Delay-induced instabilities in nonlinear feedback systems

W. Wischert, A. Wunderlin, and A. Pelster
Institut für Theoretische Physik und Synergetik, Universität Stuttgart, Pfaffenwaldring 57/4, D-70550 Stuttgart, Germany

M. Olivier and J. Groslambert
Laboratoire de Physique et Métrologie des Oscillateurs du Centre National de La Recherche Scientifique, 32, avenue de l'Observatoire, F-25000 Besançon, France
(Received 9 June 1993)

Extending general methods developed in synergetics to construct order-parameter equations we work out a systematic elimination procedure for the stable modes in nonlinear delay systems. It will be shown that in the vicinity of an instability the dynamical behavior of the infinite-dimensional delay system is approximately governed by a low-dimensional set of order-parameter equations which turn out to be of the form of ordinary differential equations, i.e., they no longer contain memory terms. The general formalism will then be compared with experimental data obtained from the nonlinear operation of a phase-locked loop where the finite propagation time of the signal in the feedback loop is taken into account.

PACS number(s): 05.45.+b, 02.30.Ks

INTRODUCTION

In recent years much effort has been devoted to understand the origin of dynamic instabilities in nonlinear systems which are induced by the finite propagation time of signals in feedback loops. Although delay effects leading to an oscillatory behavior have been well known for a long time, especially in radio engineering sciences, it was only in 1979 that it was emphasized by Ikeda [1] in the domain of optical bistable devices and Kislov, Zalogin, and Myasin [2] in the engineering sciences that delay-induced instabilities can lead to a more complex behavior. Meanwhile numerous experimental and theoretical studies mainly on acousto-optic and electro-optic bistable devices have clearly demonstrated that instabilities leading to a chaotic behavior can easily be induced in a broad class of nonlinear delay systems. Apart from a period-doubling route to chaos quasiperiodicity, intermittency, and locking behavior have been observed [3–18]. In the chaotic domain studies on the statistical properties of nonlinear delay equations have demonstrated that the dimension of a delay-induced chaotic attractor is directly proportional to the time delay involved [19–21]. This observation seems to be independent of the precise form of the system under investigation. The resulting possibility to generate high-dimensional chaotic attractors by simply increasing the delay time makes nonlinear delay systems promising candidates to contribute to the still unresolved relationship between high-dimensional chaotic attractors and the phenomena of turbulence [22–24].

Delay-induced instabilities also provide a powerful tool in the investigation of irregular behavior observed in other scientific disciplines. In physiological control systems the Mackey-Glass model, a scalar nonlinear delay-differential equation, has been applied successfully to describe anomalies in the regeneration of white blood cells due to the finite propagation time of chemosensitive substances in the blood circulation [25]. Another example from physiology which allowed for the first time a systematic investigation of the influence of time delays in human control systems through a noninvasive technique is the human pupil light reflex exhibiting oscillatory or more complicated behavior [26,27]. More recently, delay effects have also found applications in economic systems [28] and the cognitive sciences [29–31]. In order to analyze instabilities attributed to time delays it appears of great value to employ the concepts of synergetics which have been developed over the past decades to describe the origin of processes of self-organization in nature [32,33]. These methods are based on the fact that near dynamical instabilities self-organized pattern formation can be described by a small number of modes which contribute to the macroscopically observed structure formation processes. This important result is due to the fact that in the vicinity of instabilities the high-dimensional set of nonlinear evolution equations modeling a complex system can be reduced to a low-dimensional set of order-parameter equations describing the evolving pattern formation on a macroscopic time scale. Often this can already be achieved by an adiabatic elimination procedure [32–35]. However, when dealing with delay effects, an application of this method has to be performed with special care. Due to the infinite-dimensional character of a delay system a naive application of the adiabatic elimination procedure can no longer be used, a fact which has already been emphasized in [36–38] by comparing the dynamics of a simple nonlinear delay-differential equation with its discrete map obtained from a naive adiabatic approximation. The reason is that the state vector characterizing a nonlinear delay system evolves in a finite-dimensional state space whereas the dynamics has to be formulated rigorously in an infinite-dimensional extended state space.

In this article we point out that the concepts of synergetics can still be applied for a general class of nonlinear delay systems when the evolution equations are formulated in an appropriate infinite-dimensional function space. Following ideas developed by Hale [39], we are able to generalize the concept of order parameters in such a way that the adiabatic elimination procedure can systematically be applied to nonlinear delay systems in a more general setting.

In order to compare our results with experimental data we have analyzed the dynamics of a first-order phase-locked loop where delay lines have been introduced in the feedback loop. In varying the time delay the loop undergoes a hierarchy of instabilities from an oscillatory to a chaotic motion of the phase error signal. When the loop exceeds its synchronous operation range a cycle slipping motion of the phase error occurs. Quite surprisingly this behavior is associated with a $1/f$ spectrum. Apart from the practical interest of these results we note the close relation of our system to acousto-optic bistable devices which among others played the role of a pacemaker in the physical studies of delay-induced instabilities. Especially we emphasize the rich dynamical behavior of our system which is modeled by an elementary evolution equation for the nonlinear operation of the loop. We therefore consider our control system as a suitable candidate to examine delay-induced instabilities theoretically as well as experimentally in great detail.

In order to keep this article self-contained we divide it into two parts. In the first part we elaborate the mathematical framework. As a main result we formulate a generalized elimination procedure for nonlinear delay systems which allows us to systematically construct order-parameter equations. These order-parameter equations describe the macroscopic behavior of a delay system close to instabilities. We shall proceed as follows. In Sec. I A we introduce basic properties of delay-differential equations by formulating their dynamics in an extended infinite-dimensional function space. We then discuss in Sec. I B the eigenvalue problem of the linearized system. The results are used in Sec. I C to present a systematic procedure which allows us to derive order-parameter equations by the application of fundamental ideas which have been proposed in synergetics.

In the second part of the article the formalism is applied to a first-order phase-locked loop (PLL) where the finite propagation time of signals in the feedback loop is taken into account. In Sec. II A we derive the scalar nonlinear delay-differential equation modeling this control system with time delay. The model equation then serves as a concrete example to demonstrate how a correct elimination procedure has to be performed (Sec. II B). In Sec. II C the experimentally observed delay-induced instabilities of the PLL are compared with theoretical results. A conclusion is presented in Sec. II D.

I. THEORETICAL CONSIDERATIONS

A. Basic properties of delay-differential equations

In this section we summarize important notions concerning delay-differential equations. Generalizing a given finite-dimensional state space to an infinite-dimensional extended state space we succeed in embedding delay-differential equations in the context of functional differential equations. This procedure is based on ideas of Hale [39].

1. Definition of the problem

The dynamical behavior of a nonlinear system is generally described by a state vector $\mathbf{q}(t)$ in an n-dimensional state space Γ. Its components are assumed to be related to experimentally accessible physical quantities which completely characterize the underlying dynamics of the system. In the situation where the finite propagation time of the signal in a feedback loop is taken into account, the evolution of the state vector $\mathbf{q}(t)$ becomes nonlocal in time. Such delay effects are often formulated by an autonomous delay-differential equation which reads in its simplest form

$$\dot{\mathbf{q}}(t)=\mathbf{N}(\mathbf{q}(t),\mathbf{q}(t-\tau),\{\sigma_i\}) \ . \tag{1}$$

Here $\mathbf{N}$ denotes a nonlinear vector field which depends on the vector $\mathbf{q}$ at time t as well as time $t-\tau$, where τ stands for the time delay. Additionally the set of parameters $\{\sigma_i\}$ serves for measuring external influences on the system. For the time being we assume that these control parameters $\{\sigma_i\}$ are kept fixed and we can omit them in our notation.

If we are interested in solutions of Eq. (1) at times $t\geq 0$ it becomes necessary to define the state vector $\mathbf{q}(t)$ in the entire interval $[-\tau,0]$. To this end one has to consider Eq. (1) together with the initial condition

$$\mathbf{q}(\Theta)=\mathbf{g}(\Theta) \ , \quad -\tau\leq\Theta\leq 0 \ , \tag{2}$$

where $\mathbf{g}$ is a given continuous initial vector-valued function in a suitable function space $\mathcal{C}$.

2. Evolution in an extended state space

From a more formal point of view the initial value problem given by Eqs. (1) and (2) represents an unsatisfactory situation in the sense that the vector-valued function $\mathbf{g}$ is mapped onto a trajectory in the n-dimensional state space Γ. Such a mapping from an infinite-dimensional function space onto a finite-dimensional vector space is accompanied by a considerable loss of information. Different initial vector-valued functions can lead to crossings of corresponding trajectories in Γ. This means that the uniqueness of solutions cannot be assured when we restrict our attention to the state space Γ.

To get rid of this problem we have to reformulate our description of the delay system. This can be performed by extending the finite-dimensional state space Γ to an infinite-dimensional function space $\mathcal{C}$ where the initial vector-valued function $\mathbf{g}$ is defined. This point of view enables us to describe the state of the delay system at time t by an extended state vector $\mathbf{q}_t\in\mathcal{C}$ and to refer to $\mathcal{C}$ as the extended state space. We therefore construct $\mathbf{q}_t$ by adjusting the trajectory $\mathbf{q}(t)\in\Gamma$ in the interval $[t-\tau,t]$ according to the prescription (compare Fig. 1)

$$\mathbf{q}_t(\Theta)=\mathbf{q}(t+\Theta) \ , \quad -\tau\leq\Theta\leq 0 \ . \tag{3}$$

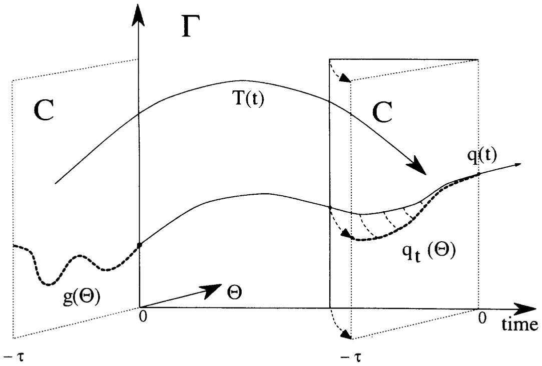

FIG. 1. Folding mechanism relating a history of $\mathbf{q}(t) \in \Gamma$ to a single element $\mathbf{q}_t \in \mathcal{C}$.

The dynamics of the delay system, however, can now be satisfactorily described in $\mathcal{C}$. To define the mapping from the initial state vector $\mathbf{g}$ to the extended state vector $\mathbf{q}_t$ at time t we introduce the nonlinear solution operator $\mathcal{T}(t)$ by the relation

$$\mathbf{q}_t(\Theta)=(\mathcal{T}(t)\mathbf{g})(\Theta)\ , \quad -\tau\le\Theta\le 0\ . \tag{4}$$

The uniqueness of this mapping implies that $\mathcal{T}(t)$ has the usual properties of a semigroup, that is,

$$\mathcal{T}(t+s)=\mathcal{T}(t)\mathcal{T}(s)\ , \quad t,s\ge 0\ , \quad \text{and } \mathcal{T}(0)=\mathcal{I}\ . \tag{5}$$

$\mathcal{I}$ denotes the identity operator in $\mathcal{C}$.

