PHYSICAL REVIEW E VOLUME 49, NUMBER 3 MARCH 1994

Dependence of drag on a Galilean invariance-breaking parameter in lattice Boltzmann flow simulations

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(Received 27 August 1993)

We present two-dimensional lattice Boltzmann simulations of flow past a cylinder which show that the drag coefficient is proportional to the factor multiplying the convective term in the Navier-Stokes-like equations obtained for Boolean lattice gases and for certain lattice Boltzmann models. With the correct expression for the drag coefficient, we show that the results of previous Boolean lattice-gas studies of drag agree with experiment and with each other.

PACS number(s): 47.15.−x, 47.60.+i, 51.10.+y

INTRODUCTION

There have been several studies of drag on obstacles in lattice-gas simulations of fluid flow. Some have yielded good agreement with experimental results, others have not. We are able to reconcile the disparity in the results by demonstrating that the appropriate expression for the drag coefficient is different from what has often been previously used. When the correct expression for drag is used, both Boolean lattice gases and lattice Boltzmann models yield good agreement with experimental results. We also discuss some numerical issues which affect the drag.

We will first describe the lattice Boltzmann models used to investigate the effect on the drag of Galilean invariance breaking. We then present and discuss our results. Finally, we compare the present study with previous ones.

MODEL DESCRIPTION

Recently, lattice Boltzmann models have been proposed [1–5] which do not suffer from either the intrinsic noise or the velocity-dependent pressure of the Boolean Frisch-Hasslacher-Pomeau (FHP) lattice gases [6,7]. Lattice Boltzmann models are a natural tool for investigating what effects the departures of lattice-gas equations of motion from the usual Navier-Stokes have on drag coefficient. The reasons are that they are not noisy, allowing accurate determination of the drag, and that the Galilean invariance-breaking parameter can be changed without changing either the viscosity or the density of the lattice gas. They are thus much easier to handle than Boolean models.

We briefly review how the coarse-grained equations of motion are obtained. One begins by performing a multiscale expansion of the site populations, f_i, near their equilibrium values, f_i^0, using the explicit form of the equilibrium populations for a triangular lattice in two dimensions,

$$f_i^0 = \frac{\rho}{b} + \frac{\rho D}{c^2 b} c_{i\alpha} u_\alpha + \frac{4\rho}{bc^4} g Q_{i\alpha\beta} u_\alpha u_\beta \ , \qquad (1)$$

with $\Sigma_i f_i = \rho$, $\Sigma_i \mathbf{c}_i f_i = \rho \mathbf{u}$, b the number of nonzero $\mathbf{c}_i$ (here 6), ρ the density, g an arbitrary constant, and $Q_{i\alpha\beta} = c_{i\alpha} c_{i\beta} - c^2 \delta_{\alpha\beta}/2$. The $\mathbf{c}_i$ are the microscopic velocities, whose magnitude is $c = 1$. Note that if for g one substitutes $g(\rho) = (b-2\rho)(b-\rho)^{-1}$, one obtains exactly the equilibrium population of the Boolean FHP model [7], obtained by expanding the Fermi-Dirac expression for population in powers of u.

The resulting equation of motion for the momentum in the usual limit is

$$\frac{\partial(\rho \mathbf{u})}{\partial t} = -\rho g \mathbf{u} \cdot \nabla \mathbf{u} - \nabla p + \rho \nu \nabla^2 \mathbf{u}, \qquad (2)$$

with the pressure p given by

$$p = c^2 \rho/2 + g \rho u^2, \qquad (3)$$

and the kinematic viscosity ν determined by the choice of collision rules. For collisions of the time-irreversible form of the BGK (Bhatnagar-Gross-Krook) model [3], $\Omega_i = (f_i - f_i^0)/\tau$, one obtains $\nu = (\tau - 1/2)/4$.

Equations (2) and (3) differ from the Navier-Stokes equations in two ways. The factor g multiplying the convective term in general breaks Galilean invariance, and the pressure has an unphysical dependence on velocity. Both of these defects can be separately remedied [3,5]. In the lattice Boltzmann approach, one regains Galilean invariance by simply choosing $g = 1$. This is, however, not in general possible for Boolean lattice gases in which g is determined by ρ, as given above.

