sec. IV A this peak is easily understood: in the interior of the lattice (where the BC has little influence) the lattice orders into a periodic

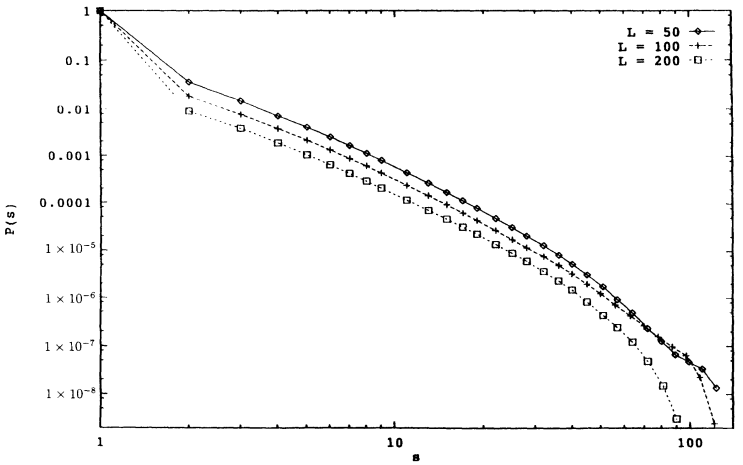

FIG. 4. Integrated distribution $P(s)$ of avalanche sizes for $\alpha = 0.05$ and $L = 50$, 100, and 200. For each of the lattices, the first $2000 \times L^2$ avalanches were discarded, and at least $40\,000 \times L^2$ next avalanches were used for averaging.

state with all avalanches having $s = 1$. It is only near the boundary that this state cannot be reached. Since the boundary sites do not get contributions from all four neighbors (because some of these neighbors are absent), they do not return to their critical stress at the same time as the interior sites, preventing the periodic state. Thus the main effect of the boundary is to create an inhomogeneity which prevents the ordering.

Although a thorough investigation of this phenomenon is difficult due to extremely long transient times on large lattices (for small lattices we verify that transient times are rather short, as found in [20]), we have a number of indications that this scenario is indeed correct. It suggests that for $\alpha < \alpha_c$ avalanches are mainly near the boundary. Occasionally, an avalanche will penetrate deeper into the lattice, but the disorder it produces will soon heal, and the bulk of the lattice will stay ordered for most of the time. In contrast, when $\alpha > \alpha_c$, the whole lattice is prevented from ordering and large avalanches will also be triggered deep inside the lattice. This difference is of course only seen on sufficiently large lattices. Hence it is not even clearly seen in the present simulations (where L was up to 400) and was impossible to detect in those of [20] where $L \leq 50$.

The first indication for the correctness of this scenario comes from observing the mean avalanche size during the transient period. Again we start with all F_i randomly in $[0, 1]$. As observed in [20], the mean avalanche size decreases first sharply, in order to approach the asymptotic value slowly from below (Fig. 5). This is easily explained by the fact that the attraction towards the periodic state in the interior of the lattice is much faster than the establishment of the very delicate long-range correlations that will give the scale invariant avalanches. Indeed, the initial decrease of the avalanche size is exactly the same as for periodic BC's.

That large avalanches appear first near the boundary is also observed when watching the evolution on-screen for small α. It explains the observation that $P(s) \sim 1/L$ in Fig. 4 for $s \geq 2$, and the extreme slowness of the approach towards the final state. From Fig. 5(b) we see that even 4×10^8 avalanches are not enough to reach stationarity for $L = 200$, $\alpha = 0.1$. It is these long