In the following we intend to reformulate the initial value problem given by Eqs. (1) and (2) in the finite-dimensional state space Γ by deducing an equivalent initial value problem in the infinite-dimensional extended state space $\mathcal{C}$. To construct an evolution equation for $\mathbf{q}_t$ we formally differentiate Eq. (4) with respect to time t,

$$\frac{d}{dt}\mathbf{q}_t(\Theta)=(\mathcal{A}\mathbf{q}_t)(\Theta)\ , \quad -\tau\le\Theta\le 0\ . \tag{6}$$

Here $\mathcal{A}$ stands for the infinitesimal generator corresponding to the solution operator $\mathcal{T}(t)$ and is defined by

$$(\mathcal{A}\mathbf{q}_t)(\Theta)=\lim_{\epsilon\to 0}\frac{1}{\epsilon}[(\mathcal{T}(\epsilon)\mathbf{q}_t)(\Theta)-\mathbf{q}_t(\Theta)]\ . \tag{7}$$

To perform the limit one has to take into account that the action of $\mathcal{T}(\epsilon)$ on $\mathbf{q}_t$ is distinct in different regimes. Indeed we find, from Eqs. (1)–(4),

$$(\mathcal{T}(\epsilon)\mathbf{q}_t)(\Theta)=\begin{cases}\mathbf{q}(t+\Theta+\epsilon)\ , & t+\Theta+\epsilon\le 0\\ \mathbf{q}(0)+\int_0^{t+\Theta+\epsilon}\mathbf{N}(\mathbf{q}(s),\mathbf{q}(s-\tau))ds\ , & \\ & t+\Theta+\epsilon>0\ .\end{cases} \tag{8}$$

3. *Representation of the generator*

The infinitesimal generator $\mathcal{A}$ can be explicitly constructed from the nonlinear vector field $\mathbf{N}$ in Eq. (1) in the following way. In a first step we assume that $\mathbf{N}$ can be expanded in a Taylor series around a suitable (stationary) reference point $\mathbf{q}_{\text{stat}}$ which can be chosen without loss of generality as $\mathbf{q}_{\text{stat}}=\mathbf{0}$. Using index notation, the ith component of a typical term of second order in that series reads

$$N_{ijk}^{(2)}q_j(t)q_k(t-\tau)\ , \tag{9}$$

where Einstein's summation convention has been used. In a second step we have to reexpress the state vectors $\mathbf{q}(t)$ and $\mathbf{q}(t-\tau)$ appearing in the Taylor expansion in terms of the extended state vector $\mathbf{q}_t$ by inverting Eq. (3),

$$\mathbf{q}(t)=\int_{-\tau}^{0}d\Theta\,\delta(\Theta)\mathbf{q}_t(\Theta)$$

and (10)

$$\mathbf{q}(t-\tau)=\int_{-\tau}^{0}d\Theta\,\delta(\Theta+\tau)\mathbf{q}_t(\Theta)\ .$$

Applying this procedure to each term in the series and collecting terms of the same order in the extended state vector $\mathbf{q}_t$, the nonlinear vector field $\mathbf{N}(\mathbf{q}(t),\mathbf{q}(t-\tau))$ becomes a functional of $\mathbf{q}_t$ with the explicit representation

$$\mathcal{N}[\mathbf{q}_t(\cdot)]=\sum_{k=1}\mathcal{N}^{(k)}[\mathbf{q}_t(\cdot)]\ , \tag{11}$$

with

$$\mathcal{N}_i^{(k)}[\mathbf{q}_t(\cdot)]=\int_{-\tau}^{0}d\Theta_1\cdots\int_{-\tau}^{0}d\Theta_k\,\omega_{ij_1,\ldots,j_k}^{(k)}(\Theta_1,\ldots,\Theta_k)q_{t,j_1}(\Theta_1)\cdots q_{t,j_k}(\Theta_k)\ . \tag{12}$$

In the case of delay-differential equations the matrix-valued densities $w^{(k)}$ turn out to be linear combinations of δ functions as they were introduced in Eq. (10). We note, however, that Eq. (11) could also describe more general delay systems. Matrix-valued densities $w^{(k)}$ which are piecewise continuous functions of their arguments lead, for example, to integro-differential equations which model systems with distributed delay. This observation allows us to perform our following considerations in the context of the more general class of functional differential equations.

With our reformulation Eq. (8) is transformed in a way that it solely contains elements of the extended state space $\mathcal{C}$,

$$(\mathcal{T}(\epsilon)\mathbf{q}_t)(\Theta)=\begin{cases}\mathbf{q}(t+\Theta+\epsilon)\ , & t+\Theta+\epsilon\le 0\\ \mathbf{q}(t)+\int_0^{t+\Theta+\epsilon}\mathcal{N}[\mathbf{q}_s(\cdot)]ds\ , & \\ & t+\Theta+\epsilon>0\ .\end{cases} \tag{13}$$

Inserting this result in Eq. (7) and performing the limit $\epsilon\to 0$ we finally obtain an explicit representation of the infinitesimal generator

$$(\mathcal{A}\mathbf{q}_t)(\Theta)=\begin{cases}\dfrac{d}{d\Theta}\mathbf{q}_t(\Theta)\ , & -\tau\leq\Theta<0\\ \mathcal{N}[\mathbf{q}_t(\cdot)]\ , & \Theta=0\ .\end{cases} \tag{14}$$

B. The linear problem

To analyze the dynamical behavior of the nonlinear functional differential equations (6), (11), and (14), a thorough discussion of the corresponding linearized problem appears to be of great value. Therefore we shall extend well-established concepts concerning the linear stability analysis of ordinary differential equations to linear functional differential equations. We elaborate the analogy as closely as possible so that our procedure contains the results in the theory of ordinary differential equations in the limiting case of a vanishing delay time τ.

1. Linearization

We start to formulate a linear stability analysis by assuming that the nonlinear system possesses a reference state $\mathbf{q}_{\text{stat}}\in\Gamma$. In what follows we restrict ourselves to the simplest situation where $\mathbf{q}_{\text{stat}}$ is independent of time t and obeys according to Eq. (1)

$$\mathbf{N}(\mathbf{q}_{\text{stat}},\mathbf{q}_{\text{stat}})=0\ . \tag{15}$$

In consistency with Eq. (3) the reference state $\mathbf{q}_{\text{stat}}$ can also be defined in the extended state space $\mathcal{C}$ by a vector-valued constant. Additionally we read off from Eqs. (6) and (14) that

$$\mathcal{N}[\mathbf{q}_{\text{stat}}]=0\ . \tag{16}$$

In the next step we analyze the behavior of the system in a certain neighborhood of the chosen reference state by considering small deviations $\tilde{\mathbf{q}}_t$ from $\mathbf{q}_{\text{stat}}$. Inserting

$$\mathbf{q}_t(\Theta)=\mathbf{q}_{\text{stat}}+\tilde{\mathbf{q}}_t(\Theta)\ ,\quad -\tau\leq\Theta\leq 0 \tag{17}$$

into Eqs. (6), (11), and (14) and dropping the tilde we obtain in the linear approximation

$$\frac{d}{dt}\mathbf{q}_t(\Theta)=(\mathcal{A}_L\mathbf{q}_t)(\Theta)\ ,\quad -\tau\leq\Theta\leq 0\ , \tag{18}$$

$$(\mathcal{A}_L\mathbf{q}_t)(\Theta)=\begin{cases}\dfrac{d}{d\Theta}\mathbf{q}_t(\Theta)\ , & -\tau\leq\Theta<0\\ \mathcal{L}[\mathbf{q}_t(\cdot)]\ , & \Theta=0\ ,\end{cases} \tag{19}$$

$$\mathcal{L}[\mathbf{q}_t(\cdot)]=\int_{-\tau}^{0}d\Theta\, w(\Theta)\mathbf{q}_t(\Theta)\ . \tag{20}$$

$\mathcal{A}_L$ denotes the infinitesimal generator restricted to the linear case and the vector-valued functional $\mathcal{L}$ is the linear approximation to $\mathcal{N}$ in the neighborhood of $\mathbf{q}_{\text{stat}}$. Its matrix-valued density w is given as a functional derivative of $\mathcal{N}$ evaluated at the reference state

$$w(\Theta)=\left.\frac{\delta\mathcal{N}[\mathbf{q}_t(\cdot)]}{\delta\mathbf{q}_t(\Theta)}\right|_{\mathbf{q}_t=\mathbf{q}_{\text{stat}}}\ ,\quad -\tau\leq\Theta\leq 0\ . \tag{21}$$

We note that Eq. (21) can be obtained from the explicit representation of the nonlinear functional $\mathcal{N}$ introduced in Eqs. (11) and (12).

2. Eigenvalue problem

The knowledge of the infinitesimal generator $\mathcal{A}_L$ allows us to construct vector-valued right-hand eigenfunctions $\boldsymbol{\phi}^\lambda$ of $\mathcal{A}_L$ in the extended state space $\mathcal{C}$. Inserting the ansatz

$$\mathbf{q}_t(\Theta)=\boldsymbol{\phi}^\lambda(\Theta)e^{\lambda t}\ ,\quad -\tau\leq\Theta\leq 0 \tag{22}$$

into the linear evolution equation (18) we obtain the eigenvalue problem for the infinitesimal generator $\mathcal{A}_L$,

$$(\mathcal{A}_L\boldsymbol{\phi}^\lambda)(\Theta)=\lambda\boldsymbol{\phi}^\lambda(\Theta)\ ,\quad -\tau\leq\Theta\leq 0\ . \tag{23}$$

Evaluating this eigenvalue problem in the interval $[-\tau,0)$ and taking into account the definition of the infinitesimal generator in Eq. (19) we obtain for the right-hand eigenfunctions $\boldsymbol{\phi}^\lambda$ the solution

$$\boldsymbol{\phi}^\lambda(\Theta)=\boldsymbol{\phi}^\lambda(0)e^{\lambda\Theta}\ ,\quad -\tau\leq\Theta<0 \tag{24}$$

for arbitrary values of λ. The eigenvalues are determined when we wish to guarantee that Eq. (24) satisfies the eigenvalue problem Eq. (23) for the single value $\Theta=0$. Applying again Eq. (19) we gain a transcendental eigenvalue equation of a matrix $W(\lambda)$,

$$W(\lambda)\boldsymbol{\phi}^\lambda(0)=\lambda\boldsymbol{\phi}^\lambda(0)\ , \tag{25}$$

where $W(\lambda)$ is given by

$$W(\lambda)=\int_{-\tau}^{0}d\Theta\, w(\Theta)e^{\lambda\Theta}\ . \tag{26}$$

The requirement for nontrivial right-hand eigenvectors $\boldsymbol{\phi}^\lambda(0)$ then leads to a transcendantal characteristic equation for the eigenvalues λ,

$$\det[W(\lambda)-\lambda I]=0\ . \tag{27}$$

Here I represents the unity matrix.

For a justification of the nonlinear treatment of functional differential equations (see Sec. I C) we need some information about the spectrum of the infinitesimal generator $\mathcal{A}_L$. We summarize without proof important properties of the transcendental characteristic equation (27).

(1) $\mathcal{A}_L$ possesses a pure point spectrum which consists of an infinite number of eigenvalues.

(2) In each finite strip parallel to the imaginary axes only a finite set of eigenvalues is located.

(3) The eigenvalues accumulate at $\text{Re}(\lambda)\rightarrow-\infty$.

(4) For all values of the control parameters the real part of the eigenvalues is bounded from above.

A more detailed discussion can be found in [39].

3. Adjoint eigenvalue problem

In general, the infinitesimal generator $\mathcal{A}_L$ is not self-adjoint. Therefore it becomes necessary to consider in addition to the eigenvalue problem Eq. (23) the corresponding adjoint one defined in another extended state space $\mathcal{C}^\dagger$ dual to $\mathcal{C}$. It turns out that the canonical bilinear form for ordinary systems is not appropriate to relate both state spaces in the case of delay systems. To properly take into account memory effects a modified bilinear form has to be introduced. This can be motivated from the Fredholm alternative which considers the inhomogeneous version of Eq. (23),

$$([\mathcal{A}_L-\lambda\mathcal{I}]\zeta^\lambda)(\Theta)=\chi(\Theta)\ ,\quad -\tau\le\Theta\le 0\ . \tag{28}$$

Using the definition of the infinitesimal generator $\mathcal{A}_L$ from Eq. (19) we obtain in the case $\Theta\in[-\tau,0)$ as a solution of (28)

$$\zeta^\lambda(\Theta)=\zeta^\lambda(0)e^{\lambda\Theta}+\int_0^\Theta ds\, e^{\lambda(\Theta-s)}\chi(s)\ . \tag{29}$$

In the case $\Theta=0$ Eq. (28) correspondingly reduces to

$$\int_{-\tau}^0 d\Theta\, w(\Theta)\zeta^\lambda(\Theta)-\lambda\zeta^\lambda(0)=\chi(0)\ . \tag{30}$$

Inserting Eq. (29) in (30) we obtain the solution condition

$$[W(\lambda)-\lambda I]\zeta^\lambda(0)=\chi(0)-\int_{-\tau}^0 d\Theta\int_0^\Theta ds\, e^{-\lambda(s-\Theta)}\times w(\Theta)\chi(s)\ . \tag{31}$$

We read off from Eqs. (29) and (31) that only in the case when the transcendental characteristic equation (27) is not fulfilled, does the inhomogeneous Eq. (28) have the unique solution

$$\begin{aligned}\zeta^\lambda(\Theta)&=([\mathcal{A}_L-\lambda\mathcal{I}]^{-1}\chi)(\Theta)\\ &=[W(\lambda)-\lambda I]^{-1}\\ &\quad\times\left[\chi(0)-\int_{-\tau}^0 d\Theta\int_0^\Theta ds\, e^{-\lambda(s-\Theta)}w(\Theta)\chi(s)\right]\\ &\quad\times e^{\lambda\theta}+\int_0^\Theta ds\, e^{\lambda(\Theta-s)}\chi(s)\ .\end{aligned} \tag{32}$$

Additionally the solution condition Eq. (31) suggests how to introduce both the dual state space $\mathcal{C}^\dagger$ and a bilinear form. We assume that $\mathcal{C}^\dagger$ consists of n-dimensional vector-valued functions defined on the interval $[0,\tau]$ and that the bilinear form is given by

$$(\psi^\dagger,\phi)=\langle\psi^\dagger(0)\ ,\phi(0)\rangle-\int_{-\tau}^0 d\Theta ds\langle\psi^\dagger(s-\Theta),w(\Theta)\phi(s)\rangle \tag{33}$$

for all $\phi\in\mathcal{C}$ and $\psi^\dagger\in\mathcal{C}^\dagger$, where $\langle\cdots,\cdots\rangle$ denotes the usual canonical scalar product. It should be remarked that by this choice each delay system induces its own bilinear form which in general does not fulfill the properties of a scalar product. These are only guaranteed in the very special case when the matrix-valued densities $w(\Theta)$ are symmetric and positive definite for all values of $\Theta\in[-\tau,0]$. We emphasize that in the limit of vanishing delay ($\tau\to0$) the canonical scalar product is recovered from Eq. (33) irrespective of the underlying dynamics.