Chen *et al.* [5] showed that by adding rest particles with a suitable steady-state distribution and modifying the equilibrium distribution of moving particles, a pressure free from explicit velocity dependence is possible. Specifically, again including the parameter g, one takes

$$f_r^0 = d_r + g \gamma_r u^2 \qquad (4)$$

for the equilibrium distribution of the rest particles and

$$f_i^0 = d + \frac{\rho D}{c^2 b} c_{i\alpha} u_\alpha + \frac{4\rho}{bc^4} g c_{i\alpha} c_{i\beta} u_\alpha u_\beta + g \gamma u^2 \ , \qquad (5)$$

with the constants d, d_r, γ, and γ_r related through the conservation of mass and momentum; explicitly, $\rho = d_r + db$, and $0 = g(\gamma_r + 2\rho c^{-2} + b\gamma)$. Again, g is arbitrary, with

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the particular choice $g = 1$ yielding Galilean invariance. The diagonal part of the inviscid momentum flux tensor, the pressure, is then

$$f_i^0 u_\alpha u_\alpha = p = dbc^2/2 + g(\rho/2 + bc^2\gamma/2)u^2. \tag{6}$$

By choosing γ to make the term in parentheses vanish in (6), one obtains a pressure free from explicit velocity dependence. The coarse-grained equation of motion is again (2), though now just with $p = dbc^2/2$.

By simulating flow around a cylinder using these lattice Boltzmann equations, we can investigate the effects of varying g in (1) or in (4) and (5), thus varying g in the equation of motion (2) and in the expression for pressure. Since the pressure resulting from (4) and (5) contains no explicit velocity dependence, varying g in these expressions yields results uncontaminated by any effects introduced through the unphysical velocity dependence resulting from (1). Also, since the coarse-grained equations of motion of Boolean lattice gases differ from (2) only through g being determined by local density and ν being determined by the choice of Boolean collision rules, this study is relevant for Boolean lattice gases as well as lattice Boltzmann models.

RESULTS

To investigate dependence of the drag on the Galilean invariance-breaking parameter g, we ran simulations of the six-velocity models whose equilibrium distributions are given by (1) and of the seven-velocity models whose equilibrium distributions are given by (4) and (5) at Re$=UDg/\nu = 48$ (Re is the Reynolds number). Drag was calculated by summing the momentum change caused by bounce-back collision rules on all obstacle sites. The drag coefficient is thus

$$C_D = \frac{\Sigma \Delta u_x}{\frac{1}{2}\rho U^2 D \Delta t}\,, \tag{7}$$

with U the free-stream velocity, D the obstacle diameter, and the sum taken over the surface of the obstacle, and Δu_x the change of momentum in the direction of mean flow, x. The mass of a particle is taken to be unity.

Our principal result is that C_D is proportional to the parameter g in f_i^0 for both the six- and seven-velocity cases. Figure 1 shows the effect on C_D of varying g. One interpretation of Fig. 1 leading to a g-independent drag coefficient is that the correct expression is

$$C_D' = \frac{\Sigma \Delta g u_x}{\frac{1}{2}\rho (gU)^2 D \Delta t} = \frac{C_D}{g}. \tag{8}$$

Using gu instead of u is consistent with defining the Reynolds number from the ratio of typical magnitudes of the convective and viscous terms in (2); Re$= gUD/\nu$. In addition, using the dimensionless time formed with D/gU rather than D/U has been found to give much better results for the Strouhal number [8,9] and cylinder wake formation [8]. Another reason to regard gu as a quantity more relevant than u is that the equation of motion for $\rho g u$ is Galilean invariant, while the equation of motion for ρu is not. We wish to point out that expression (7) was proposed as the correct expression for calculating drag by Rivet [10].

The unphysical velocity dependence in the pressure which results from choosing (1) for f_i^0 is responsible for the difference between the dark and light symbols in Fig. 1. Note that the difference is generally small but becomes more pronounced with increasing g. The velocity-dependent term in the pressure thus has very little effect on the drag. The inset to Fig. 1 shows the deviation of C_D vs g from a straight line with slope set by C_D at $g = 1$. To 11% precision, the dependence is linear over an order of magnitude in g.

We use the same density in the denominator for the drag coefficient in both the six- and seven-velocity models. This is because the rest particles do not directly contribute to the drag since they cannot strike the obstacle. Also, the denominator of C_D may be interpreted as a pressure times a length (in two dimensions), and the pressure at $u = 0$ is identical for both the six- and seven-velocity models. For ρ in C_D we use density per