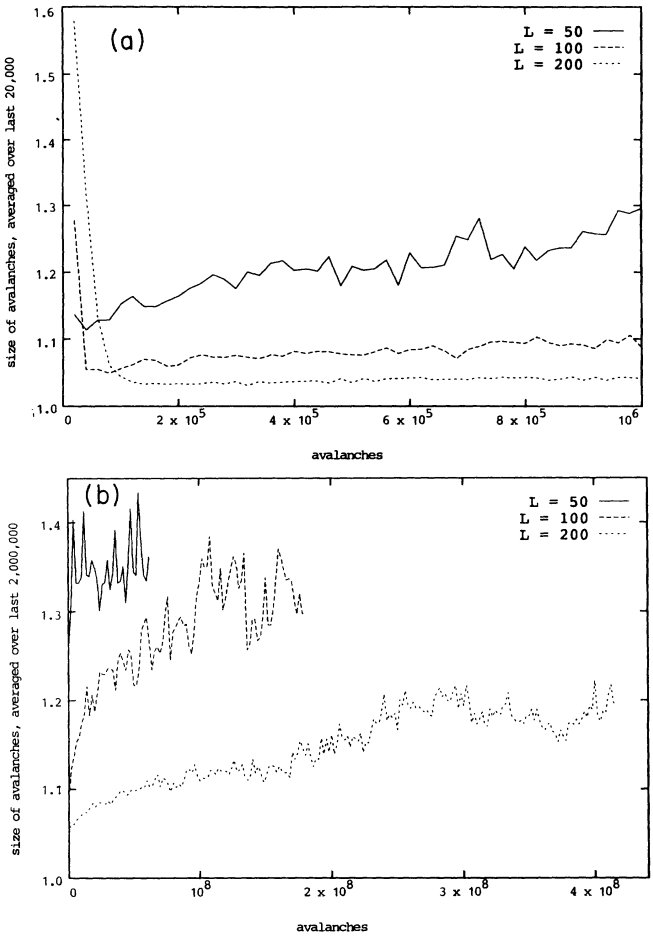

FIG. 5. Average size of avalanches in the last time bin, for bins of 2×10^4 avalanches [panel (a)] and for bins of size 2×10^6 [panel (b)]. All curves are for $\alpha = 0.1$. Notice the very slow convergence.

transient times that made a precise analysis difficult and that might be responsible for some of the deviations from scaling in Fig. 4 and in some of the following plots.

As stated above, we expect that large avalanches are no longer restricted to the surface for larger values of α. This is indeed apparent in Fig. 6, where $P(S)$ is shown

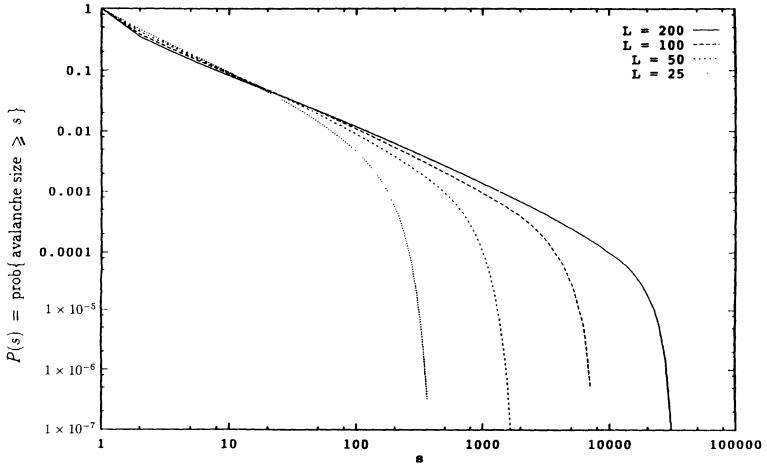

FIG. 6. Similar to Fig. 4, but for $\alpha = 0.2$. Due to the shorter transients and the somewhat lower efficiency of the algorithm, statistics is somewhat lower (ca. 10^8 avalanches for each lattice), and data are also shown for $L = 25$.

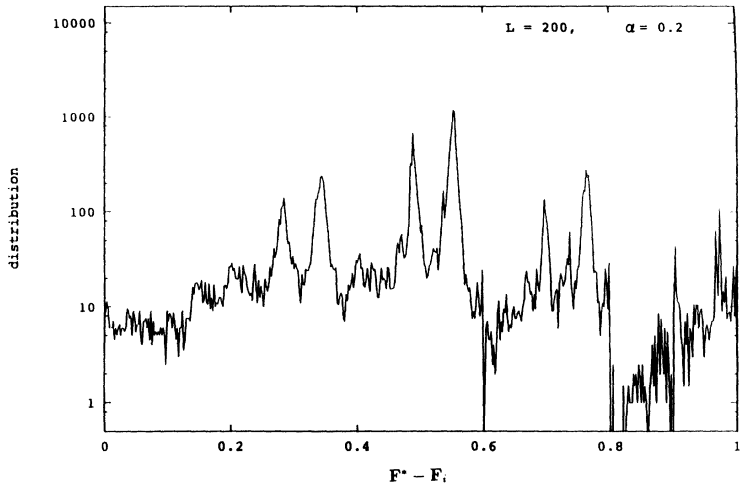

FIG. 7. Similar to Fig. 2, but for open boundary conditions. Lattice size is $L = 200$, and conservation parameter is $\alpha = 0.2$.

for several values of L, as in Fig. 4, but for $\alpha = 0.2$. We see a slight decrease of $P(s = 2)$ with L, but it is much less strong than for $\alpha = 0.05$. Similar results were observed at $\alpha = 0.15$, though this is already subcritical, which shows that $L = 200$ is still too small to see the asymptotic behavior.

Our lattice sizes are also too small for seeing the correct finite-size scaling of $P(s)$. In [20, 21] a conventional scaling ansatz for the differential distribution

$$dP(s)/ds = s^{1-\tau}\phi(s/L^D) \tag{3}$$

was made. It was found that the scaling function $\phi(x)$ is monotonic and that $D > 2$ for all values of α. Now it is easily seen that the last cannot be the correct asymptotic behavior for $\alpha < 1/4$. There, each site can discharge only a finite number of times during an avalanche. Hence $P(s)$ must be cut off at $s_{\max} \leq \text{const} \times L^2$ and thus $D \leq 2$. On the other hand, when looking at Fig. 6 we see that the cutoff $s_{\max}$ does increase faster than L^2, indicating that even our lattice sizes are not large enough to see the true asymptotics. (For very small α, $s_{\max}$ decreases with L, as seen in Fig. 4. I believe, however, that this is a transient effect whose detailed investigation is difficult due to the very slow convergence.)