With these definitions Eq. (31) can be rewritten in a concise way,

$$[W_{ij}(\lambda)-\lambda\delta_{ij}]\zeta_j^\lambda=(\alpha_i^\dagger,\chi)\ . \tag{34}$$

The vector-valued functions $\alpha_i^\dagger$ which are defined in $\mathcal{C}^\dagger$ are given by

$$\alpha_i^\dagger(s)=e^{-\lambda s}\mathbf{e}_i^\dagger\ ,\quad s\in[0,\tau]\ , \tag{35}$$

with $\mathbf{e}_i^\dagger$ the ith row of the unity matrix I. The bilinear form thus opens the possibility to enunciate Fredholm's solvability conditions formally in the same way as they can be stated when a scalar product exists.

In the following the bilinear form Eq. (33) will be applied to describe the evolution of the linearized delay system in the dual extended state space $\mathcal{C}^\dagger$. The dynamics of the delay system can be explored either by a state vector $\mathbf{q}_t\in\mathcal{C}$ or equivalently by an adjoint state vector $\mathbf{q}_t^\dagger\in\mathcal{C}^\dagger$. Whereas the elements $\mathbf{q}_t$ describe a forward evolution for $t\ge0$ the dual state vector $\mathbf{q}_t^\dagger$ has to be considered as the corresponding backward evolution for $t\le0$. Both can be related via

$$\mathbf{q}_t^\dagger(s)=\mathbf{q}_{-t}^T(-s)\ ,\quad 0\le s\le\tau\ , \tag{36}$$

where T denotes the operation of transposition. The requirement of a time-independent bilinear form according to

$$\frac{d}{dt}(\mathbf{q}_t^\dagger,\mathbf{q}_t)=0 \tag{37}$$

leads to the following evolution equation in $\mathcal{C}^\dagger$:

$$\frac{d}{dt}\mathbf{q}_t^\dagger(s)=-(\mathcal{A}_L^\dagger\mathbf{q}_t^\dagger)(s),\quad 0\le s\le\tau\ . \tag{38}$$

The explicit form of the dual infinitesimal generator $\mathcal{A}_L^\dagger$ is obtained from the bilinear form introduced above. The procedure is formally analogous to cases when a scalar product exists,

$$(\mathbf{q}_t^\dagger,\mathcal{A}_L\mathbf{q}_t)=(\mathcal{A}_L^\dagger\mathbf{q}_t^\dagger,\mathbf{q}_t)\ . \tag{39}$$

Indeed, the application of Eqs. (19) and (20) on Eqs. (33) and (39) combined with a partial integration leads to the following explicit representation of the dual infinitesimal generator $\mathcal{A}_L^\dagger$:

$$(\mathcal{A}_L^\dagger\mathbf{q}_t^\dagger)(s)=\begin{cases}-\dfrac{d}{ds}\mathbf{q}_t^\dagger(s)\ , & 0<s\le\tau\\ \mathcal{L}^\dagger[\mathbf{q}_t^\dagger(\cdot)]\ , & s=0\ ,\end{cases} \tag{40}$$

$$\mathcal{L}^\dagger[\mathbf{q}_t^\dagger(\cdot)]=\int_0^\tau ds\,\mathbf{q}_t^\dagger(s)w(-s)\ . \tag{41}$$

We are now in a position to solve the dual eigenvalue problem

$$(\mathcal{A}_L^\dagger\psi^{\dagger\lambda})(s)=\lambda\psi^{\dagger\lambda}(s)\ ,\quad 0\le s\le\tau\ . \tag{42}$$

In close analogy to the original eigenvalue problem the solution for the left-hand eigenfunctions $\psi^{\dagger\lambda}$ can be written in the form

$$\psi^{\dagger\lambda}(s)=\psi^{\dagger\lambda}(0)e^{-\lambda s}\ ,\quad 0\le s\le\tau\ , \tag{43}$$

where $\psi^{\dagger\lambda}(0)$ is the left-hand eigenvector of the matrix $W(\lambda)$,

$$\psi^{\dagger\lambda}(0)W(\lambda)=\lambda\psi^{\dagger\lambda}(0)\ . \tag{44}$$

We note that we have used different notations for the right-hand eigenfunctions ϕ^λ and the dual left-hand eigenfunctions $\psi^{\dagger\lambda}$. This is due to the fact that the infinitesimal generator $\mathcal{A}_L$ is generally not self-adjoint so that it is not possible to transform the eigenfunctions into each other by performing a transposition and by substituting $\Theta=-s$. In what follows it will become important that the eigenfunctions $\boldsymbol{\phi}^{\lambda}$ and $\boldsymbol{\psi}^{\dagger\lambda}$ of the infinitesimal generator $\mathcal{A}_L$ and its dual $\mathcal{A}_L^{\dagger}$, respectively, can be chosen to form a biorthonormal set in $\mathcal{C}$ and $\mathcal{C}^{\dagger}$.

4. Biorthonormality conditions

First, we derive the biorthogonality condition for different eigenvalues $\mu\neq\lambda$. Using the explicit expression of the bilinear form (33) and applying Eqs. (24) and (43) we obtain

$$(\boldsymbol{\psi}^{\dagger\lambda},\boldsymbol{\phi}^{\mu})=\langle\boldsymbol{\psi}^{\dagger\lambda}(0),\boldsymbol{\phi}^{\mu}(0)\rangle-\left\langle\boldsymbol{\psi}^{\dagger\lambda}(0),\left|\int_{-\tau}^{0}d\Theta\int_{0}^{\Theta}ds\,e^{-\lambda(s-\Theta)}w(\Theta)e^{\mu s}\right|\boldsymbol{\phi}^{\mu}(0)\right\rangle . \tag{45}$$

An integration with respect to s yields for the right-hand side

$$\langle\boldsymbol{\psi}^{\dagger\lambda}(0),\boldsymbol{\phi}^{\mu}(0)\rangle-\left\langle\boldsymbol{\psi}^{\dagger\lambda}(0),\left|\int_{-\tau}^{0}d\Theta\frac{[e^{(\mu-\lambda)\Theta}-1]e^{\lambda\Theta}}{\mu-\lambda}w(\Theta)\right|\boldsymbol{\phi}^{\mu}(0)\right\rangle . \tag{46}$$

Using the definition of $W(\lambda)$ in Eq. (26) and its eigenvalue equation (25) the biorthogonality results from

$$\langle\boldsymbol{\psi}^{\dagger\lambda}(0),\boldsymbol{\phi}^{\lambda}(0)\rangle-\frac{1}{\mu-\lambda}\langle\boldsymbol{\psi}^{\dagger\lambda}(0),[W(\mu)-W(\lambda)]\boldsymbol{\phi}^{\mu}(0)\rangle=0 . \tag{47}$$

To normalize the biorthogonal set of eigenfunctions, we introduce a proper normalization constant N_λ in a symmetric way,

$$\boldsymbol{\phi}^{\lambda}(0)=N_\lambda\boldsymbol{\phi}_N^{\lambda}(0) , \quad \boldsymbol{\psi}^{\dagger\lambda}(0)=N_\lambda\boldsymbol{\psi}_N^{\dagger\lambda}(0) . \tag{48}$$

From the requirement

$$(\boldsymbol{\psi}^{\dagger\lambda},\boldsymbol{\phi}^{\lambda})=1 \tag{49}$$

we determine the normalization constant N_λ by performing similar calculations as above for the case $\mu=\lambda$,

$$N_\lambda=\frac{1}{\left|\left\langle\boldsymbol{\psi}_N^{\dagger\lambda}(0),\left|I-\int_{-\tau}^{0}d\Theta\,w(\Theta)\Theta e^{\lambda\Theta}\right|\boldsymbol{\phi}_N^{\lambda}(0)\right\rangle\right|^{1/2}} . \tag{50}$$

Summarizing the results, the biorthonormality relation reads

$$(\boldsymbol{\psi}^{\dagger\lambda},\boldsymbol{\phi}^{\mu})=\delta_{\lambda\mu} . \tag{51}$$

C. Nonlinear treatment near instabilities: The concept of order parameters

In general it appears to be impossible to present a general solution method for nonlinear functional differential equations. However, in the case of nonlinear ordinary differential equations the concept of order parameters and enslaved modes [32–35] provides for a powerful tool to formulate an equivalent simpler nonlinear problem which allows for the discussion of qualitative properties close to instabilities. This motivates us to generalize this concept to functional differential equations.

1. Complete nonlinear equations

To appropriately apply our results derived for linear functional differential equations we have to reformulate the nonlinear problem given by Eqs. (6), (11), and (14). To this end we split the original vector-valued functional $\mathcal{N}[\mathbf{q}_t]$ into its linear $\mathcal{L}[\mathbf{q}_t]$ and nonlinear part $\tilde{\mathcal{N}}[\mathbf{q}_t]$,

$$\mathcal{N}[\mathbf{q}_t(\cdot)]=\mathcal{L}[\mathbf{q}_t(\cdot)]+\tilde{\mathcal{N}}[\mathbf{q}_t(\cdot)] . \tag{52}$$

Using Eqs. (14), (19), and (58) the evolution equation (6) in the extended state space $\mathcal{C}$ can be rewritten in the form

$$\frac{d}{dt}\mathbf{q}_t(\Theta)=(\mathcal{A}_L\mathbf{q}_t)(\Theta)+X_0(\Theta)\tilde{\mathcal{N}}[\mathbf{q}_t(\cdot)] ,\quad -\tau\leq\Theta\leq 0 , \tag{53}$$

where X_0 denotes a matrix-valued function defined by

$$X_0(\Theta)=\begin{cases}0 , & -\tau\leq\Theta<0\\ I , & \Theta=0 .\end{cases} \tag{54}$$

2. Motivation for a projector formalism

In the following we intend to discuss the dynamics of the nonlinear functional differential equation (53) in the vicinity of an instability. We consider a situation where the control parameters $\{\sigma_i\}$ are chosen in such a way that the spectrum of the infinitesimal generator $\mathcal{A}_L$ is bounded from above by the imaginary axes (compare subsection I B 2). When appropriately changing the control parameters $\{\sigma_i\}$ some eigenvalues cross the imaginary axes. According to the spectral properties of $\mathcal{A}_L$ it is guaranteed that only a finite number of modes become unstable. In Fig. 2 the spectrum representing this situation is schematically illustrated.

These spectral properties of the linearized problem provide a decomposition of the state space $\mathcal{C}$ into a finite-dimensional subspace $\mathcal{U}$ spanned by the unstable modes and an infinite-dimensional subspace $\mathcal{S}$ spanned by the remaining stable modes. In the following we shall investigate that it is only approximatively possible to project the nonlinear dynamics onto both subspaces $\mathcal{U}$ and $\mathcal{S}$. After a subsequent proper adiabatic elimination of the infinite number of stable modes we end up with a low-dimensional set of order-parameter equations describing the dynamical behavior of the nonlinear system on a macroscopic level. It turns out that these order-parameter

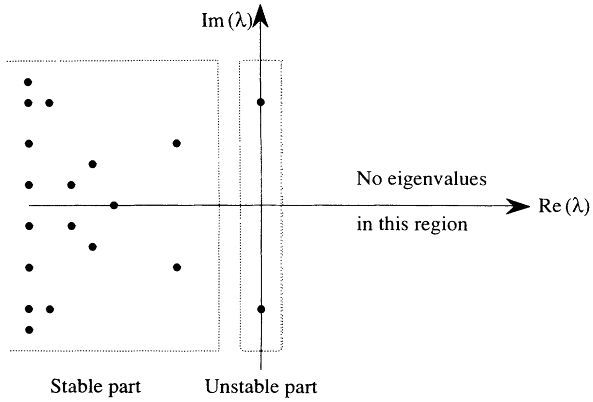

FIG. 2. Schematic representation of the spectrum of a delay-differential equation at an instability. The unstable part consists of those eigenvalues whose real part nearly vanishes whereas the infinity of eigenvalues of the stable part lie in a region separated by a finite distance from the unstable part.

equations are of the form of ordinary differential equations, i.e., they no longer contain memory terms.

