

## Correspondence in quasiperiodic and chaotic maps: Quantization via the von Neumann equation

Joshua Wilkie and Paul Brumer

*Chemical Physics Theory Group, Department of Chemistry, University of Toronto, Toronto, Ontario, Canada M5S 1A1*

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A generalized approach to the quantization of a large class of maps on a torus, i.e., quantization via the von Neumann equation, is described and a number of issues related to the quantization of model systems are discussed. The approach yields well-behaved mixed quantum states for tori for which the corresponding Schrödinger equation has no solutions, as well as an extended spectrum for tori where the Schrödinger equation can be solved. Quantum-classical correspondence is demonstrated for the class of mappings considered, with the Wigner-Weyl density  $\rho(p, q, t)$  going to the correct classical limit. An application to the cat map yields, in a direct manner, nonchaotic quantum dynamics, plus the exact chaotic classical propagator in the correspondence limit.

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### I. INTRODUCTION

Quantization of systems with classical analogs proceed by a well known procedure. Specifically, one replaces coordinates or momenta by operators in the Hamiltonian, constructs the Schrödinger equation and imposes boundary conditions which arise directly from the classical physics. In this paper we show for the case of maps on a torus, model systems in nonlinear dynamics, that modifications to this procedure are essential to allow the quantum dynamics to unambiguously approach the correct classical limit.

Area-preserving mappings on the torus are model systems which have long been used to illustrate the essence of regular or chaotic classical dynamics because they are of low dimensionality and can be readily propagated. These maps are essentially systems which are confined to finite coordinate- and momentum-space regions  $0 \leq q < a$ ,  $0 \leq p < b$ , with rules of dynamic evolution which confine the system to this region through a dependence on  $q \bmod a$ ,  $p \bmod b$ . In addition, time evolution is in discrete steps. Much of the current interest [1-6] in quantizing these mappings stems from the expectation that they will prove useful in understanding the role of classical chaos in quantum dynamics and in understanding quantum-classical correspondence. Indeed, little is known about quantum-classical correspondence for systems whose classical analog is chaotic [7]. While the correspondence principle requires that quantum mechanics should produce the classical laws in the limit that Planck's constant approaches zero [8], the characteristics of the limit are such as to make this verification extremely difficult [7].

Central to the issues of classical-quantum correspondence is the specific prescription used to quantize a given classical system. In this paper we show that an alternative extended quantization procedure allows a method of quantizing mappings on a torus which yields solutions having a number of desirable features which do not arise

in the traditional quantization via the Schrödinger equation.

Motivation for reconsidering the quantization procedure for maps on a torus stems from an examination of the results of previous efforts in this area. Hannay and Berry [2] and Balazs and Voros [5] chose to directly quantize the Schrödinger propagator while Ford and co-workers [1] quantize certain maps by introducing a kicked-oscillator Hamiltonian ( $-\infty < q, p < \infty$ )

$$H(t) = p^2/2\mu + \epsilon q^2/2 \sum_{s=-\infty}^{\infty} \delta(s - t/T). \quad (1)$$

The dynamics generated by this Hamiltonian stroboscopically produces the classical dynamics of the map. They then construct the propagator for this system. The restriction of the classical dynamics to a torus  $0 \leq q < a$ ,  $0 \leq p < b$  is introduced by imposing periodic boundary conditions on the wave functions in both the position and momentum representations. The results of this procedure have been the subject of considerable controversy. Specifically, Ford and co-workers [1] showed that although classical cat map dynamics is algorithmically complex, the quantum cat does not show this type of random behavior in the classical limit. Berry and co-workers, however, argue [7] that such measures are less useful indicators of correspondence failure than believed by Ford and co-workers. Further, Graffi and co-workers [9] have recently shown that the quantum cat is mixing in a specific limit,  $N \rightarrow \infty$  along the subsequence of  $N$  prime, where  $\hbar = 1/N$ . This is in contrast with Ford's demonstration of quasiperiodic dynamics when taking a somewhat different approach to the classical limit.

Clearly the traditional quantization procedure leads to a quantum cat map whose behavior as  $\hbar \rightarrow 0$  is, at best, subtle. In addition, the traditional approach leads to two other features which we regard as undesirable: (a) only a restricted class of classical tori, those with  $ab = \hbar N$ ,  $N$  integer, can be quantized [10]. Indeed, some mappings

can only be quantized for  $N$  even [1,5]; and (b) the resultant wave functions are highly singular, being a set of  $\delta$  functions.

In this paper we show that introducing a translationally invariant Hamiltonian, quantizing the associated von Neumann (quantum Liouville) equation  $\partial\hat{\rho}/\partial t = [H, \hat{\rho}]/i\hbar$  for the density matrix  $\hat{\rho}$ , and obtaining solutions satisfying Bloch boundary conditions, allows quantization of tori for all  $ab$ , is physically complete, leads to smooth well behaved solutions and is measure preserving. Further, we show that the correct classical propagator and the exact classical dynamics emerge directly from the quantum propagator in the classical limit, with the limit taken as the complete sequence  $N \rightarrow \infty$ . Hence the classical limit of our quantum propagator for the cat map generates algorithmically complex dynamics, the system is mixing, etc.

Note that our approach yields density-matrix solutions for cases (e.g.,  $ab \neq hN$ ) for which there are no pure states. The fact that density-matrix solutions exist when wave functions do not is of considerable interest and is emphasized below.

This paper is organized as follows: Sec. II summarizes aspects of the traditional quantization procedure and treats the case of  $\epsilon = 0$  in Eq. (1) explicitly. The extended quantization procedure is then introduced in Sec. III where it is also applied to the case of Eq. (1) with  $\epsilon = 0$ . Examining this case allows us to emphasize the different spectra and solutions obtained by this approach and to show that both eigenfunctions and eigenvalues of the von Neumann equation directly approach the classical limit. In Sec. IV we generalize the discussion to the case of  $\epsilon \neq 0$  and use the von Neumann quantization scheme to obtain the quantum propagator. Classical and quantum map dynamics are expressed in a form suitable for the study of correspondence. Section V then explores the quantum dynamics in the correspondence limit where we show that all maps treated reduce to the correct classical limit. In the application to the quantum cat we show that its quantum dynamics is not ergodic, but that the chaotic classical cat is nonetheless recovered in the  $\hbar \rightarrow 0$  limit. Throughout this treatment we make extensive use of the Wigner-Weyl representation [11,12], a  $(p, q)$ -dependent representation of quantum mechanics whose form encourages a direct comparison with classical dynamics. In Sec. VI we address the question of why there are no pure states for the quantum cat, showing that the system best resembles a discretized Langevin process. Finally, Sec. VII contains a summary and remarks on future work.

## II. TRADITIONAL QUANTIZATION

Constructing the appropriate quantum generalization of a given classical system requires a correspondence rule for replacing classical observables by quantum operators, a choice of a dynamical state equation and a statement of the associated boundary conditions. The appropriate dynamical state equations are normally assumed to be the Schrödinger equation for pure-state dynamics and the

von Neumann equation (or quantum Liouville equation) for mixed-state dynamics and, customarily, the choice of boundary condition is dictated by features of the physical system. Quantum mappings on the torus have no known physical realization, and so some freedom would appear to exist for the choice of boundary conditions. Below we discuss the limitations of quantization via the Schrödinger equation, both for (previously adopted) periodic boundary conditions as well as for Bloch boundary conditions. We later show that a physically complete theory of quantum mappings on a torus does result from a von Neumann based quantization procedure in concert with Bloch type boundary conditions. Of general interest is the different way in which the boundary conditions, implemented in these two procedures and both apparently consistent with the classical map, affect the quantum picture.