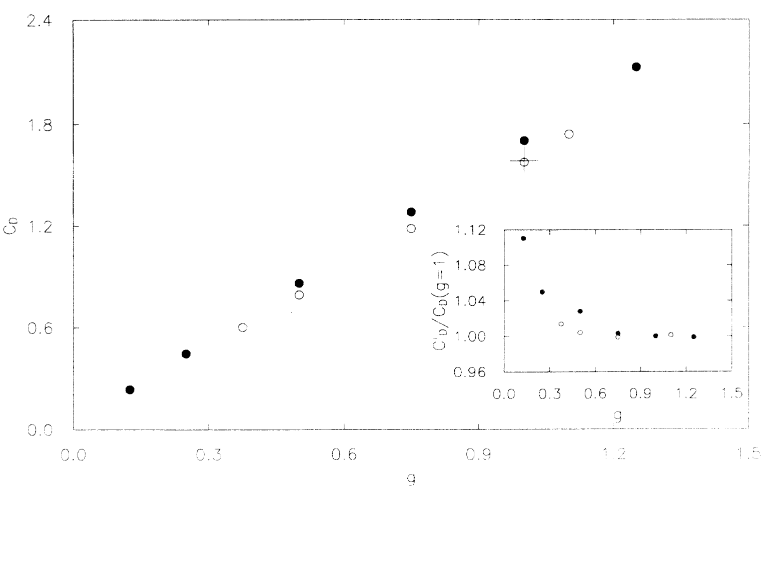

FIG. 1. Drag coefficient C_D, defined by (7) at Re$= gUD/\nu = 48$ versus g. Outlined symbols are results for seven-velocity model with f_i^0 and f_r^0 defined by (4) and (5), filled symbols are results for six-velocity models with f_i^0 given by (1). The cross represents the experimental result of Wieselberger [12]. Experimental results of Tritton [11] are smaller by roughly 8%. Inset shows deviation from a straight line with slope defined by the value at $g = 1$. C_D' is defined by (8).

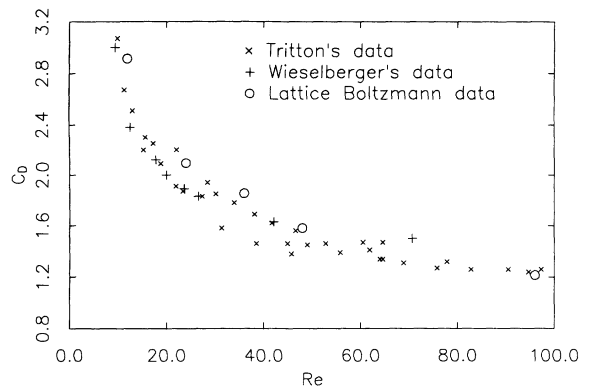

FIG. 2. C_D plotted against Re for lattice Boltzmann equations defined by (4) and (5) with $g = 1$. Also plotted are the experimental results of Tritton and Wieselberger.

unit area, not density per site (the two are related by a factor of $\frac{\sqrt{3}}{2}$).

The value at $g = 0$ is not accessible to simulation, since as $g \to 0$, it is impossible to keep Re= gUD/ν finite, due to the instability in the lattice Boltzmann equations at small ν. Simply setting $g = 0$ does not yield $C_D = 0$.

We also present in Fig. 2 a comparison of C_D measured with the seven-velocity model with $g = 1$ over a Re range 12–96 with the experimental results of Tritton[11] and Wieselberger [12] (Fig. 2). The lattice Boltzmann data agree with the experimental results nearly as well as the results of the experimental studies agree with each other, that is, to 15%.

COMPARISON WITH BOOLEAN STUDIES OF DRAG

There have been several previous studies of drag on cylinders made with Boolean lattice gases [8,9,13-15]. We show that after accounting for finite-size effects, these results also appear to be consistent with a g dependence of C_D. We also discuss some potential sources of error in lattice-gas simulations.

We first address the effect of the ratio Λ of cylinder diameter to system width on C_D. The inset of Fig. 3 shows our data, taken with periodic boundary conditions in the direction perpendicular to the flow. Clearly $\Lambda < 0.1$ is necessary for the drag to be independent of Λ to 5%.

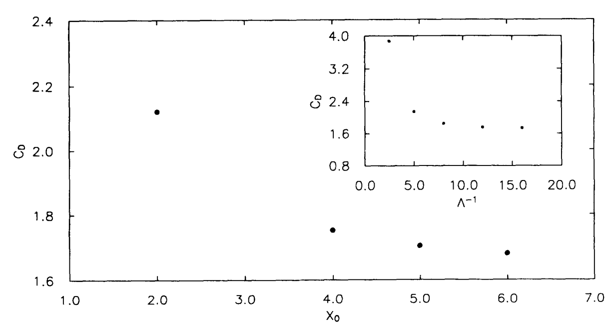

FIG. 3. C_D plotted against distance of obstacle center from inlet, X_0. Data are for f_i^0 given by (1), and at Re= 48, with $g = 1$. The ratio Λ of obstacle diameter to system width is 1/12. Inset: C_D plotted against the ratio Λ^{-1} of system width to obstacle diameter. Data are as for the main figure, but with $X_0 = 4D$.