3. Definition of projectors

First we formalize the decomposition of the infinite-dimensional extended state space $\mathcal{C}$ into an m-dimensional unstable subspace $\mathcal{U}$ corresponding to the unstable modes and the remaining infinite-dimensional stable subspace $\mathcal{S}$. Due to the finite dimensionality of $\mathcal{U}$, the set of vector-valued eigenfunctions $\boldsymbol{\phi}^{\lambda_1},\ldots,\boldsymbol{\phi}^{\lambda m}$ represent a basis in $\mathcal{U}$. For a concise notation we collect these eigenfunctions in an $n\times m$ matrix

$$\Phi_u(\Theta)=(\boldsymbol{\phi}^{\lambda_1}(\Theta)\ \cdots\ \boldsymbol{\phi}^{\lambda_m}(\Theta)) \tag{55}$$

and the corresponding adjoint eigenfunctions in an $m\times n$ matrix

$$\Psi_u^\dagger(s)=\begin{bmatrix}\boldsymbol{\psi}^{\dagger\lambda_1}(s)\\ \vdots\\ \boldsymbol{\psi}^{\dagger\lambda_m}(s)\end{bmatrix}. \tag{56}$$

We note that in the matrix notation the biorthonormality relation (51) takes the form

$$(\Psi_u^\dagger,\Phi_u)=I\ . \tag{57}$$

With these definitions the eigenvalue problem Eq. (23) and its adjoint Eq. (42) can be rewritten in a matrix notation

$$(\mathcal{A}_L\Phi_u)(\Theta)=\Phi_u(\Theta)\Lambda_u\ ,\quad (\mathcal{A}_L^\dagger\Psi_n^\dagger)(s)=\Lambda_u\Psi_u^\dagger(s)\ , \tag{58}$$

where Λ_u is an $m\times m$ matrix of canonical form. This enables us to introduce the operator $\mathcal{P}_u$ which projects the extended state space $\mathcal{C}$ onto the finite-dimensional subspace $\mathcal{U}$:

$$\mathcal{P}_u\cdot=\Phi_u(\Theta)(\Psi_u^\dagger,\cdot)\ . \tag{59}$$

The corresponding projection onto the complementary subspace $\mathcal{S}$ is achieved by the operator

$$\mathcal{Q}_s\cdot=(\mathcal{I}-\mathcal{P}_u)\cdot\ . \tag{60}$$

4. Investigation of the projection

With the above-mentioned definitions we are now in a position to decompose the extended state vector $\mathbf{q}_t$ into two different parts,

$$\mathbf{q}_t(\Theta)=\mathbf{U}_t(\Theta)+\mathbf{S}_t(\Theta)\ , \tag{61}$$

where we specialize $\mathbf{U}_t$ and $\mathbf{S}_t$ in the following. In analogy to ordinary differential equations we first consider a linear motion where $\mathbf{U}_t$ can be chosen as

$$\mathbf{U}_t(\Theta)=(\mathcal{P}_u\mathbf{q}_t)(\Theta)=\Phi_u(\Theta)e^{\Lambda_u t}\tilde{\mathbf{u}}(0) \tag{62}$$

and $\tilde{\mathbf{u}}(0)$ is determined by the initial conditions. This hypothesis implies that $\mathbf{U}_t$ and $\mathbf{S}_t$ are elements of the subspaces $\mathcal{U}$ and $\mathcal{S}$, respectively. In addition it is possible to deduce from the dynamics of $\mathbf{U}_t$ in the extended state space $\mathcal{C}$ a corresponding dynamics in the state space Γ. This relation is established by defining the state vector in Γ according to $\mathbf{U}(t)=\mathbf{U}_t(0)$ and by verifying the property

$$\mathbf{U}_t(\Theta)=\mathbf{U}(t+\Theta)\ . \tag{63}$$

In the nonlinear case, however, we have to allow for varying $\tilde{\mathbf{u}}$ in Eq. (62), which provides us with two conflicting ambitions. On the one hand, a variation of $\tilde{\mathbf{u}}(0)$ should allow us to remain within the linearly invariant subspace $\mathcal{U}$. This suggests the ansatz

$$\mathbf{U}_t(\Theta)=\Phi_u(\Theta)e^{\Lambda_u t}\tilde{\mathbf{u}}(t)\ . \tag{64}$$

The fact that this motion indeed is confined to $\mathcal{U}$ is expressed by the relation

$$(\mathcal{P}_u\mathbf{U}_t)(\Theta)=\mathbf{U}_t(\Theta)\ , \tag{65}$$

which can be immediately confirmed from the definition of the projector $\mathcal{P}_u$ in Eq. (59). On the other hand, the ansatz (64) destroys the relation between both state spaces Γ and $\mathcal{C}$ since condition (63) is obviously not satisfied. Therefore we perform an extension of the ansatz (64) by

$$\mathbf{U}_t(\Theta)=\Phi_u(\Theta)e^{\Lambda_u t}\tilde{\mathbf{u}}_t(\Theta)\ ,\quad \text{with } \tilde{\mathbf{u}}_t(\Theta)=\tilde{\mathbf{u}}(t+\Theta)\ . \tag{66}$$

However, it turns out that this motion is no longer restricted to the linear subspace $\mathcal{U}$. Indeed, this can be immediately examined by noting

$$(\mathcal{P}_u\mathbf{U}_t)(\Theta)\neq\mathbf{U}_t(\Theta)\ . \tag{67}$$

A compromise between both conflicting ambitions can be generated by a reasonable approximation which is suggested by the following consideration. Regarding the ansatz (66) the difference between $\mathcal{P}_u\mathbf{U}_t$ and $\mathbf{U}_t$ only depends on the deviation $\tilde{\mathbf{u}}_t(\Theta)-\tilde{\mathbf{u}}_t(0)$ in the interval $[-\tau,0]$. In the neighborhood of an instability $\tilde{\mathbf{u}}_t$ will be a slowly varying vector in both of its variables. We therefore assume that the control parameters can be chosen in such a way that the following condition is fulfilled:

$$\tau\left|\frac{d}{d\Theta}\tilde{\mathbf{u}}_t(\Theta)\right| \ll \left|\tilde{\mathbf{u}}_t(\Theta)\right| . \tag{68}$$

In this case condition (65) is approximatively valid so that we can describe the slow motion of a system in the subspace of unstable modes by an ordinary differential equation. Following the standard procedures developed in [32,33,39,40], we introduce generalized order parameters $\mathbf{u}$ by defining

$$\mathbf{U}_t(\Theta)=(\mathcal{P}_u\mathbf{q}_t)(\Theta)=\Phi_u(\Theta)\mathbf{u}(t) , \tag{69}$$

$$\mathbf{u}(t)=(\Psi_u^\dagger,\mathbf{q}_t) . \tag{70}$$

They describe the macroscopic behavior of the delay system close to an instability. Assuming the validity of the approximation (68) there still remains the nontrivial problem to formulate a similar projection for the motion in the subspace $\mathcal{S}$. Indeed, for the remaining stable mode amplitudes an inequality similar to (68) will not hold. Furthermore, we cannot be sure that the linear vector-valued eigenfunctions determined above completely span the function space $\mathcal{C}$. In order to avoid a concrete representation of the remaining part of the extended state vector $\mathbf{q}_t$ in $\mathcal{S}$ we define the stable modes by

$$\mathbf{S}_t(\Theta)=(\mathcal{Q}_s\mathbf{q}_t)(\Theta)=\mathbf{s}_t(\Theta) . \tag{71}$$

Using Eqs. (61), (70), (71), and the biorthonormality relation (57) we can approximatively project Eq. (53) onto the subspaces $\mathcal{U}$ and $\mathcal{S}$. We end up with a coupled set of nonlinear evolution equations for the order parameters $\mathbf{u}(t)$ and the stable modes $\mathbf{s}_t$,

$$\frac{d}{dt}\mathbf{u}(t)=\Lambda_u\mathbf{u}(t)+\Psi_u^\dagger(0)\tilde{\mathcal{N}}[\Phi_u(\cdot)\mathbf{u}(t)+\mathbf{s}_t(\cdot)] , \tag{72}$$

$$\frac{d}{dt}\mathbf{s}_t(\Theta)=(\mathcal{A}_L\mathbf{s}_t)(\Theta)+\{X_0(\Theta)-\Phi_u(\Theta)\Psi_u^\dagger(0)\}\times\tilde{\mathcal{N}}[\Phi_u(\cdot)\mathbf{u}(t)+\mathbf{s}_t(\cdot)] . \tag{73}$$

5. *Adiabatic elimination procedure*

In this form the coupled set of Eqs. (72) and (73) allows for an application of the adiabatic elimination procedure [32–35]. In the vicinity of an instability the real part of the eigenvalues of the order parameters $\mathbf{u}$ nearly vanishes whereas the eigenvalues of the stable modes $\mathbf{s}_t$ still remain negative. Interpreting the real part of these eigenvalues as damping constants we can introduce relaxation times τ_u and τ_s through the relation

$$\tau_{u,s}\sim\frac{1}{|\mathrm{Re}(\lambda_{u,s})|} , \tag{74}$$

which leads to the condition

$$\tau_u \gg \tau_s . \tag{75}$$

Equation (75) expresses the fact that a time scale hierarchy is established between the order parameters and the stable modes. The stable modes evolve on a time scale which is much faster than the time scale of the order parameters. The asymptotic dynamics of the stable modes will then be completely prescribed by the order parameters. As was shown in [41,42] there exists a finite-dimensional center manifold $\mathbf{h}(\Theta,\mathbf{u}(t))$ with the usual properties $\mathbf{h}(\Theta,0)=0$ and $d\mathbf{h}(\Theta,0)/d\mathbf{u}=0$ on which the asymptotic dynamics of the nonlinear delay system evolves,

$$\mathbf{s}_t(\Theta)=\mathbf{h}(\Theta,\mathbf{u}(t)) . \tag{76}$$

Inserting Eq. (76) into Eqs. (72) and (73), we obtain an implicit equation for the center manifold $\mathbf{h}$,

$$\frac{d\mathbf{h}(\Theta,\mathbf{u}(t))}{d\mathbf{u}}\{\Lambda_u\mathbf{u}(t)+\Psi_u^\dagger(0)\tilde{\mathcal{N}}[\Phi_u(\cdot)\mathbf{u}(t)+\mathbf{h}(\cdot\mathbf{u}(t))]\}$$
$$=(\mathcal{A}_L\mathbf{h}(\cdot\mathbf{u}(t)))(\Theta)+\{X_0(\Theta)-\Phi_u(\Theta)\Psi_u^\dagger(0)\}\tilde{\mathcal{N}}[\Phi_u(\cdot)\mathbf{u}(t)+\mathbf{h}(\cdot\mathbf{u}(t))] . \tag{77}$$

Having solved Eq. (77) the dynamics of the nonlinear delay system on the center manifold is completely described by the finite-dimensional closed set of order-parameter equations

$$\frac{d}{dt}\mathbf{u}(t)=\Lambda_u\mathbf{u}(t)+\Psi_u^\dagger(0)\tilde{\mathcal{N}}[\Phi_u(\cdot)\mathbf{u}(t)+\mathbf{h}(\cdot,\mathbf{u}(t))] . \tag{78}$$

To approximatively determine the center manifold $\mathbf{h}$ in lowest order, we first assume that the nonlinear functional $\tilde{\mathcal{N}}$ starts with terms of order $O(|\mathbf{q}_t|^r)$. We therefore restrict our further considerations to the approximative functional

$$\tilde{\mathcal{N}}_i[\Phi_u(\cdot)\mathbf{u}(t)+\mathbf{h}(\cdot,\mathbf{u}(t))]\approx\tilde{\mathcal{N}}_i^{(r)}[\Phi_u(\cdot)\mathbf{u}(t)] . \tag{79}$$