### A. Boundary conditions

Traditional quantization procedures [1,2] define the dynamics on the full phase space,  $-\infty < p, q < \infty$ , but impose periodic boundary conditions on the wave function in both the coordinate  $[\psi(q)]$  and momentum representations  $[\bar{\psi}(p)]$ , i.e.,

$$\psi(q) = \psi(q + a) \equiv T_q(a)\psi(q), \quad (2)$$

$$\bar{\psi}(p) = \bar{\psi}(p + b) \equiv T_p(b)\bar{\psi}(p), \quad (3)$$

where Eqs. (2) and (3) implicitly define the coordinate and momentum translation operators

$$T_q(a) = \exp(i\hat{p}a/\hbar), \quad T_p(b) = \exp(-i\hat{q}b/\hbar). \quad (4)$$

As a consequence of Eqs. (2) and (3), dynamics within each  $(q, p)$  unit cell is expected to reflect the character of torus dynamics [i.e., dynamics with variables  $(q \bmod a)$  and  $(p \bmod b)$ ]. From Eq. (3) it follows that  $\{1 - e^{-ibq/\hbar}\}\psi(q) = 0$  which implies that  $\psi(q) = 0$  unless  $q = nh/b$  where  $n \in \mathbb{Z}$ . Similarly, from Eq. (2) it follows that  $\{1 - e^{iap/\hbar}\}\bar{\psi}(p) = 0$  which implies  $\bar{\psi}(p) = 0$  unless  $p = mh/a$  where  $m \in \mathbb{Z}$ . Imposing both conditions simultaneously gives  $\{1 - e^{iab/\hbar}\}\psi(q) = 0$  and  $\{1 - e^{-iab/\hbar}\}\bar{\psi}(p) = 0$  so that we must have  $ab = hN$  where  $N \in \mathbb{Z}$ . Fixing  $N$  one then obtains the class of allowed wave functions

$$\psi(q) = \sqrt{a} \sum_{j=-\infty}^{\infty} \psi_j \delta(q - ja/N), \quad (5)$$

with  $\psi_{j+N} = \psi_j$ . This condition implies that

$$\psi_j = \sum_{l=1}^N \alpha_l e^{2\pi ijl/N}, \quad (6)$$

where the  $\alpha_l$  coefficients are arbitrary. In the momentum representation these states take the form

$$\bar{\psi}(p) = \sqrt{b} \sum_{j=-\infty}^{\infty} \sum_{k=1}^N \bar{\psi}_k \delta(p - (j + k/N)b), \quad (7)$$

where  $\bar{\psi}_k = \sqrt{N}\alpha_k$ . Thus, in general (independent of the Hamiltonian  $H$ ) periodic boundary conditions in  $p, q$  lead to a discrete spectrum with non-normalizable wave functions, comprised of sets of  $\delta$  functions. The traditional interpretation of  $|\psi|^2$  as a probability is inapplicable. Note that it is impossible to quantize any mapping on a torus by this route unless  $ab = hN$ , a consequence of the Fourier transform relationship between  $p$  and  $q$  and of the requirement to simultaneously fit waves in both the  $q$  and  $p$  directions. With  $ab \neq hN$  neither the Hamiltonian, nor any other operator, has eigenvalues. The quantization procedure is thus incomplete insofar as only a small number of classical tori can be quantized [10]. Finally, note that these conclusions are independent of the system and so apply to any mapping confined to a torus.

The Wigner-Weyl picture [11,12] provides a  $(p, q)$ -dependent quantum representation in the density-matrix formulation which allows a direct comparison with classical mechanics in phase space. One can readily demonstrate that in this picture the dynamics lies on points in phase space which are rational multiples of  $a$  and  $b$ . Ford and co-workers show [1] that this behavior survives in the classical limit and is responsible for nonalgorithmically complex dynamics for the classical limit of the quantum cat. It is obviously also the case for any map quantized by this procedure.

The situation is similar if Bloch type boundary conditions are adopted. That is, Bloch boundary conditions on the wave function

$$\psi(q+a) = e^{ip_0 a/\hbar} \psi(q), \quad (8)$$

$$\bar{\psi}(p+b) = e^{-iq_0 b/\hbar} \bar{\psi}(p),$$

with constant  $q_0, p_0$  again yield states

$$\psi(q) = \sqrt{a} \sum_{j=-\infty}^{\infty} \psi_j \delta(q - q_0 - ja/N) \quad (9)$$

only when  $ab = hN$ . Here

$$\psi_j = e^{ip_0 a j/N} \sum_{l=1}^N \alpha_l e^{2\pi i j l/N},$$

where the  $\alpha_l$  coefficients are arbitrary. In the momentum representation

$$\bar{\psi}(p) = \sqrt{b} e^{-ipq_0/\hbar} \sum_{j=-\infty}^{\infty} \sum_{k=1}^N \bar{\psi}_k \delta(p - p_0 - (j+k/N)b), \quad (10)$$

where  $\bar{\psi}_k = \sqrt{N}\alpha_k$ . Bloch boundary conditions will, however, prove useful in the extended quantization scheme discussed below.

### B. The case of $\epsilon = 0$

Consider now, for illustration purposes, the case of  $\epsilon = 0$ . Here Eq. (1) gives  $N = ab/h$  independent Hamil-

tonian eigenfunctions [13(a)]

$$\bar{\psi}_l(p) = \sqrt{b/N} \sum_{j=-\infty}^{\infty} \delta(p - (j+l/N)b) \quad (11)$$

and associated discrete spectrum  $E_l = (lb/N)^2/2\mu$ ,  $l = 1, \dots, N$ . We also consider, for comparison with results below, the solution from the perspective of the von Neumann (or quantum Liouville) equation:

$$\partial \hat{\rho} / \partial t = -i[H, \hat{\rho}] / \hbar \equiv -iL_q \hat{\rho}. \quad (12)$$

Here  $\hat{\rho}$  is the density operator and Eq. (12) defines the quantum Liouville operator  $L_q$ . The eigenvalues  $\lambda_{k,l}$  and eigenfunctions  $\hat{\rho}_{k,l}$  of the  $L_q$  satisfy

$$L_q \hat{\rho}_{k,l} = \frac{1}{\hbar} [H, \hat{\rho}_{k,l}] = \lambda_{k,l} \hat{\rho}_{k,l}. \quad (13)$$

Both may be obtained directly from the Hamiltonian solutions and the latter conveniently expressed in the Wigner-Weyl representation. Specifically  $(k, l = 1, \dots, N)$ :

$$\lambda_{k,l} = (E_k - E_l) / \hbar = (k^2 - l^2)b^2 / (2\mu N^2) \quad (14)$$

and  $\hat{\rho}_{k,l} = |\psi_k\rangle\langle\psi_l|$  so that the Wigner-Weyl representation of  $\hat{\rho}_{k,l}$ , denoted  $\rho_{k,l}(p, q)$ , is given by [11,12]

$$\begin{aligned} \rho_{k,l}(p, q) &= h^{-1} \int_{-\infty}^{\infty} dq' \langle q - q'/2 | \psi_k \rangle \\ &\quad \times \langle \psi_l | q + q'/2 \rangle e^{ipq'/\hbar} \\ &= h^{-1} \int_{-\infty}^{\infty} dp' \langle p + p'/2 | \psi_k \rangle \\ &\quad \times \langle \psi_l | p - p'/2 \rangle e^{iqp'/\hbar}. \end{aligned} \quad (15)$$

Using Eq. (11) we have

$$\begin{aligned} \rho_{k,l}(p, q) &= 2a^{-1} \sum_{n,m=-\infty}^{\infty} e^{2\pi i(n-m)qN/a} e^{2\pi i(k-l)q/a} \\ &\quad \times \delta\left(2p - \left(n + m + \frac{k+l}{N}\right)b\right). \end{aligned} \quad (16)$$

A simple change in the integers of summation in this expression for  $\rho_{k,l}(p, q)$  and use of the identity [14]

$$\sum_{j=-\infty}^{\infty} e^{2\pi i j x} = \sum_{j=-\infty}^{\infty} \delta(x - j) \quad (17)$$

gives

$$\begin{aligned} \rho_{k,l}(p, q) &= (2N)^{-1} e^{2\pi i(k-l)q/a} \\ &\quad \times \sum_{n,m=-\infty}^{\infty} (-1)^{mn} \delta(q - ma/2N) \\ &\quad \times \delta\left(p - \left(n + \frac{k+l}{N}\right)b/2\right). \end{aligned} \quad (18)$$

Thus the eigenfunctions of  $L_q$  in this representation are  $\delta$  functions localized on a discrete number of points of the torus which are rational multiples of  $a$  and  $b$ , i.e., a

$\delta$  function “brush” in phase space. Note that Eq. (17) is repeatedly used throughout this paper although it is not always cited.