Some workers have used no-slip instead of periodic boundaries, with parabolic inflow and outflow velocity profiles instead of flat ones. The effect this choice has on the drag depends on the aspect ratio. At $\Lambda = 0.2$, we find $C_D(\text{no-slip})/C_D(\text{periodic}) = 0.743$, while at $\Lambda = 0.1$, $C_D(\text{no-slip})/C_D(\text{periodic}) = 0.824$ with Re defined from the peak of the Poiseuille velocity profile. Thus drag obtained from simulations run with no-slip boundary conditions is not identical to drag obtained from periodic boundary conditions at reasonable aspect ratios.

Another boundary condition which may affect the drag is the distance of the center of the obstacle from the inflow, X_0. Figure 3 shows our data which make clear that X_0 should be at least $5D$ for drag to be independent of X_0 to 2%. Of course, if the outflow is too close to the obstacle, C_D will also be affected. We only note that an obstacle center to outflow distance of $12D$ is sufficient to prevent such contamination.

Lastly, there is the possibility of a Knudsen-layer effect on the drag. This effect will be strongest for small Re, since there the ratio of diameter to mean free path (proportional to D/ν) will be smallest. We found a negligible difference between $D/\nu = 320$ and $D/\nu = 160$ at Re= 12, suggesting that even with the small obstacle sizes which lattice Boltzmann equations allow, Knudsen-layer effects are negligible.

We now consider previous studies ordered by increasing values of $g(\rho)$. We consider first the results of Hayot and Wagner [9], obtained for Re= 100 at $g(\rho) = 0.25$. They found $C_D = 0.575$ at $\Lambda = 0.1$ and $X_0 = 4D$, using the density per site in the definition of the drag coefficient instead of the density per unit area. Correcting for the effect of the small X_0 and using the density per area in the drag, one finds an amended $C_D = 0.408$ which yields for the value to be compared with experiment $C_D/g = 1.63$. This value is 15% above Wieselberger's experimental value. The agreement with experiment is thus much better than what was concluded in [9]. Moreover, one must conclude that the estimation of drag from the velocity profile in [9] is problematic because of the noise level of the Boolean simulation and possibly because the method requires a channel longer than is computationally feasible to be accurate.

Hayot and Lakshmi [8] found for Re= 30 $C_D = 1.24$, at $\Lambda = 0.31$, and for Re= 60 $C_D = 1.06$ at $\Lambda = 0.36$, both at $g(\rho) = 0.25$ with periodic boundary conditions. After correcting for large aspect ratio (using the data of Fig. 3) and the use of density per site, the amended values are, for Re= 30, $C_D = 0.50$ and for Re= 60, $C_D = 0.44$. Again, the amended values are consistent with the dependence suggested by Fig. 1; that is, $C_D' = C_D/g$ gives a much better agreement with experiment.

We turn to the studies of Duarte and Brosa [13] , of Kohring [14], and of Vogeler and Wolf-Gladrow [15], who measured drag on circular and hexagonal obstacles in the range 5 <Re< 80, with $\Lambda = 0.2$ and Poiseuille in-

TABLE I. Summary of comparison with Boolean studies. The crudeness of the corrections (described in the text) allows only a relative comparison between the agreement of C_D and C'_D with experiment. The result from [13] is not corrected for X_0. Experimental C_D is from Wieselberger [12].

Reynolds number	Raw C_D	Geometry corrected C_D	Geometry corrected C'_D	Experimental C_D	reference
100	0.575	0.408	1.63	1.43	[9]
46.3	1.34	1.27	1.59	1.61	[13]
30	1.24	0.50	2.0	1.79	[8]

flow. They worked with $g(\rho) = 0.8$. Thus the effect of $g(\rho)$ on the results of these workers would be much less pronounced than for the preceding studies. The effects of large aspect ratio and Poiseuille inflow nearly cancel for these studies—the correction to free-stream drag results from these two effects being 0.95. Without knowing the value of X_0 used in all their simulations, it is not possible to explicitly compare their results with those of the present study. We note only that all three of these studies did not include $g(\rho)$ in their definitions of C_D or Re. The changes caused by including g in both quantities contribute in opposite ways to the value of C_D and are of similar magnitude over the range of Re investigated. Thus the effects of the two omissions nearly cancel each other, which explains why the drag results of these authors agree with experiment. Table I summarizes the comparison with Boolean simulations.

Both the simulation results presented in this study and previous Boolean drag data support using expression (8) for lattice-gas studies which break Galilean invariance. When expression (8) is used for C_D, results in the literature are seen to be consistent with each other and with experiment. Furthermore, the effect on C_D of the unphysical velocity dependence of the pressure is small.

ACKNOWLEDGMENTS

The author thanks F. Hayot for invaluable guidance, and D. d'Humières for helpful discussion. This work was supported by the Department of the Navy, Office of Naval Research, under Grant No. N00014-92-J-1271, and benefitted from computer time at the Ohio Supercomputer Center.

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