Using the notation from Eqs. (11) and (12) we get

$$\tilde{\mathcal{N}}_i^{(r)}[\Phi_u(\cdot)\mathbf{u}(t)]=\tilde{N}^{(r)}_{ij_1,\ldots,j_r}u_{j_1}(t)\cdots u_{j_r}(t) , \tag{80}$$

where the $n\times m\times\cdots\times m$ tensor $\tilde{N}^{(r)}$ has the explicit representation

$$\tilde{N}^{(r)}_{ij_1,\ldots,j_r}=\int_{-\tau}^{0}d\Theta_1\cdots\int_{-\tau}^{0}d\Theta_r w^{(r)}_{ij_1',\ldots,j_r'}(\Theta_1,\ldots,\Theta_r)\Phi_{u,j_1'j_1}(\Theta_1)\cdots\Phi_{u,j_r'j_r}(\Theta_r) . \tag{81}$$

Up to order $O(|\mathbf{u}|^r)$ the center manifold $\mathbf{h}$ can be obtained by performing the ansatz

$$h_i(\Theta,\mathbf{u}(t)) \approx H_{ij_1,\ldots,j_r}(\Theta)u_{j_1}(t)\cdots u_{j_r}(t) , \tag{82}$$

where the unknown $n\times m\times\cdots\times m$ tensor H is assumed to be fully symmetric in its last r indices. Inserting this ansatz into Eq. (77) and taking into account the approximation (79), the tensor H can be determined by

$$H_{ij_1,\ldots,j_r}(\Theta)=\sum_{l=1}^{n}\tilde{N}^{(r)}_{lj_1,\ldots,j_r}\left[[\mathcal{A}_L-(\lambda_{j_1}+\cdots+\lambda_{j_r})\mathcal{I}]^{-1}\left[\sum_{l'=1}^{m}\Phi_{u,il'}(\cdot)\Psi^{\dagger}_{u,l'l}(0)-X_{0,il}(\cdot)\right]\right](\Theta) , \tag{83}$$

where Einstein's summation convention has *not* been used and λ_j stands for the jth eigenvalue of Λ_u. We note that the operator $[\mathcal{A}_L-(\lambda_{j_1}+\cdots+\lambda_{j_r})\mathcal{I}]^{-1}$ appearing in Eq. (83) acts only on elements defined in $\mathcal{S}$. Therefore it can be evaluated by applying the definition in Eq. (32). Not to overload the formalism we only give an explicit expression for the center manifold $\mathbf{h}$ for the special case of a scalar delay-differential equation (see Sec. II B).

II. APPLICATIONS

In this section we apply the formalism developed above to a concrete example of a nonlinear feedback system with time delay. We have chosen a first-order phase-locked loop where analog and numerical delay lines have been introduced. We show that this nonlinear delay system is susceptible to a hierarchy of delay-induced instabilities destroying the synchronous operation of the loop. In applications such delay effects are important since many control systems are based on the phase-locked principle. In phase-locked frequency synthesizers, e.g., delay-induced instabilities resulting from the finite propagation time of signals through frequency dividers are the origin of phase and frequency instabilities of the whole device [43,44]. Apart from possible applications, a delayed first-order PLL has the important property that all observable instabilities are purely a result of delays. Additionally, the simple form of the corresponding model equation (see Sec. II A) allows for a quantitative comparison of the theory with experimental data.

A. Derivation of a model equation

A PLL generally consists of a reference oscillator, a voltage-controlled oscillator (VCO) whose output signal has to be synchronized with the reference signal, a low-pass filter, and a phase detector. In the case of a first-order PLL, the low-pass filter only serves to eliminate higher-frequency components. The finite propagation time of signals is taken into account by delay lines. The experimental apparatus is shown in Fig. 3.

The reference signal $y_1(t)$ is supposed to be an oscillation with angular frequency ω_0,

$$y_1(t)=A\sin\Theta(t) , \quad \text{with } \dot{\Theta}(t)=\omega_0 . \tag{84}$$

The voltage-controlled oscillator produces a signal $y_4(t)$ whose angular frequency $\dot{\Psi}(t)$ depends linearly on the incoming voltage $y_3(t)$,

$$y_4(t)=B\cos\Psi(t) , \quad \text{with } \dot{\Psi}(t)=\omega+K_1y_3(t) . \tag{85}$$

ω is the central angular frequency of the VCO and K_1 its sensitivity. The delay line placed at the output of the VCO takes into account the finite propagation time in the feedback loop. After a round-trip the signal returns at the mixer input with a time delay τ,

$$y_5(t)=y_4(t-\tau) . \tag{86}$$

The mixer acting as a phase detector is the only nonlinear element in the circuit. If it has a sinusoidal characteristic its output signal is proportional to the product of the two incoming signals, i.e.,

$$\begin{aligned} y_2(t)&=\beta y_1(t)y_5(t)\\ &=\frac{\beta AB}{2}\{\sin[\Theta(t)-\Psi(t-\tau)]\\ &\qquad+\sin[\Theta(t)+\Psi(t-\tau)]\} . \end{aligned} \tag{87}$$

$\beta AB/2=\mu$ is the sensitivity of the mixer. Its output signal consists of both a low-frequency component and a high-frequency term. However, the latter term is eliminated by a low-pass filter in such a way that at the filter output we have the experimentally accessible signal

$$y_3(T)=\mu\sin[\Theta(t)-\Psi(t-\tau)] . \tag{88}$$

The dynamical variable of interest is the phase difference ϕ between the two oscillators which is defined by

$$\phi(t)=\Theta(t)-\Psi(t-\tau) . \tag{89}$$

In order to derive an evolution equation for the phase difference we differentiate ϕ with respect to time t. Using Eqs. (85) and (88) we obtain

$$\dot{\phi}(t)=-K\sin[\phi(t-\tau)]+\omega_0-\omega , \tag{90}$$

where $K=K_1\mu$ represents the so-called open-loop gain of the PLL. Equation (90) is the model equation of a de-

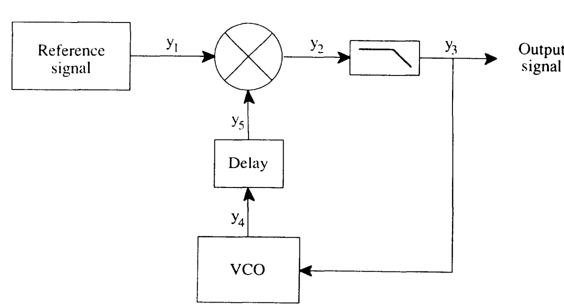

FIG. 3. First-order phase-locked loop with time delay.

layed first-order phase-locked loop. It has the same form as Eq. (1) and represents a first-order scalar nonlinear delay-differential equation for the phase error signal ϕ.

It turns out that the frequency deviation $\omega-\omega_0$ appearing in Eq. (90) qualitatively influences the dynamics of the nonlinear delay system only when it is operated in its fully nonlinear domain (compare Sec. II C). In what follows we are interested in discussing the oscillatory behavior of the loop. Therefore we can assume without loss of generality that the central frequency of the VCO is tuned to the reference frequency, i.e., the frequency deviation $\omega-\omega_0$ vanishes. In this case an additional scaling in Eq. (90) expressed by $t=\tau\tilde{t}$ and the introduction of a new variable $\tilde{\phi}(\tilde{t})=\phi(\tau\tilde{t})$ allows us to transform Eq. (90) into the more condensed form

$$\dot{\tilde{\phi}}(\tilde{t})=-K\tau\sin[\tilde{\phi}(\tilde{t}-1)] \ . \tag{91}$$

From Eq. (91) we observe that the dynamics is only affected by one effective control parameter i.e., the product between the open-loop gain K and the time delay τ. This scaling property means that we can restrict our analysis of Eq. (90) to the case where only one of both parameter (e.g., K) is varied whereas the other one (e.g., τ) can be assumed to be fixed.

B. Examination of the model equation

For the illustration of our formalism developed in Sec. I, we apply in detail the elimination procedure to the scalar nonlinear delay-differential equation (90) modeling a first-order PLL with time delay under the assumption of a vanishing frequency deviation $\omega-\omega_0$. In the notation of Sec. I the evolution equation of this nonlinear delay system in the state space Γ reads

$$\dot{q}(t)=-K\sin[q(t-\tau)] \ , \tag{92}$$

with $K>0$. A corresponding evolution in the extended state space $\mathcal{C}$ is given by Eqs. (6), (11), (12), and (14) in the special case that the densities $w^{(j)}$ are chosen as

$$w^{(2k-1)}(\Theta_1,\ldots,\Theta_{2k-1})=K\frac{(-1)^k}{(2k-1)!}\delta(\Theta_1+\tau)\cdots \times\delta(\Theta_{2k-1}+\tau) \ , \tag{93}$$

$$w^{(2k)}(\Theta_1,\ldots,\Theta_{2k})=0 \ , \tag{94}$$

where $k=1,2,\ldots$.

1. Linear stability analysis

The stationary solutions of the nonlinear delay system are defined by Eqs. (15) and (16) in Γ and $\mathcal{C}$, respectively. In our case we obtain

$$q_{\text{stat}}=l\pi \ , \tag{95}$$

where l denotes an arbitrary integer. In the vicinity of these stationary solutions the linearized problem is given by Eqs. (18)–(20) where the density w is calculated as a functional derivative according to (21),

$$w(\Theta)=(-1)^{l+1}K\delta(\Theta+\tau) \ . \tag{96}$$

Specializing Eq. (26) to our example

$$W(\lambda)=(-1)^{l+1}Ke^{-\lambda\tau} \ , \tag{97}$$

the transcendental characteristic equation (27) for the eigenvalues λ takes the form

$$\lambda+(-1)^lKe^{-\lambda\tau}=0 \ . \tag{98}$$

We immediately deduce that the stationary solutions $q_{\text{stat}}=\pi$ (mod2π) are unstable for all values of K and τ, since there exists one real eigenvalue which turns out to be positive. For the remaining stationary solutions $q_{\text{stat}}=0$ (mod2π) we have to investigate whether such eigenvalues with positive real part already exist for small values of K and τ. Applying a generalization of the Routh-Hurwitz criterion due to Pontryagin (see, e.g., [39]), one can prove that the real parts of all eigenvalues are in the left half of the complex plane only for $K\tau<\pi/2$. This means that the stationary state q_{stat} loses its stability at

$$K\tau=\pi/2 \ . \tag{99}$$

In accordance with the scaling arguments leading to Eq. (91) the instability condition (99) depends only on the product of K and τ. Keeping τ fixed, we can study the type of instability by expanding the eigenvalue $\lambda=\lambda(K)$ up to the first order in the deviations from the critical value $K_c=\pi/2\tau$ according to the ansatz

$$\lambda(K)=a(K-K_c)+i[b+c(K-K_c)]+O((K-K_c)^2) \ . \tag{100}$$

Here we implicitly assumed that the real part of the eigenvalue $\lambda=\lambda(K)$ vanishes at the instability $K=K_c$. Inserting the ansatz (100) into the characteristic equation (98) (for the case of even l) and comparing coefficients up to the first order in $K-K_c$ we obtain the solutions

$$\lambda(K)=\frac{K_c\tau}{1+(K_c\tau)^2}(K-K_c) \pm i\left[K_c+\frac{1}{1+(K_c\tau)^2}(K-K_c)\right]+O((K-K_c)^2) \ . \tag{101}$$

At the instability $K=K_c$ two complex conjugate eigenvalues lie on the imaginary axes. They correspond to oscillatory solutions with the frequencies

$$\Omega_{\pm}=\pm K_c \ . \tag{102}$$

From the deviations $K-K_c$ in Eq. (101) we read off that the transversality condition $[d\lambda(K)/dK]|_{K=K_c}\neq 0$ is fulfilled. This means that a Hopf bifurcation occurs at $K=K_c$.

The biorthonormal set of eigenfunctions corresponding to the linearized infinitesimal generator $\mathcal{A}_L$ and its adjoint $\mathcal{A}_L^\dagger$ can be directly obtained by specializing Eqs. (24) and (43),

$$\phi^\lambda(\Theta)=N_\lambda e^{\lambda\Theta} \ , \quad \psi^{\dagger\lambda}(s)=N_\lambda e^{-\lambda s} \ . \tag{103}$$

Using Eqs. (96) and (98) the normalization constants N_λ from Eq. (50) explicitly read

$$N_\lambda = \frac{1}{\sqrt{1+\lambda\tau}} \ . \tag{104}$$

We note that in the case of a scalar delay system the eigenfunctions ϕ^λ and $\psi^{\dagger\lambda}$ can be transformed into each other by substituting $\Theta=-s$. The dynamics of the linear evolution equation is related to its adjoint one by a simple time inversion.