The results in this subsection are used later below for comparison with the extended quantization results.

### III. EXTENDED QUANTIZATION

An examination of a different approach to map quantization procedure is hence well motivated. The extended quantization procedure which we advocate entails three new features: (a) the introduction of “mod operators” which produce a properly symmetrized Hamiltonian, (b) solving the von Neumann [Eq. (12)], rather than Schrödinger, equation and (c) use of Bloch boundary conditions on the density matrix. The latter is motivated immediately below. [Throughout this paper, as above, we denote quantum density operators by  $\hat{\rho}$  and the Wigner-Weyl representation by  $\rho(p, q)$ . In addition, analog quantities in classical mechanics, such as the phase-space density, will carry a “c” superscript.]

#### A. Boundary conditions

Consider then the von Neumann equation with either of the two choices of boundary condition, i.e., periodic and Bloch type. From the perspective of the density matrix, periodic boundary conditions on the wave function imply  $T\hat{\rho} = \hat{\rho}$ ,  $\hat{\rho}T = \hat{\rho}$  for  $T = T_q(a)$  and  $T = T_p(b)$ . In the Wigner-Weyl representation [11]

$$\rho(p, q) = h^{-1} \int_{-\infty}^{\infty} dq' \langle q - q'/2 | \hat{\rho} | q + q'/2 \rangle e^{ipq'/\hbar} \quad (19)$$

these conditions take the form

$$e^{ipa/\hbar} \rho(p, q \pm a/2) = e^{-ipa/\hbar} \rho(p, q \pm a/2) = \rho(p, q), \quad (20)$$

$$e^{iqb/\hbar} \rho(p \pm b/2, q) = e^{-iqb/\hbar} \rho(p \pm b/2, q) = \rho(p, q).$$

The following requirements on the density result from Eq. (20): (a)  $(1 - e^{i2qb/\hbar})\rho(p, q) = 0$  which implies  $q = hl/2b$  where  $l \in \mathbb{Z}$ , (b)  $(1 - e^{i2pa/\hbar})\rho(p, q) = 0$  which implies  $p = hk/2a$  where  $k \in \mathbb{Z}$ , and (c)  $(1 - e^{iab/\hbar})\rho(p, q) = 0$  which implies that  $ab = hN$  where  $N \in \mathbb{Z}$ . The resulting states are mixtures of the  $\delta$ -type wave functions of Eq. (5) localized on the rational points of the torus and they exist only when  $ab = hN$ . Hence this combination of dynamic law and boundary condition regenerate the same difficulty as that associated with the Schrödinger approach and is, once again, independent of the system. This is the case since Eq. (20) is an exact statement of the requirement of periodicity on the wave function, transferred to the density matrix. It results in precisely the same restriction as it did in its application in the wave function picture [15].

A similar situation does not, however, arise with the application of Bloch boundary conditions to the density matrix  $\rho$ . Specifically, here Eq. (8) implies that

$$\begin{aligned} \langle q + a | \hat{\rho}_{jk} | q + a \rangle &= \langle q' + a | \psi_j \rangle \langle \psi_k | q + a \rangle \\ &= \langle q' | \psi_j \rangle \langle \psi_k | q \rangle \\ &= \langle q' | \hat{\rho}_{jk} | q \rangle \end{aligned} \quad (21)$$

with a similar result in the momentum-space picture. Thus the phase  $\exp(ip_0a/\hbar)$  associated with the wave function drops out, resulting in a quite different boundary condition in the density-matrix picture than in the wave function picture. Specifically, the conditions  $T^{-1}\hat{\rho}T = \hat{\rho}$  imply Bloch boundary conditions on the wave function  $\psi$  if  $\hat{\rho} = |\psi\rangle\langle\psi|$ , but densities  $\hat{\rho}$  can exist which satisfy  $T^{-1}\hat{\rho}T = \hat{\rho}$  and which do not take the form of a weighted sum over pure states satisfying Bloch boundary conditions on the wave function. These boundary conditions will prove extremely useful in conjunction with the introduction of mod operators, as described below.

#### B. Mod operators

We consider solutions to the von Neumann equation with Bloch type boundary conditions. To do so we first introduce the operators  $(\widehat{p \bmod b})$  and  $(\widehat{q \bmod a})$ . These are obtained as a direct extension of the classical Fourier expansions for  $(q \bmod a)$  and  $(p \bmod b)$ , i.e., as extensions of

$$(q \bmod a)/a = Q \bmod 1 = \frac{1}{2} + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{i}{2\pi m} f_{0,m}, \quad (22)$$

$$(p \bmod b)/b = P \bmod 1 = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{i}{2\pi n} f_{n,0},$$

where  $f_{n,m}(P, Q) = \exp[2\pi i(nP + mQ)]$  and where we have introduced the convenient dimensionless variables  $Q = q/a, P = p/b$ . Specifically, the quantum operators, analogs of Eq. (22), are

$$(\widehat{p \bmod b})/b = \widehat{P \bmod 1} = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{i}{2\pi n} \hat{f}_{n,0}, \quad (23)$$

with eigenvalues  $P \bmod 1$  and conjugate variable [17(a)]

$$(\widehat{q \bmod a})/a = \widehat{Q \bmod 1} = \frac{1}{2} + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{i}{2\pi m} \hat{f}_{0,m} \quad (24)$$

with eigenvalues  $Q \bmod 1$ . Here the unitary operators  $\hat{f}_{n,m}$  are obtained by Weyl quantizing [11]  $f_{n,m}(P, Q) = \exp[2\pi i(nP + mQ)]$ , i.e.,

$$\begin{aligned} \hat{f}_{n,m} &= h^{-1} \int dp \int dq e^{2\pi i(nP + mQ)} \int dv e^{ipv/\hbar} |q + v/2\rangle \\ &\quad \times \langle q - v/2| \\ &= \int_{-\infty}^{\infty} dq e^{2\pi imQ} |q - n\alpha a/2\rangle \langle q + n\alpha a/2| \\ &= \int_{-\infty}^{\infty} dp e^{2\pi inP} |p + m\alpha b/2\rangle \langle p - m\alpha b/2|, \end{aligned} \quad (25)$$

where  $\alpha = h/ab$  is a dimensionless form of Planck's constant. Although Eq. (25) provides forms most useful for computations, applying the coordinate translation operator inside the integral allows us to rewrite Eq. (25) in a far more attractive form:

$$\hat{f}_{n,m} = \exp\{2\pi i(n\hat{P} + m\hat{Q})\}, \quad (26)$$

where  $(\hat{Q}, \hat{P}) = (\hat{q}/a, \hat{p}/b)$  are the scaled coordinate and momentum operators. The  $\hat{f}_{n,m}$  operators [16], whose properties are discussed in Appendix A, will allow us to write the quantum operator analog of the Fourier transforms necessary to treat periodic systems on tori.