Our sizes are large enough to refute also another claim of [21], namely that $P(s)$ shows ordinary finite-size scaling. According to the latter, it should be possible to superimpose the four curves in Fig. 6 by a simple shift. Trying this (e.g., by making a copy on a transparency), one sees that this does not work: with increasing L the cutoff in Fig. 6 becomes sharper. This is indeed expected from the fact that $s_{\max}$ approaches its upper bound with increasing L. I conjecture that the cutoff of $P(s)$ becomes a step function for $L \to \infty$, so that dP/ds develops a δ-function peak. Essentially the same behavior was found in a forest fire model [13] and in simulations of the Abelian sandpile model if sand was thrown only onto the central site [36, 37] or if averages were taken only over avalanches starting near the center of the lattice [37].

Finally, I show in Figs. 7 and 8 stress distributions similar to those shown in figs. 2 and 3 for periodic BC's. Only the case $\alpha > \alpha_c$ is shown. While the distribution of the stress difference between neighbors (shown in Fig. 8) is hardly changed when comparing to Fig. 3, the distribution of $F^* - F^i$ is very much different. Thus we see that global ordering is completely destroyed by large avalanches, while local ordering is still maintained.

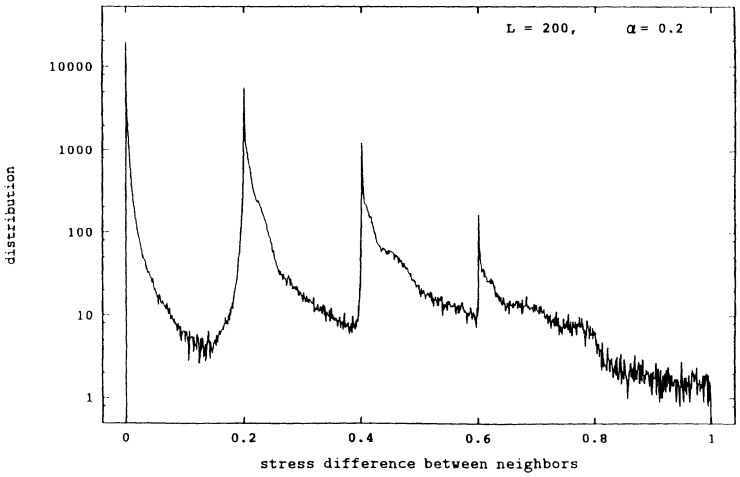

FIG. 8. Similar to Fig. 3, but for open boundary conditions and for the same parameters as in Fig. 7.

V. DISCUSSION

We have seen that the periodic boundary conditions and the much larger lattices studied in the present paper considerably modify the conclusions of [20]. Thus the use of an efficient algorithm can be crucial in obtaining the correct large-scale behavior of driven systems. Straightforward modifications of the present algorithm could be applied to other models with steadily and uniformly increasing stress [11, 16, 17, 19, 38].

In spite of the fact that stress is not conserved, I verified the existence of scale invariance for the specific model of [20]. This is surprising since scaling is tied to conservation in sandpile models [6]. The crucial difference between sandpile models and the present model seems to be that the latter is noiseless. It is driven uniformly (instead of stochastically), and in contrast to the model of [17, 19] it is not chaotic either.

The latter implies that the model can order into a temporally periodic state. It does this for low levels of stress conservation, but temporal ordering is prevented at high conservation. The transition between these states is not sharp but is clearly visible. An unusual feature is that the state with higher temporal order has lower spatial order and vice versa.

Ordering is also prevented by using nonperiodic boundary conditions. They are crucial for the scale invariant avalanches observed in [20], as neither the periodic nor the aperiodic states on a periodic lattice show any avalanche scaling. Seen from that point of view, the open boundary conditions used in [20] essentially play the role of an inhomogeneity. I conjecture that similar avalanche scaling should be observed with other sources of spatial inhomogeneity. A model with inhomogeneity provided by frozen randomness in the thresholds F^* was studied in [16]. Indeed, these authors observed scaling even in one dimension where the homogeneous model of [20] would be trivial. Since boundaries between tectonic plates are not homogeneous, this could mean that simulations with nonperiodic boundary conditions [20] of the present model are indeed more relevant for real earthquakes than simulations with periodic boundary conditions. But in view of the importance of inhomogeneities for the scaling, it would seem necessary to model them more carefully for the model to be realistic.

Note added. After submission, a paper by Socolar *et al.* appeared in Phys. Rev. E **47**, 2366 (1993), where it was also found that the model of [20] shows no scaling for periodic BC's. These authors claim that *all* initial configurations lead to trivial periodic states in this case, which seems to result from studying too small lattices.

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[35] Though this fit is much better than would be an exponential ansatz, it is far from perfect. The best fit seems to be obtained by a stretched exponential with exponent ≈ 0.2, together with a peak at $s = 1$. We do not have any theoretical argument in favor of a stretched exponential.

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