2. Order-parameter equations

In the vicinity of this instability two modes become unstable. Their corresponding eigenvalues (101) are denoted by $\lambda_u^\pm$ with the property $\lambda_u^+=(\lambda_u^-)^*$. The infinite-dimensional extended state space $\mathcal{C}$ therefore decomposes into a two-dimensional unstable subspace $\mathcal{U}$ and a remaining infinite-dimensional subspace $\mathcal{S}$. According to (55) and (56) a biorthonormal basis of $\mathcal{U}$ can be represented by the matrices

$$\Phi_u(\Theta)=\left[\frac{e^{\lambda_u^+\Theta}}{\sqrt{1+\lambda_u^+\tau}}, \frac{e^{\lambda_u^-\Theta}}{\sqrt{1+\lambda_u^-\tau}}\right] ,$$
$$\Psi_u^\dagger(s)=\begin{bmatrix}\dfrac{e^{-\lambda_u^+ s}}{\sqrt{1+\lambda_u^+\tau}}\\ \dfrac{e^{-\lambda_u^- s}}{\sqrt{1+\lambda_u^-\tau}}\end{bmatrix} . \tag{105}$$

We denote the components of the order parameter $\mathbf{u}$ by $u(t)$ and its complex conjugate $u^*(t)$ and collect the corresponding eigenvalues into the diagonal matrix Λ_u according to

$$\mathbf{u}(t)=\begin{bmatrix}u(t)\\ u^*(t)\end{bmatrix} , \quad \Lambda_u=\begin{bmatrix}\lambda_u^+ & 0\\ 0 & \lambda_u^-\end{bmatrix} . \tag{106}$$

It is our aim to derive the order-parameter equations (78) in the case of the Hopf bifurcation of our model equation up to lowest order in the nonlinearities. According to Eq. (94) the lowest order of the nonlinearity is given by $r=3$. As a consequence of (82) the center manifold h also starts with terms of order $r=3$. Therefore we can approximate the nonlinearity in the order-parameter equations (78) in the same way as in (79),

$$\tilde{\mathcal{N}}[\Phi_u(\cdot)\mathbf{u}(t)+\mathbf{h}(\cdot,\mathbf{u}(t))]\approx\tilde{\mathcal{N}}^{(3)}[\Phi_u(\cdot)\mathbf{u}(t)] . \tag{107}$$

This means that the stable modes do not influence the order-parameter equations in lowest order. In our case Eq. (79) reduces to

$$\tilde{\mathcal{N}}^{(3)}[\Phi_u(\cdot)\mathbf{u}(t)]=\tilde{N}_{111}^{(3)}u^3(t)+3\tilde{N}_{112}^{(3)}u^2(t)u^*(t) +3\tilde{N}_{122}^{(3)}u(t)u^{*2}(t)+\tilde{N}_{222}^{(3)}u^{*3}(t) . \tag{108}$$

The coefficients are calculated from Eqs. (81), (94), and (105),

$$\tilde{N}_{111}^{(3)}=-\frac{1}{6K^2}\frac{\lambda_u^{+3}}{(1+\lambda_u^+\tau)^{3/2}} , \quad \tilde{N}_{222}^{(3)}=\tilde{N}_{111}^{(3)*} ,$$
$$\tilde{N}_{112}^{(3)}=-\frac{1}{6K^2}\frac{\lambda_u^{+2}\lambda_u^-}{(1+\lambda_u^+\tau)(1+\lambda_u^-\tau)^{1/2}} , \quad \tilde{N}_{122}^{(3)}=\tilde{N}_{112}^{(3)*} . \tag{109}$$

With the above-mentioned approximation (107) and Eqs. (105), (106), and (108) the first of both order-parameter equations (78) reads

$$\frac{d}{dt}u(t)=\lambda_u^+u(t) +\frac{1}{\sqrt{1+\lambda_u^+\tau}}\{\tilde{N}_{111}^{(3)}u^3(t)+3\tilde{N}_{112}^{(3)}u^2(t)u^*(t) +3\tilde{N}_{122}^{(3)}u(t)u^{*2}(t) +N_{222}^{(3)}u^{*3}(t)\} , \tag{110}$$

whereas the second one for u^* is the complex conjugate to (110). In order to derive the normal form of the order-parameter equations we apply the rotating-wave approximation [32,33]. From the linear part of Eqs. (110) and (102) we read off that the order parameters vary near an instability as

$$u(t)\sim e^{iK_ct} , \quad u^*(t)\sim e^{-iK_ct} . \tag{111}$$

Keeping only nonlinear terms which oscillate in phase with the respective order parameter, Eq. (110) can be transformed to the normal form of a Hopf bifurcation,

$$\frac{d}{dt}u(t)=\lambda_u^+u(t)-b|u(t)|^2u(t) . \tag{112}$$

The complex-valued coefficient b is calculated from Eqs. (109) and (110),

$$b=\frac{K_c^2\tau}{2[1+(K_c\tau)^2]^{3/2}}+i\frac{K_c}{2[1+(K_c\tau)^2]^{3/2}}+O(K-K_c) . \tag{113}$$

In order to adequately discuss the periodic orbit corresponding to the normal form (112) we introduce polar coordinates according to

$$u(t)=r(t)e^{i\phi(t)} . \tag{114}$$

Taking into account the expansions (101) and (113) in the deviations $K-K_c$ the steady state is characterized by

$$r_0^2=\frac{\mathrm{Re}(\lambda_u^+)}{\mathrm{Re}(b)}=2\sqrt{1+(K_c\tau)^2}\frac{K-K_c}{K_c}+O((K-K_c)^2) , \tag{115}$$

$$\frac{d}{dt}\phi(t)=\mathrm{Im}(\lambda_u^+)-r_0^2\mathrm{Im}(b)=K_c\{1+O((K-K_c)^2)\} . \tag{116}$$

We note that in our case the amplitude-dependent frequency shift arising from the nonlinearities compensates the frequency dependence of the order $K-K_c$ due to the linear stability analysis.

3. *Center manifold*

The results of the preceding subsection have shown that the center manifold h does not contribute to the order-parameter equations in its lowest order. However, for the sake of completeness, we shall give its explicit representation for the chosen model equation. In lowest order ($r=3$) Eq. (82) reduces to

$$h(\Theta,u(t),u^*(t))=H_{111}(\Theta)u^3(t)+3H_{112}(\Theta)u^2(t)u^*(t)+3H_{122}(\Theta)u(t)u^*(t)^2+H_{222}(\Theta)u^*(t)^3 \; . \tag{117}$$

Since the center manifold h represents a real function the coefficients have the properties $H_{122}=H^*_{112}$ and $H_{222}=H^*_{111}$. Therefore it is sufficient to determine H_{111} and H_{112}. Specializing Eq. (83) we obtain the result that in our case of a scalar delay equation the coefficients $H_{j_1j_2j_3}$ are proportional to the coefficients $\tilde{N}^{(3)}_{j_1j_2j_3}$ given by Eq. (109),

$$H_{j_1j_2j_3}(\Theta)=\tilde{N}^{(3)}_{j_1j_2j_3}G_{j_1j_2j_3}(\Theta) \; . \tag{118}$$

The weighting tensor $G_{j_1j_2j_3}$ has to be calculated from

$$G_{j_1j_2j_3}(\Theta)=([\mathcal{A}_L-(\lambda_{j_1}+\lambda_{j_2}+\lambda_{j_3})\mathcal{I}]^{-1}\times[\Phi_u(\cdots)\Psi_u^\dagger(0)-X_0(\cdots)])(\Theta) \; , \tag{119}$$

and the eigenvalues λ_j have to be identified with λ_u^+ and λ_u^- for $j=1,2$, respectively. From Eqs. (32), (54), and (58) we obtain

$$G_{j_1j_2j_3}(\Theta)=\{\Phi_u(\Theta)[\Lambda_u-\lambda I]^{-1}\Psi_u^\dagger(0)-[W(\lambda)-\lambda I]^{-1}e^{\lambda\Theta}\}_{\lambda=\lambda_{j_1}+\lambda_{j_2}+\lambda_{j_3}} \; . \tag{120}$$

Applying Eqs. (97), (103), and (105) we obtain for the weighting tensor the result

$$G_{j_1j_2j_3}(\Theta)=\left[\frac{e^{\lambda_u^+\Theta}}{(1+\lambda_u^+\tau)(\lambda_u^+-\lambda)}+\frac{e^{\lambda_u^-\Theta}}{(1+\lambda_u^-\tau)(\lambda_u^--\lambda)}+\frac{e^{\lambda\Theta}}{\lambda+Ke^{-\lambda\tau}}\right|_{\lambda=\lambda_{j_1}+\lambda_{j_2}+\lambda_{j_3}} \; . \tag{121}$$

Because we want to evaluate some components of the weighting tensor at the instability, we note that we have in this case $\lambda_u^{\pm}=\pm iK_c$ and $K=K_c$. Taking into account the transcendental characteristic equation (98), Eq. (121) reduces for $j_1=j_2=j_3=1$ to

$$G_{111}(\Theta)=\frac{i}{4K_c}\left|\frac{2e^{iK_c\Theta}}{1+iK_c\tau}+\frac{e^{-iK_c\Theta}}{1-iK_c\tau}-e^{3iK_c\Theta}\right| \; . \tag{122}$$

Equivalently we evaluate $G_{j_1j_2j_3}$ for $j_1=j_2=1$, $j_3=2$. However, this choice leads to $\lambda=2\lambda_1+\lambda_2=iK_c$, which represents an eigenvalue of Λ_u. As can be seen from Eq. (121) this creates a singularity in two terms. Therefore a limiting procedure $\epsilon\to 0$ has to be performed for $\lambda=iK_c+\epsilon$,

$$\begin{aligned}G_{112}(\Theta)&=\lim_{\epsilon\to 0}\left|-\frac{e^{-iK_c\Theta}}{\epsilon(1+iK_c\tau)}+\frac{i}{2K_c}\frac{e^{-iK_c\Theta}}{1-iK_c\tau}+\frac{e^{iK_c\Theta}(1+\epsilon\Theta)}{\epsilon(1+iK_c\tau)}\right|\\&=\frac{\Theta e^{iK_c\Theta}}{1+iK_c\tau}+\frac{i}{2K_c}\frac{e^{-iK_c\Theta}}{1-iK_c\tau} \; .\end{aligned} \tag{123}$$

Combining the expressions in (23), (118), (122), and (123) one could determine the unknown coefficients H_{111} and H_{112} of the center manifold for all values of Θ.

C. Experimental results

Our investigations of the dynamical behavior of phase-locked loops under the influence of time delays were motivated by recent experimental and theoretical studies of the chaotic dynamics of second-order phase-locked loops driven by external signals [45–48]. These systems are modeled by a three-dimensional nonlinear evolution equation and consequently show chaotic behavior in some parameter ranges. In contrast to that, we are not aware of similar studies with PLL's influenced by time delays although in this case one could use its simplest representative, namely, a first-order PLL to induce irregular behavior of the output signal. In order to elucidate the complex phenomena which arise when we take into account time delays in nonlinear feedback loops, we report on experimental measurements performed on such first-order phase-locked loops where the finite propagation time of signals was achieved by inserting delay lines in the feedback loop [49,50].

For small values of the control parameters the PLL operates in its synchronous range, producing a dc-output signal. Increasing the control parameters we observe an oscillatory instability (Sec. II C 1). However, no loss of synchronization is associated with this instability. The synchronization even is retained up to still higher values of the control parameters where the loop exhibits a period-doubling route to chaos (Sec. II C 2). In the chaotic regime the output signal begins to jump in an irregular way, indicating a loss of synchronization. With a modified experimental setup we were able to detect the corresponding cycle slipping motion.

In our experiments we induced the above-mentioned instabilities by implementing analog and numerical delay lines into the feedback loop. In the case of the analog delay lines (Thomson Sintra LR H807), the time lag is fixed to 15 and 30 μsec. In order to guarantee a constant time delay over a broad frequency range, we have to operate the loop at a reference frequency of 135 MHz. The chosen phase detector (HP 10514A Mixer) with nearly sinusoidal characteristic represents the only nonlinear element in the circuit. As we use a low-pass filter with a cutoff frequency of approximately 200 kHz, its influence on the dynamics can be neglected. The time delays provided by the numerical delay line vary between 70 and 500 μsec. Because of its relatively low sampling rate the numerical delay line has to be implemented behind the low-pass filter and not at the VCO output. According to the considerations in Sec. II A 1 this experimental rearrangement does not modify the model equation for the phase error signal.