The introduction of these operators allows us to revise the Hamiltonian in Eq. (1) to read

$$H = (\widehat{p \bmod b})^2/2\mu + \epsilon(\widehat{q \bmod a})^2/2 \sum_{s=-\infty}^{\infty} \delta(s - t/T). \quad (27)$$

This new Hamiltonian satisfies  $[H, T_p(b)] = [H, T_q(a)] = 0$ , i.e.,  $H$  reflects the desired cell-like character of the full phase space. This character is also demanded of the eigenfunctions  $\hat{\rho}_{i,j}$  of the von Neumann equation, i.e., by requiring the generalized Bloch boundary conditions on  $\hat{\rho}$ ,

$$\begin{aligned} \hat{\rho} &= T_p^{-1}(b)\hat{\rho}T_p(b), \\ \hat{\rho} &= T_q^{-1}(a)\hat{\rho}T_q(a). \end{aligned} \quad (28)$$

These conditions have the form, in the Wigner-Weyl representation, of

$$\begin{aligned} \rho(p, q + a) &= \rho(p, q), \\ \rho(p + b, q) &= \rho(p, q). \end{aligned} \quad (29)$$

Hence these conditions correspond precisely to the boundary conditions one would impose on a classical phase-space density if the density (rather than the wave function) were periodic in  $p, q$ .

States satisfying Eqs. (29) must, by Fourier's theorem, be of the form

$$\begin{aligned} \rho(p, q) &= \frac{1}{ab} \sum_{n,m=-\infty}^{\infty} \rho_{n,m} f_{n,m} \\ &= \frac{1}{ab} \sum_{n,m=-\infty}^{\infty} \rho_{n,m} \exp\{2\pi i(nP + mQ)\} \end{aligned} \quad (30)$$

in the Wigner-Weyl representation and so distributions satisfying Bloch boundary conditions are expected to satisfy the analogous quantum Fourier expansion:

$$\begin{aligned} \hat{\rho} &= \frac{1}{ab} \sum_{n,m=-\infty}^{\infty} \rho_{n,m} \hat{f}_{n,m} \\ &= \frac{1}{ab} \sum_{n,m=-\infty}^{\infty} \rho_{n,m} \exp\{2\pi i(n\hat{P} + m\hat{Q})\}. \end{aligned} \quad (31)$$

We utilize this approach below to examine the case of  $\epsilon = 0$ . Specifically, we solve the von Neumann equation [Eq. (12)] using the Hamiltonian of Eq. (27) coupled with the Bloch boundary conditions which require Eq. (31). The general case  $\epsilon \neq 0$  is considered in Sec. III.

### C. The case of $\epsilon = 0$

Consider the extended quantization procedure applied to the case of  $\epsilon = 0$ . The relevant Hamiltonian is then [13(b)]

$$H = (\widehat{p \bmod b})^2/2\mu = b^2(\widehat{P \bmod 1})^2/2\mu \quad (32)$$

and the associated von Neumann equation

$$L_q \hat{\rho} = \frac{b^2}{2\mu\hbar} [(\widehat{P \bmod 1})^2, \hat{\rho}] = i\partial\hat{\rho}/\partial t. \quad (33)$$

Eigenfunctions of the Liouville operator satisfy

$$L_q \hat{\rho} = \lambda \hat{\rho}. \quad (34)$$

Inserting Eqs. (31) and (33) into Eq. (34) we obtain

$$\begin{aligned} \frac{b^2}{2\mu\hbar} \sum_{n,m=-\infty}^{\infty} \rho_{n,m} [(\widehat{P \bmod 1})^2, \hat{f}_{n,m}] \\ = \lambda \sum_{n,m=-\infty}^{\infty} \rho_{n,m} \hat{f}_{n,m}. \end{aligned} \quad (35)$$

Using Eq. (23) it can be readily shown that, for arbitrary  $P_0$  (i.e., arbitrary  $p_0/b$ ),

$$\begin{aligned} \sum_{n=-\infty}^{\infty} e^{-2\pi i n P_0} [(\widehat{P \bmod 1})^2, \hat{f}_{n,m}] \\ = \{[(P_0 + m\alpha/2) \bmod 1]^2 - [(P_0 - m\alpha/2) \bmod 1]^2\} \\ \times \sum_{n=-\infty}^{\infty} e^{-2\pi i n P_0} \hat{f}_{n,m}. \end{aligned} \quad (36)$$

Hence we can choose  $\rho_{n,m} = \delta_{m,j} e^{-2\pi i n P_0}$ . The eigenfunctions and eigenvalues of  $L_q$  with Bloch boundary conditions are then given by

$$\hat{\rho}_{j,p_0} = (ab)^{-1} \sum_{n=-\infty}^{\infty} e^{2\pi i [n(\hat{p}-p_0)/b + j\hat{q}/a]} \quad (37)$$

or, alternatively, using the momentum representation [Eq. (25)] of  $\hat{f}_{n,m}$  and Eq. (17)

$$\hat{\rho}_{j,p_0} = a^{-1} \sum_{n=-\infty}^{\infty} |p_0 + nb + \pi j\hbar/a\rangle \langle p_0 + nb - \pi j\hbar/a| \quad (38)$$

with eigenvalues

$$\lambda_{j,p_0} = \frac{b^2}{2\mu\hbar} \left\{ \left[ \left( \frac{p_0}{b} + \frac{\pi j \hbar}{ab} \right) \bmod 1 \right]^2 - \left[ \left( \frac{p_0}{b} - \frac{\pi j \hbar}{ab} \right) \bmod 1 \right]^2 \right\}, \quad (39)$$

where  $j$  is an integer,  $0 \leq p_0 < b$ , and the arguments of the bras and kets in Eq. (38) correspond to eigenvalues of  $\hat{p}$ . The Wigner-Weyl representation  $\rho_{j,p_0}(p, q)$  of these solutions makes evident their character:

$$\rho_{j,p_0}(p, q) = a^{-1} e^{2\pi i j q/a} \sum_{n=-\infty}^{\infty} \delta(p - p_0 - nb). \quad (40)$$

Stationary states ( $\lambda_{j,p_0} = 0$ ) are of two types, arising from either  $j = 0$  or  $p_0 = 0$ . The stationary states with  $j = 0$  are uniform in  $q$  and belong to the point spectrum, whereas the  $p, q$  integral over the stationary states with  $j \neq 0$  and  $p_0 = 0$  are zero, characteristic of elements of the continuous spectrum. The overall spectrum of  $L_q$  is continuous, due to the continuous  $p_0$  label.

Several aspects of these solutions are worth emphasizing. First, Eqs. (38) and (39) provide solutions for all values of  $a$  and  $b$ . Second, the eigenfunctions in Eq. (38) are, in general, mixed states, since wave functions exist only for  $ab = hN$  whereas Eq. (38) applies to all  $ab$ . If the condition  $ab = hN$  is satisfied then it is possible to recombine some of the degenerate Liouville eigenstates to produce pure states. Even then, however, the pure states are only a small fraction of all of the possible solutions.

To permit a comparison of Eqs. (16), (38), and (39) to the classical limit, consider solutions to the classical Liouville equation for motion on a torus with the appropriate classical boundary conditions [which are the same as Eq. (29)]. That is, we consider the classical dynamics via the Liouville equation [12]:

$$\partial \rho^{(c)} / \partial t = \{H, \rho^{(c)}\} \equiv -iL_c \rho^{(c)}. \quad (41)$$

Here  $\{, \}$  denotes the Poisson bracket,  $\rho^{(c)}$  is the classical phase-space density, and this equation defines the classical Liouville operator  $L_c$ . Eigenfunctions and eigenvalues of  $L_c$  are given by the solutions to the classical Liouville eigenvalue problem

$$L_c \rho^{(c)} = -i(p \bmod b/\mu) \partial \rho^{(c)} / \partial q = \lambda^{(c)} \rho^{(c)}. \quad (42)$$

Solutions to Eq. (42) are expected to satisfy Eq. (29). Hence  $\rho^{(c)}$  must take the form of Eq. (30) and so (with classical Fourier coefficients being denoted  $\rho_{n,m}^{(c)}$ )

$$\sum_{n,m=-\infty}^{\infty} \rho_{n,m}^{(c)} \left[ \frac{2\pi(p \bmod b)m}{\mu a} - \lambda^{(c)} \right] e^{2\pi i(np/b + mq/a)} = 0. \quad (43)$$

Using Eq. (23) and taking matrix elements in the Fourier basis

$$\left[ \lambda^{(c)} - \frac{\pi m b}{\mu a} \right] \rho_{n,m}^{(c)} = -i \frac{m b}{\mu a} \sum_{k \neq n}^{\infty} \frac{\rho_{k,m}^{(c)}}{k - n}. \quad (44)$$

We assume a solution of the form  $\rho_{n,m}^{(c)} = \Omega_m e^{-in\theta_m}$  and substitute this expression into Eq. (44). We obtain

$$\left[ \lambda^{(c)} - \frac{\pi m b}{\mu a} \right] \Omega_m e^{-in\theta_m} = -i \frac{m b}{\mu a} \Omega_m e^{-in\theta_m} \sum_{k \neq 0}^{\infty} \frac{e^{-ik\theta_m}}{k} \quad (45)$$

or

$$\lambda^{(c)} - \frac{\pi m b}{\mu a} = -\frac{2mb}{\mu a} \sum_{k=1}^{\infty} \frac{\sin(k\theta_m)}{k}. \quad (46)$$

Using the identity [19]

$$\sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k} = \frac{\pi - \theta}{2} \quad (47)$$

we have

$$\theta_m = \frac{\mu a \lambda^{(c)}}{m b}. \quad (48)$$

Defining  $\theta_m/2\pi = p_0/b$ ,  $p_0 \in [0, b)$ , and choosing  $\Omega_m = \delta_{m,j}$  so that  $\rho_{n,m}^{(c)} = \delta_{m,j} e^{-2\pi i n p_0/b}$  gives resultant classical eigensolutions

$$\begin{aligned} \rho_{j,p_0}^{(c)}(p, q) &= (ab)^{-1} e^{2\pi i j q/a} \sum_{n=-\infty}^{\infty} e^{2\pi i n(p-p_0)/b} \\ &= a^{-1} e^{2\pi i j q/a} \sum_{n=-\infty}^{\infty} \delta(p - p_0 - nb), \end{aligned} \quad (49)$$

with associated eigenvalues

$$\lambda_{j,p_0}^{(c)} = 2\pi j p_0 / \mu a \quad (50)$$

with  $0 \leq p_0 < b$ . Here the second identity results from an application of Eq. (17).

A comparison of the classical [Eqs. (49) and (50)] and quantum [Eqs. (37) and (39)] results shows that the quantum and classical eigenfunctions are identical and the quantum eigenvalues go to the proper classical limit as  $\hbar \rightarrow 0$ . This is not the case with the results of the traditional quantization procedure, as discussed by Ford and co-workers [1].

#### D. Remarks on extended quantization

The procedure above involves two modifications to the standard quantization approach: (a) the introduction of modular variables and operators into both the classical and quantum problem, and (b) the use of the von Neumann equation in quantum mechanics. The need for the latter stems from the fact that the solutions which we find are, in fact, not pure states and are not comprised of averages over pure states. That is, we find that solutions to the von Neumann equation exist where Hamiltonian eigenfunctions do not. The need for modular variables and operators is also worth emphasizing. Specif-

ically, note that the solutions Eq. (38) and Eq. (40) are not eigenfunctions of their respective quantum and classical Liouville operators if these operators are written in terms of nonmodular  $p, q$  variables. Indeed eigenfunctions of the Liouville operators with Cartesian  $p, q$  variables and periodic boundary conditions do not exist for mappings on the torus. This is easily seen as follows. Consider the Liouville problem in  $p, q$ . Since the Hamiltonian in Eq. (1) with  $\epsilon = 0$  is quadratic it gives rise to a quantum Liouville operator which, in the Wigner-Weyl representation, is identical to the classical Liouville operator. Thus, to obtain both the quantum and classical Liouville eigenfunctions for  $\epsilon = 0$  we would need only consider the classical Liouville eigenequation  $L_c \rho^{(c)} = \lambda^c \rho^{(c)}$  where  $L_c = -i(p/\mu)\partial/\partial q$ . Taking matrix elements in the Fourier basis would again yield Eq. (44) which as we have seen has the solutions given by Eq. (40). But consideration of these solutions shows that they are eigensolutions of  $L_c = -i(p \bmod b/\mu)\partial/\partial q$ , and not of  $L_c = -i(p/\mu)\partial/\partial q$ . Thus the procedure, starting with Cartesian  $p, q$  variables, generates an inconsistency whose origin is readily apparent. That is, although the Hamiltonian is of the form given by Eq. (1), the Hilbert space chosen to solve the problem is that spanned by the Fourier basis functions. But the Hamiltonian in Eq. (1) and its associated Liouville operator cannot be decomposed on this Fourier basis, i.e., they do not satisfy the translational invariances of the Fourier basis. Hence mathematical inconsistency and an incorrect form for the quantum propagator result.

The advantages of the von Neumann approach over the conventional Hamiltonian quantization are now evident. Equations (38) and (39) provide a solution for any value of  $ab$ , are integrable over any of the periodic unit cells, and are well behaved. Further, in the classical limit ( $\hbar \rightarrow 0$ ),  $\lambda_{j,p_0} \rightarrow \lambda_{j,p_0}^{(c)}$  [compare Eqs. (39) and (50)] so that the classical limit is properly reached by both eigenfunctions and eigenvalues.

#### IV. CLASSICAL AND QUANTUM MAPPING DYNAMICS ( $\epsilon \neq 0$ )

Here we focus on the more general case of  $\epsilon \neq 0$ , obtain the propagator, and show that it readily yields the correct classical limit.

##### A. Classical mechanics

Consider then the Hamiltonian [Eq. (1)] in reduced modular variables ( $\eta = Tb/\mu a \in \mathbb{Z}$  and  $\xi = -\epsilon Ta/b \in \mathbb{Z}$ ):

$$H(P, Q, t) = \frac{ab}{2T} \left\{ \eta(P \bmod 1)^2 - \xi(Q \bmod 1)^2 \sum_{s=-\infty}^{\infty} \delta(s - t/T) \right\}. \quad (51)$$

The classical dynamics of the map is defined by the clas-

sical propagator

$$\Lambda_c(T) = \mathcal{T} \exp \left\{ -i \int_{MT}^{(M+1)T} dt L_c(t) \right\}, \quad (52)$$

where  $L_c(t)$  is the classical Liouville operator associated with Eq. (51) and where the integral is over a time interval beginning just after the  $M$ th kick to just after the  $(M+1)$ st kick. Then the dynamics is given by

$$\Lambda_c(T) \begin{pmatrix} Q \bmod 1 \\ P \bmod 1 \end{pmatrix} = \phi^{-1} \begin{pmatrix} Q \\ P \end{pmatrix} \bmod 1, \quad (53)$$

where

$$\begin{aligned} \begin{pmatrix} Q_{n+1} \\ P_{n+1} \end{pmatrix} &= \phi \begin{pmatrix} Q_n \\ P_n \end{pmatrix} \bmod 1 \\ &= \begin{pmatrix} 1 & \eta \\ \xi & 1 + \eta\xi \end{pmatrix} \begin{pmatrix} Q_n \\ P_n \end{pmatrix} \bmod 1. \end{aligned} \quad (54)$$

These linear mappings  $\phi$  are such that  $I = \xi Q^2 + \eta\xi QP - \eta P^2$  is a constant of the motion when  $|2 + \eta\xi| \leq 2$ . However, when  $|2 + \eta\xi| > 2$ ,  $I$  is a hyperbola which is wrapped densely around the torus by the “mod 1” operation and so effectively destroyed. We examine the quantum dynamics of the map in both these regimes and obtain results for all  $\eta, \xi$ , including the Arnold cat map [18] ( $\eta = \xi = 1$ ), and systems like the quasiperiodic discretized particle in a box [20] ( $\xi = 0$ ).