1. Oscillatory instability

In a first step we are interested in the examination of the locking behavior of the loop at fixed time delay τ by choosing the open loop gain K as the control parameter. Experimentally K can be altered by varying either the amplitudes of the two oscillators or the sensitivity of the VCO. At low values of the control parameter the loop settles down to its fixed points [compare Eq. (95)], producing a dc current at the output of the mixer. This well-known behavior of a first-order phase-locked loop keeps until we reach a critical value of the control parameter at $K=81$ kHz where the phase error signal begins to oscillate. Performing the same investigation with different delay times we could assure that this instability is uniquely due to delay effects. In Fig. 4 the experimentally observed frequency at the onset of the oscillation is plotted versus different values of the time delay. We notice from Fig. 4 that the theoretical results [compare Eqs. (99), (102)] are in good agreement with the experimental data.

Since the locking range of the loop is proportional to the open-loop gain, we see from the instability condition $K\tau=\pi/2$ that especially in applications where signals have to be tracked over a wide frequency range, even small time delays can induce such an instability. It should be noted, however, that the oscillatory motion of the phase error signal is not associated with a loss of synchronization of the loop. Indeed, varying the frequency deviation of the two oscillators, a dc component will be superimposed to the oscillating signal, indicating that the loop still synchronizes.

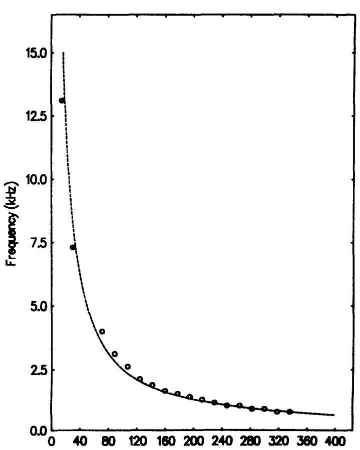

FIG. 4. First oscillatory instability of a first-order PLL with time delay. Bullets represent experimental data obtained with analog delay lines, circles correspond to the numerical delay line, and the dashed line illustrates the theoretical result.

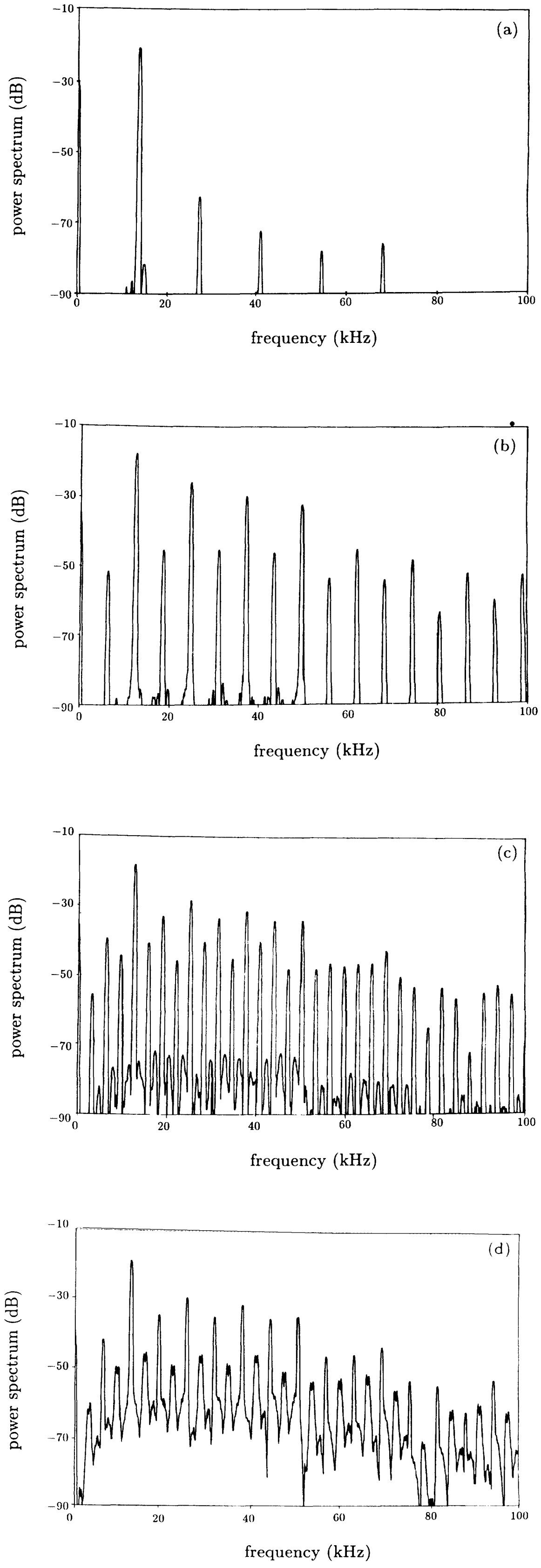

FIG. 5. Power spectrum of a first-order PLL with time delay at the onset of a chaotic motion of the phase error signal for different values of K. (a) Periodic solution, (b) period-2 solution, (c) period-4 solution, and (d) chaotic solution.

2. *Chaotic instabilities*

When we wish to operate the loop in its full nonlinear domain we have to increase the control parameters further. Due to the appearance of higher harmonics the numerical delay line can no longer be used in this regime. Therefore we have restricted our following investigations to the time delay $\tau = 15$ μsec prescribed by the chosen analog delay line.

A period-doubling route to chaos can be experimentally detected at control parameter values exceeding $K > 180$ kHz. We have resolved the doubling route up to period 8. A further increase of the control parameter leads to a broadband spectrum, indicating a chaotic motion of the phase error signal of the loop. The power spectra corresponding to the experimentally observed instabilities at the onset of the chaotic motion are represented in Fig. 5.

The experimentally measured critical values of the open-loop gain for these instabilities are summarized in Table I.

Since no analytical expressions are available to compare the experimentally observed hierarchy of instabilities with theoretical results, we simulate Eq. (90) by using the electrical simulation program PSPICE. In order to appropriately model the nonlinear delay system we have to remark that in the chaotic domain the experiments yield a nonvanishing dc component (≈ 70 mV) of the phase error signal although the two oscillators are tuned. This additional dc component could not be removed by replacing the phase detector. Consequently we have to regard this drift as being due to a "spurious" frequency deviation. Indeed, taking into account a nonvanishing $\omega - \omega_0$ in the numerical simulations we observe the same transition to chaos via a period-doubling route to chaos. Figure 6 represents the numerical simulations of the dynamics of the phase error signal and its power spectrum in the period-2 regime.

When the open-loop gain exceeds $K = 211$ kHz the loop desynchronizes. This desynchronization is associated with irregular jumps of the phase error signal in multiples of $\pm 2\pi$, a so-called cycle slipping motion. We postpone the discussion of this very important phenomenon to the next subsection, since a modified experimental apparatus is necessary to investigate these phase jumps.

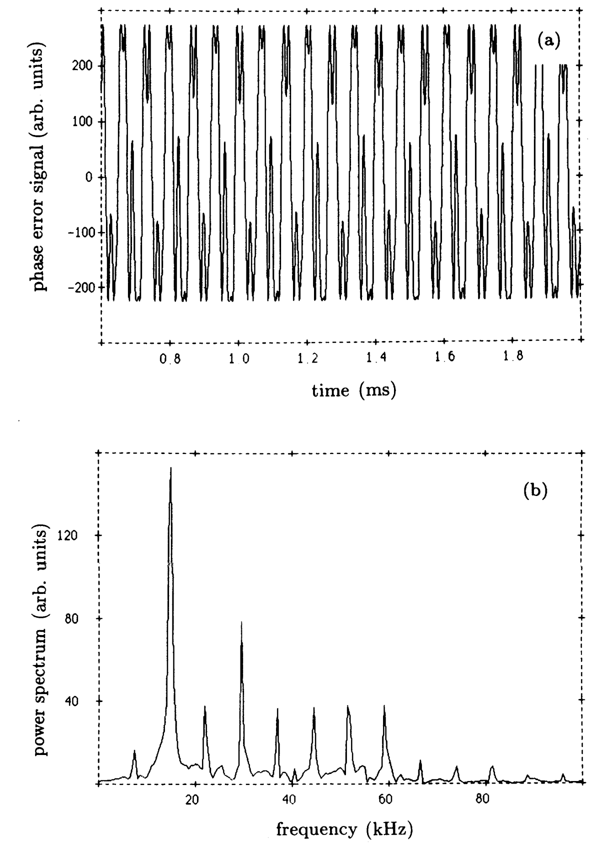

FIG. 6. Numerical simulations of the model equation for a nonvanishing frequency deviation at the onset of chaos. (a) Period-2 solution; (b) corresponding power spectrum.

3. *Cycle slipping behavior*

Experimentally, phase jumps in multiples of $\pm 2\pi$ cannot be detected at the output of a phase detector. This is due to the fact that a mixer is insensitive to signals whose phase is shifted by $\pm 2\pi$. Therefore we were led to modify our experimental setup in such a way that a detection of the cycle slipping motion became possible (Fig. 7).

The idea consists in the application of frequency divid-

TABLE I. Experimentally observed instabilities of the phase error signal for different values of the open-loop gain. The time delay was fixed at $\tau = 15$ μsec. For the amplitudes of the two oscillators we have chosen $A = B = +5$ dBm.

Open-loop gain K (kHz)	Instability	Locking condition
<81	dc output (stationary state)	Synchronous operation
81	Oscillatory instability	
180	Period-2 bifurcation	
187	Period-4 bifurcation	
188	Period-8 bifurcation	
>188	Chaos	
<211	Cycle slipping motion	Out of lock

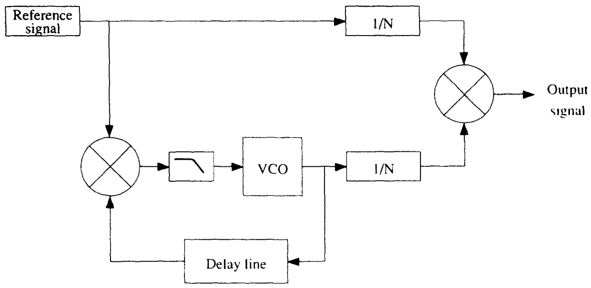

FIG. 7. Experimental apparatus to detect phase jumps in a first-order PLL with time delay.

ers in order to reduce the phase jumps to values lower than $\pm 2\pi$. At the output of an additionally introduced phase detector these jumps can be detected. In the experimental setup we chose digital frequency dividers with a division rate of $N=10$ which lead to phase jumps in multiples of $\pm 2\pi/10$. The jumps occur in a very irregular fashion. "Laminar" regions corresponding to a synchronized motion of the loop are interrupted by "bursts" destroying the synchronous operation (Fig. 8).

Quite surprisingly, a spectral analysis clearly demonstrates that the cycle slipping motion is associated with a $1/f$ spectrum [49,50]. We can exclude that this spectral distribution is due to a fluctuating environment, since white or $1/f^2$ noise sources introduced at the output of the first phase detector leave the spectrum unchanged. Therefore we conclude that the cycle slipping process is due to an intrinsic deterministic effect of the nonlinear dynamics of the loop. Recent investigations of the chaotic dynamics in Josephson junctions [51] leading to a similar cycle slipping motion of the quantum phase difference across the junction seem to confirm this conjecture.

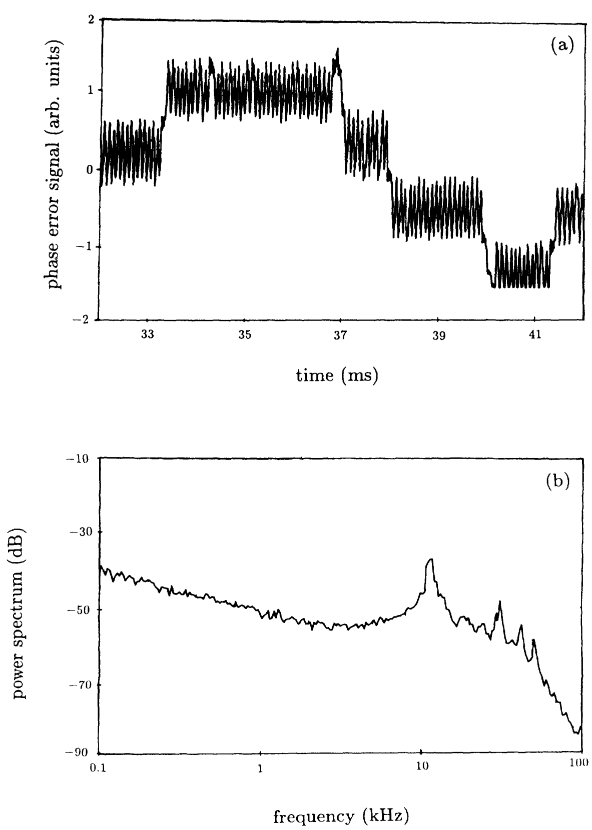

FIG. 8. Cycle slipping of a first-order PLL with time delay. (a) Temporal domain; (b) power spectrum.