To obtain the time evolution of the phase-space density which at time zero is  $\rho^{(c)}(P, Q, 0)$ , we Fourier expand in  $f_{n,m}$ . At time zero, and later time  $MT$ , we have

$$\rho^{(c)}(P, Q, 0) = \frac{1}{ab} \sum_{n,m=-\infty}^{\infty} \rho_{n,m}^{(c)}(0) f_{n,m}, \quad (55)$$

$$\begin{aligned} \rho^{(c)}(P, Q, MT) &= \frac{1}{ab} \sum_{n,m=-\infty}^{\infty} \rho_{n,m}^{(c)}(MT) f_{n,m} \\ &= \Lambda_c(T)^M \rho^{(c)}(P, Q, 0). \end{aligned} \quad (56)$$

A straightforward analysis of the effect of  $\Lambda_c(T)$  on  $\rho^{(c)}(P, Q, 0)$  gives

$$\Lambda_c(T) \rho^{(c)}(P, Q, 0) = \frac{1}{ab} \sum_{n,m} \rho_{\phi'(nm)}^{(c)} f_{n,m}, \quad (57)$$

where  $\phi'(nm)$  is the transpose of the matrix in Eq. (54) multiplying the column vector with elements  $n, m$ . Clearly, evolution of the map, from this perspective, corresponds to the interchange of coefficients in the Fourier basis  $f_{n,m}$ . We define the classical propagator  $G_c$  of the Fourier coefficients, for particular reference with the quantum result obtained later below, as

$$\rho_{n,m}^{(c)}(T) = \sum_{k,l=-\infty}^{\infty} G_c(n, m; k, l) \rho_{k,l}^{(c)}(0). \quad (58)$$

Comparison of Eq. (56) for  $M = 1$  and Eq. (57) reveals that

$$G_c(n, m; k, l) = \delta_{(k,l), \phi'(nm)}. \quad (59)$$

Generalization to the step from time  $MT$  to time

$(M + 1)T$  gives

$$\rho_{n,m}^{(c)}((M + 1)T) = \sum_{k,l} G_c(n, m; k, l) \rho_{k,l}^{(c)}(MT) \quad (60)$$

so that

$$\begin{aligned} \rho^{(c)}(P, Q, (M + 1)T) \\ = \frac{1}{ab} \sum_{n,m} \left[ \sum_{k,l} G_c(n, m; k, l) \rho_{k,l}^{(c)}(MT) \right] f_{n,m}. \end{aligned} \quad (61)$$

Equation (61) provides a general expression for the time evolution of the classical phase-space density in terms of the classical propagator  $G_c$  matrix elements as coefficients in a Fourier expansion.

### B. Quantum mechanics

Consider now the quantum dynamics with the Hamiltonian, the analog of Eq. (51) with the appropriate introduction of mod operators,

$$\begin{aligned} \hat{H}(t) = \frac{ab}{2T} \left\{ \eta(\widehat{P \bmod 1})^2 \right. \\ \left. - \xi(\widehat{Q \bmod 1})^2 \sum_{s=-\infty}^{\infty} \delta(s - t/T) \right\}. \end{aligned} \quad (62)$$

The associated time evolution operator is

$$\hat{U}(T) = \mathcal{T} e^{-i \int_{MT}^{(M+1)T} dt \hat{H}(t)/\hbar}, \quad (63)$$

where the integration is performed from just after the  $M$ th kick to just after the  $(M + 1)$ st kick. The resultant  $\hat{U}(T)$  is  $M$  independent.

The goal of our calculation below is to obtain the quantum analog of  $G_c$ . To do so is complicated in execution but simple in its essence. Specifically, we introduce the Fourier operator basis [Eq. (26)] for the quantum distribution  $\hat{\rho}$  at time  $MT$  as

$$\hat{\rho}(MT) = \frac{1}{ab} \sum_{n,m=-\infty}^{\infty} \rho_{n,m}(MT) \hat{f}_{n,m} \quad (64)$$

and endeavor to express the time evolution of the Fourier coefficients  $\rho_{n,m}$  as the quantum analog of Eq. (60), i.e., as

$$\rho_{n,m}((M + 1)T) = \sum_{k,l=-\infty}^{\infty} G(n, m; k, l) \rho_{k,l}(MT). \quad (65)$$

Doing so will allow an analysis of the quantum dynamics in the Fourier basis as well as a direct comparison with the classical result. Indeed this comparison with the classical limit benefits considerably from use of the Wigner-Weyl representation. In this representation Eq. (64) assumes the form

$$\rho(P, Q, MT) = \frac{1}{ab} \sum_{n,m=-\infty}^{\infty} \rho_{n,m}(MT) f_{n,m} \quad (66)$$

so that, given Eq. (65),

$$\begin{aligned} \rho(P, Q, (M + 1)T) \\ = \frac{1}{ab} \sum_{n,m} \left[ \sum_{k,l} G(n, m; k, l) \rho_{k,l}(MT) \right] f_{n,m}. \end{aligned} \quad (67)$$

A comparison of Eq. (67) with Eq. (61) shows that classical-quantum correspondence requires [21]

$$\lim_{\alpha \rightarrow 0} \rho_{k,l}(MT) = \rho_{k,l}^{(c)}(MT) \quad (68)$$

and

$$\lim_{\alpha \rightarrow 0} G(n, m; k, l) = G_c(n, m; k, l). \quad (69)$$

Equation (68) holds if Eq. (69) holds and if  $\rho_{k,l}(0) = \rho_{k,l}^{(c)}(0)$ . However, the latter equality is assured, as seen by comparing Eq. (66) with Eq. (55), at time zero. Thus to prove the correspondence limit for the fundamental dynamical entity  $\rho(P, Q, MT)$  requires that we show Eq. (69). This is done in Sec. V after we obtain a general expression for the quantum  $G(n, m; k, l)$ . Readers uninterested in the details of this derivation should proceed to Eq. (95).

### Quantum propagator

The time evolved density operator satisfies

$$\hat{\rho}((M + 1)T) = \hat{U}(T) \hat{\rho}(MT) \hat{U}^\dagger(T). \quad (70)$$

We transform to the Wigner-Weyl representation and focus on the initial step from time 0 to  $T$ ; the same algebra pertains to the general step from time  $MT$  to  $(M + 1)T$ . Equation (70) in the Wigner-Weyl representation is

$$\rho(P, Q, T) = \frac{1}{ab} \sum_{n,m=-\infty}^{\infty} \rho_{n,m} \int_{-\infty}^{\infty} dv e^{i2\pi PV/\alpha} \langle q - v/2 | \hat{K} \hat{U}_0(T) \hat{f}_{n,m} \hat{U}_0^\dagger(T) \hat{K}^\dagger | q + v/2 \rangle, \quad (71)$$

where

$$\hat{K} = \exp\{i\pi\xi(\widehat{Q \bmod 1})^2/\alpha\}, \quad (72)$$

$$\hat{U}_0(T) = \exp\{-i\pi\eta(\widehat{P \bmod 1})^2/\alpha\},$$



so that  $\hat{U}(T) = \hat{K}\hat{U}_0$ . [Here again we employ a convention where capital letters denote scaled variables, i.e.,  $P = p/b$ ,  $Q = q/a$ ,  $V = v/a$  and below  $X = x/a$ .] The matrix element in Eq. (71) can be written as

$$\begin{aligned} \langle q - v/2 | \hat{K}\hat{U}_0(T) \hat{f}_{n,m} \hat{U}_0^\dagger(T) \hat{K}^\dagger | q + v/2 \rangle \\ = \int_{-\infty}^{\infty} dq' e^{2\pi i m Q'} \langle q - v/2 | \hat{K}\hat{U}_0(T) | q' - n\alpha a/2 \rangle \langle q' + n\alpha a/2 | \hat{U}_0^\dagger(T) \hat{K}^\dagger | q + v/2 \rangle, \end{aligned} \quad (73)$$

which involves matrix elements over  $\hat{K}\hat{U}_0(T)$ . These matrix elements may be rewritten as (see Appendix B)

$$\langle x | \hat{K}\hat{U}_0(T) | x' \rangle = \exp\{i\pi\xi(X \bmod 1)^2/\alpha\} \sum_{k=-\infty}^{\infty} g_k(\eta) \delta(x - x' - k\alpha a), \quad (74)$$

where

$$g_k(z) = \int_0^1 d\nu e^{-i\pi z \nu^2/\alpha} e^{2\pi i k \nu}. \quad (75)$$

Inserting Eq. (74) into Eq. (73) gives

$$\begin{aligned} \langle q - v/2 | \hat{K}\hat{U}_0(T) \hat{f}_{n,m} \hat{U}_0^\dagger(T) \hat{K}^\dagger | q + v/2 \rangle = \sum_{k,l=-\infty}^{\infty} g_k(\eta) g_l^*(\eta) e^{i\pi\xi[(Q-V/2) \bmod 1]^2/\alpha} e^{-i\pi\xi[(Q+V/2) \bmod 1]^2/\alpha} \\ \times e^{2\pi i m(Q - (k+l)\alpha/2)} \delta(v - (n+l-k)\alpha a). \end{aligned} \quad (76)$$

Inserting Eq. (76) into Eq. (71) gives, after some algebra,

$$\rho(P, Q, T) = \frac{1}{ab} \sum_{n,m=-\infty}^{\infty} \rho_{n,m}(0) e^{2\pi i(nP+mQ)} \kappa_{n,m}(P, Q) \quad (77)$$

with

$$\kappa_{n,m}(P, Q) = \sum_{k,l=-\infty}^{\infty} g_k(\eta) g_l^*(\eta) e^{2\pi i(l-k)P} e^{-i\pi\alpha(k+l)m} e^{i\pi\xi[(Q-\alpha(n-k+l)/2) \bmod 1]^2/\alpha} e^{-i\pi\xi[(Q+\alpha(n-k+l)/2) \bmod 1]^2/\alpha}. \quad (78)$$

Equations (77) and (78) provide an expression for propagation of the quantum map in phase space. However, extensive algebraic manipulation is necessary to extract the propagator in the Fourier basis. We begin by noting that we can write

$$\rho_{n,m} = \int_0^b dp_0 \int_0^a dq_0 \rho(P_0, Q_0, 0) e^{-2\pi i(nP_0+mQ_0)} \quad (79)$$

so that Eq. (77) can be written in the form

$$\begin{aligned} \rho(P, Q, T) = \int_0^b dp_0 \int_0^a dq_0 \rho(P_0, Q_0, 0) \\ \times \kappa(P, Q, T; P_0, Q_0, 0), \end{aligned} \quad (80)$$

where

$$\begin{aligned} \kappa(P, Q, T; P_0, Q_0, 0) \\ = \frac{1}{ab} \sum_{n,m=-\infty}^{\infty} e^{2\pi i[n(P-P_0)+m(Q-Q_0)]} \kappa_{n,m}(P, Q) \end{aligned} \quad (81)$$

is the kernel of the Liouville propagator in the Wigner-

Weyl representation. Since

$$\rho(P, Q, T) = \frac{1}{ab} \sum_{n,m} \rho_{n,m}(T) e^{2\pi i(nP+mQ)} \quad (82)$$

the Fourier coefficients of the propagated Wigner functions are

$$\rho_{n,m}(T) = \int_0^b dp_0 \int_0^a dq_0 \rho(P_0, Q_0, T) e^{-2\pi i(nP_0+mQ_0)}. \quad (83)$$

Given Eqs. (80)–(82), the goal is to rewrite the propagation of  $\rho_{n,m}(0)$  in terms of Eq. (65), so as to extract  $G(n, m; k, l)$ , i.e., in the first iteration we want

$$\rho_{n,m}(T) = \sum_{k,l=-\infty}^{\infty} G(n, m; k, l) \rho_{k,l}(0). \quad (84)$$

We begin by noting that Eq. (81) can be manipulated, in accord with Appendix C, into the form

$$\begin{aligned} \kappa(P, Q, T; P_0, Q_0, 0) &= \frac{1}{ab} \sum_{n,m=-\infty}^{\infty} e^{2\pi i[nP+mQ]} \sum_{k,l=-\infty}^{\infty} g_k(\eta) g_l^*(\eta) e^{-2\pi i[(n+k-l)P_0+mQ_0]} e^{-i\pi\alpha(k+l)m} \\ &\quad \times e^{i\pi\xi\{[Q_0-\alpha(n-k-l)/2]\bmod 1\}^2/\alpha} e^{-i\pi\xi\{[Q_0+\alpha(n+k+l)/2]\bmod 1\}^2/\alpha} \end{aligned} \quad (85)$$

and so Eq. (80) can be written as

$$\begin{aligned} \rho(P, Q, T) &= \frac{1}{ab} \sum_{n,m=-\infty}^{\infty} e^{2\pi i[nP+mQ]} \int_0^b dp_0 \int_0^a dq_0 \rho(P_0, Q_0, 0) \\ &\quad \times \sum_{k,l=-\infty}^{\infty} g_k(\eta) g_l^*(\eta) e^{-2\pi i[(n+k-l)P_0+mQ_0]} e^{-i\pi\alpha(k+l)m} \\ &\quad \times e^{i\pi\xi\{[Q_0-\alpha(n-k-l)/2]\bmod 1\}^2/\alpha} e^{-i\pi\xi\{[Q_0+\alpha(n+k+l)/2]\bmod 1\}^2/\alpha}. \end{aligned} \quad (86)$$

Comparing Eq. (86) to Eq. (82) we identify

$$\begin{aligned} \rho_{n,m}(T) &= \int_0^b dp_0 \int_0^a dq_0 \rho(P_0, Q_0, 0) \sum_{k,l=-\infty}^{\infty} g_k(\eta) g_l^*(\eta) e^{-2\pi i[(n+k-l)P_0+mQ_0]} e^{-i\pi\alpha(k+l)m} \\ &\quad \times e^{i\pi\xi\{[Q_0-\alpha(n-k-l)/2]\bmod 1\}^2/\alpha} e^{-i\pi\xi\{[Q_0+\alpha(n+k+l)/2]\bmod 1\}^2/\alpha}. \end{aligned} \quad (87)$$

Using the identity (see Appendix B for a proof)

$$e^{-i\pi z(z \bmod 1)^2/\alpha} = \sum_{j=-\infty}^{\infty} g_j(z) e^{-2\pi i z j}, \quad (88)$$

one obtains

$$\rho_{n,m}(T) = \sum_{i,j,k,l=-\infty}^{\infty} g_i(\xi) g_j^*(\xi) g_k(\eta) g_l^*(\eta) \rho_{n+k-l, m+i-j}(0) e^{-i\pi\alpha(k+l)(m+i-j)} e^{-i\pi\alpha(i+j)n}. \quad (89)$$