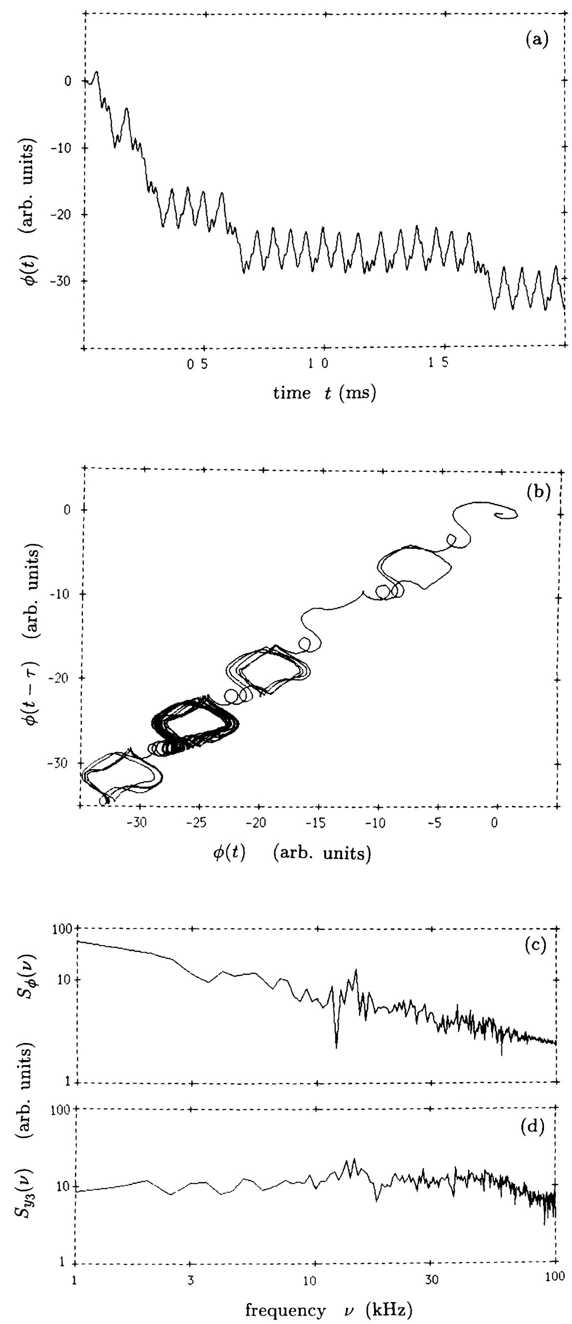

FIG. 9. Numerical simulations of the model equation (90) in the cycle slipping domain. (a) Temporal domain; (b) reconstruction of the attractors corresponding to the cycle slipping motion in the phase plane from a time series; (c) Fourier transform $S_\phi(V)$ of the phase error signal ϕ; (d) Fourier transform $S_{y_3}(V)$ of the output signal y_3.

From the numerical simulations of Eq. (90), as shown in Figs. 9(a)–9(d) we deduce that the cycle slipping motion occurs when the amplitude of the "bursts" crosses a separatrix between different attractors separated by $|\Delta\phi|=2\pi$. In this case the phase error signal no longer evolves in the attracting region of one attractor (synchronous operation) but jumps irregularly between different attractors (out-of-lock condition). In Fig. 9(b) the attractors corresponding to the cycle slipping motion are represented in the phase space spanned by $\phi(t-\tau)$ and $\phi(t)$. In Figs. 9(c) and 9(d) the spectral distribution of the phase error signal ϕ is compared to the one obtained at the output of the first phase detector y_3. In the latter case the $1/f$ spectrum is replaced by a nearly white continuous spectrum corresponding to the situation where phase jumps no longer appear.

In applications cycle slippings are an annoying feature since they influence significantly the quality properties of a control system. Therefore much effort has been made to tackle this problem. In most cases a drastic reduction of the number of cycle slippings per second can be achieved by minimizing the influences from a fluctuating environment. However, in the case where time delays are involved, there is no need for fluctuations to induce these phase jumps. Delay-induced instabilities leading to a cycle slipping motion are likely to occur, especially in cases where signals have to be tracked over a wide frequency range.

D. Conclusion

Our discussion of delay-induced instabilities has been based on several goals. In a first step we performed a formulation of delay equations as a special class of functional differential equations in an extended infinite-dimensional state space. From these considerations we were able to derive an elimination technique for the stable modes in the vicinity of an instability which goes far beyond "naive" adiabatic elimination techniques. The resulting order-parameter equations which determine the macroscopic behavior of the system close to an instability have been shown to be of the form of nonlinear ordinary differential equations. In developing these systematic methods we were guided by the fundamental concepts of synergetics, that is, the order-parameter concept and the so-called slaving principle. In a second step we described an experimental setup which could be modeled by a comparably simple nonlinear delay-differential equation. The first instability of this device has been extensively treated by our rigorous mathematical methods. Theoretical results have been successfully compared with experimental data. The experiments, however, exhibit a lot of further instabilities which cannot be handled analytically yet. This gap has partly been bridged by numerical simulations. In a future work we plan to further reduce the gap between the feasible analytical tools and observed delay-induced instabilities.

[1] K. Ikeda, Opt. Commun. **30**, 257 (1979).

[2] V. Ya. Kislov, N. N. Zalogin, and Ye. A. Myasin, Radio Eng. Electron. Phys. (USSR) **24**, 92 (1979).

[3] H. M. Gibbs, F. A. Hopf, D. L. Kaplan, and R. L. Shoemaker, Phys. Rev. Lett. **46**, 474 (1981).

[4] K. Ikeda, K. Kondo, and O. Akimoto, Phys. Rev. Lett. **49**, 1467 (1982).

[5] F. A. Hopf, D. L. Kaplan, H. M. Gibbs, and R. L. Shoemaker, Phys. Rev. A **25**, 2172 (1982).

[6] J. Y. Gao, L. M. Narducci, H. Sadiky, M. Squicciarini, and J. M. Yuan, Phys. Rev. A **30**, 901 (1984).

[7] R. Vallée and C. Delisle, Phys. Rev. A **31**, 2390 (1985).

[8] H. M. Gibbs, *Optical Bistability: Controlling Light with Light* (Academic, Orlando, 1985).

[9] P. Nardone, P. Mandel, and R. Kapral, Phys. Rev. A **33**, 2465 (1986).

[10] U. an der Heiden and M. C. Mackey, in *Temporal Disorder in Human Oscillatory Systems,* edited by L. Rensing, U. an der Heiden, and M. C. Mackey (Springer, New York, 1987).

[11] M. Le Berre, E. Ressayre, and A. Tallet, J. Opt. Soc. Am. B **5**, 1051 (1988).

[12] M. Le Berre, E. Ressayre, and A. Tallet, Opt. Commun. **72**, 123 (1989).

[13] K. Kaiser and D. Merkle, Phys. Lett. A **139**, 133 (1989).

[14] G. Giacomelli, M. Calzavara, and F. T. Arecchi, Opt. Commun. **74**, 97 (1989).

[15] E. Boe and H. C. Chang, Int. J. Bifurcation Chaos **1**, 67 (1991).

[16] F. T. Arecchi, G. Giacomelli, A. Lapucci, and R. Meucci, Phys. Rev. A **43**, 4997 (1991).

[17] E. V. Grigorieva, S. A. Kashchenko, N. A. Loiko, and A. M. Samson, Physica D **59**, 297 (1992).

[18] I. Hamilton, Phys. Rev. A **45**, 1259 (1992).

[19] J. D. Farmer, Physica D **4**, 366 (1982).

[20] M. Le Berre, E. Ressayre, A. Tallet, H. M. Gibbs, D. L. Kaplan, and M. H. Rose, Phys. Rev. A **35**, 4020 (1987).

[21] B. Dorizzi, B. Grammaticos, M. Le Berre, Y. Pomeau, E. Ressayre, and A. Tallet, Phys. Rev. A **35**, 328 (1987).

[22] K. Ikeda and O. Akimoto, in *Chaos and Statistical Methods,* edited by Y. Kuramoto (Springer, New York, 1984).

[23] K. Ikeda and K. Matsumoto, J. Stat. Phys. **44**, 955 (1986).

[24] M. Le Berre, E. Ressayre, A. Tallet, and Y. Pomeau, Phys. Rev. A **41**, 6635 (1990).

[25] L. Glass and M. C. Mackey, *From Clocks to Chaos* (Princeton University Press, Princeton, NJ, 1988).

[26] L. Glass, A. Beuter, and D. Larocque, in *Nonlinearity in Biology and Medicine,* edited by A. S. Perelson, B. Goldstein, M. Dembo, and J. A. Jadquez (Elsevier, New York, 1988).

[27] A. Longtin, J. G. Milton, J. E. Bos, and M. C. Mackey, Phys. Rev. A **41**, 6992 (1990).

[28] P. Chen, Syst. Dynamics Rev. **4**, 81 (1988).

[29] E. Niebur, H. G. Schuster, and D. M. Kammen, Phys. Rev. Lett. **67**, 1753 (1991).

[30] B. Dorizzi and B. Grammaticos, Phys. Rev. A **44**, 6958 (1991).

[31] F. Chapeau-Blondeau and G. Chauvet, Neural Networks **5**, 735 (1992).
[32] H. Haken, *Synergetics. An Introduction* (Springer, Berlin, 1977).
[33] H. Haken, *Advanced Synergetics* (Springer, Berlin, 1983).
[34] A. Wunderlin and H. Haken, Z. Phys. B **44**, 135 (1981).
[35] H. Haken and A. Wunderlin, Z. Phys. B **47**, 179 (1982).
[36] M. Le Berre, E. Ressayre, A. Tallet, and H. M. Gibbs, Phys. Rev. Lett. **56**, 274 (1986).
[37] P. Bergé, *Le Chaos,* Collection CEA, Série Synthèses (Eyrolles, Paris, 1988), p. 379.
[38] B. J. Hawdon, J. O'Gorman, and D. M. Hefferman, Z. Naturforsch., Teil A **46**, 686 (1991).
[39] J. K. Hale, *Theory of Functional Differential Equations* (Springer, New York, 1977).
[40] S. N. Chow and J. Mallet-Paret, J. Differ. Equ. **26**, 112 (1977).
[41] J. Carr, *Applications of Centre Manifold Theory* (Springer, New York, 1981).
[42] S. N. Chow, J. K. Hale, and W. Huang, Proc. R. Soc. Edinburgh, Sect. A **120**, 223 (1992).
[43] A. Blanchard, *Phase-Locked Loops* (Wiley, New York, 1976).
[44] W. F. Egan, *Frequency Synthesis by Phase Lock* (Wiley, New York, 1981).
[45] D. D'Humieres, M. R. Beasley, B. A. Huberman, and A. Libchaber, Phys. Rev. A **26**, 3483 (1982).
[46] M. Olivier and J. J. Gagnepain, in *Noise in Physical Systems and* $1/f$ *Fluctuations,* edited by M. Savelli, G. Lecoy, and J. P. Nougier (Elsevier, New York, 1983), p. 319.
[47] T. Endo and L. O. Chua, IEEE Trans. Circuits Syst. **35**, 987 (1988).
[48] T. Endo, L. O. Chua, and T. Narita, IEEE Trans. Circuits Syst. **35**, 255 (1989).
[49] W. Wischert, M. Olivier, and J. Groslambert, in *Noise in Physical Systems and* $1/f$ *Fluctuations,* edited by P. H. Handel and A. L. Chung (AIP, New York, 1993), p. 333.
[50] W. Wischert, *Anwendung synergetischer Konzepte auf Selbstorganisationsprozesse in zeitlich verzögerten Systemen* (Shaker, Aachen, 1993).
[51] W. J. Yeh and Y. H. Kao, Appl. Phys. Lett. **42**, 299 (1983).