This can immediately be generalized to the iterative form giving  $\rho_{n,m}((M+1)T)$  in terms of  $\rho_{n,m}$  at time  $MT$ . However, although this equation correctly describes the manner in which the Fourier coefficients of a Wigner distribution are mapped under the quantum Liouville propagator it is too complicated to be of use. Let  $k' = n+k-l$  and  $i' = m+i-j$ . Under this change in the integers of summation Eq. (89) becomes

$$\rho_{n,m}(T) = \sum_{i',j,k',l=-\infty}^{\infty} g_{i'+j-m}(\xi) g_j^*(\xi) g_{k'+l-n}(\eta) g_l^*(\eta) \rho_{k',i'}(0) e^{-i\pi\alpha(k'+2l-n)i'} e^{-i\pi\alpha(i'+2j-m)n}. \quad (90)$$

But the sums over  $j$  and  $l$  may be done explicitly. To begin with

$$\sum_{j=-\infty}^{\infty} g_j^*(\xi) g_{i'+j-m}(\xi) e^{-2\pi i(\alpha n)j} = \sum_{j=-\infty}^{\infty} \int_0^1 d\nu \int_0^1 d\nu' e^{i\pi\xi\nu^2/\alpha} e^{-2\pi i j \nu} e^{-i\pi\xi\nu'^2/\alpha} e^{2\pi i(i'+j-m)\nu'} e^{-2\pi i(\alpha n)j} \quad (91)$$

and use of identity (17) yields

$$\sum_{j=-\infty}^{\infty} g_j^*(\xi) g_{i'+j-m}(\xi) e^{-2\pi i(\alpha n)j} = \int_0^1 d\nu \int_0^1 d\nu' e^{i\pi\xi\nu^2/\alpha} e^{-i\pi\xi\nu'^2/\alpha} e^{2\pi i(i'-m)\nu'} \delta(\nu' - (\nu + \alpha n) \bmod 1) \quad (92)$$

$$= \int_0^1 d\nu e^{-i\pi\xi[(\nu+\alpha n) \bmod 1]^2/\alpha} e^{i\pi\xi\nu^2/\alpha} e^{2\pi i(i'-m)(\nu+\alpha n)}. \quad (93)$$

Similar calculations give

$$\sum_{l=-\infty}^{\infty} g_l^*(\xi) g_{k'+l-n}(\xi) e^{-2\pi i(\alpha n)j} = \int_0^1 d\nu' e^{-i\pi\xi[(\nu'+\alpha i') \bmod 1]^2/\alpha} e^{i\pi\xi\nu'^2/\alpha} e^{2\pi i(k'-n)(\nu'+\alpha i')}. \quad (94)$$

Insertion of these results into Eq. (90), setting  $k' = k$  and  $i' = l$ , and simplifying, gives the desired result, i.e., Eq. (84) with

$$G(n, m; k, l) = e^{i\pi\alpha(kl-nm)} \int_0^1 d\nu \int_0^1 d\nu' e^{-i\pi\xi[(\nu+\alpha n)\bmod 1]^2/\alpha} e^{i\pi\xi\nu^2/\alpha} e^{2\pi i(l-m)\nu} e^{-i\pi\eta[(\nu'+\alpha l)\bmod 1]^2/\alpha} \times e^{i\pi\eta\nu'^2/\alpha} e^{2\pi i(k-n)\nu'} \quad (95)$$

as the propagator of the Fourier coefficients [22]. This can be immediately generalized to give Eq. (65) with the same propagator  $G$ . The crucial difference between the classical  $G_c$  [Eqs.(58),(59)] and the quantum propagator  $G$  is clear insofar as the quantum propagator mixes contributions from all  $k, l$  components to produce each  $n, m$  whereas  $G_c$  does not.

Note that propagator (95) moves distributions forward in time. To obtain its inverse, i.e., the propagator which moves distributions backward in time, simply take the complex conjugate of Eq. (95) and interchange  $n$  with  $k$ , and  $m$  with  $l$  (see Appendix D for a proof).

As a simple check on the validity of the quantization procedure we redo the “free particle” case, i.e.,  $\xi = 0$ . In this case  $G$  is given by

$$G(n, m; k, l) = \delta_{l,m} e^{i\pi\alpha m(k-n)} \int_0^1 d\nu e^{-i\pi\eta[(\nu+\alpha m)\bmod 1]^2/\alpha} e^{i\pi\nu^2/\alpha} e^{2\pi i(k-n)\nu} \quad (96)$$

and the eigenequation is

$$\sum_{k,l=-\infty}^{\infty} G(n, m; k, l) \rho_{k,l} = e^{-i\lambda T} \rho_{n,m}. \quad (97)$$

Consider a solution of the form  $\rho_{k,l} = \Omega_l e^{-ik\theta_l}$ . Substitution and use of Eq. (17) yields

$$e^{-i\pi\alpha mn} \int_0^1 d\nu e^{-i\pi\eta[(\nu+\alpha m)\bmod 1]^2/\alpha} e^{i\pi\nu^2/\alpha} e^{-2\pi i\nu} \delta\left(\nu - \left(\frac{\theta_m}{2\pi} - \alpha m/2\right) \bmod 1\right) = e^{-i\lambda T} e^{-in\theta_m} \quad (98)$$

and this implies that

$$\lambda = \frac{\pi\eta}{\alpha T} \left\{ \left[ \left( \frac{\theta_m}{2\pi} + \alpha m/2 \right) \bmod 1 \right]^2 - \left[ \left( \frac{\theta_m}{2\pi} - \alpha m/2 \right) \bmod 1 \right]^2 \right\}. \quad (99)$$

Choosing  $\frac{\theta_m}{2\pi} = \frac{p_0}{b}$  where  $p_0 \in [0, b)$  gives the same solutions [Eqs. (38) and (39)] which we obtained previously.

## V. CORRESPONDENCE

Given Eq. (95) it is straightforward to examine the dynamics in the classical limit ( $\alpha \rightarrow 0$ ). We do so at fixed finite time so that, with respect to the issue of ergodic properties, we are taking the correct order of limits,  $h \rightarrow 0$  prior to any long time limit [7].

For each set of values  $\{\nu, \nu', n, l\}$  there exists a sufficiently small  $\alpha$  such that  $(\nu + \alpha n) \bmod 1 = \nu + \alpha n$  and  $(\nu' + \alpha l) \bmod 1 = \nu' + \alpha l$ . Thus in the  $\alpha \rightarrow 0$  limit

$$\begin{aligned} G(n, m; k, l) &\rightarrow \int_0^1 d\nu \int_0^1 d\nu' e^{2\pi i(l-m-\xi n)\nu} \\ &\times e^{2\pi i(k-n-\eta l)\nu'} \\ &= \delta_{l,m+\xi n} \delta_{k,n+\eta l} = \delta_{l,m+\xi n} \delta_{k,n+\eta m+\eta\xi n} \\ &= \delta_{(l,k), \phi'(nm)}, \end{aligned} \quad (100)$$

which is the correct classical limit, Eq. (59). Note that

this result holds for all maps of the form given by Eq. (54), including the classical chaotic cat map.

Having demonstrated that these quantum maps give the correct classical limit it is necessary to obtain its dynamical characteristics in the case of  $h \neq 0$ . It is straightforward to demonstrate that, independent of the value of  $\eta, \xi$ , the quantum propagator has at least two eigenfunctions with unit eigenvalue and hence that the quantum dynamics is nonergodic. Two such eigenfunctions are the uniform distribution and the Schrödinger propagator. To see this consider the uniform distribution  $1_{k,l}(0) = \delta_{k,0} \delta_{l,0}$  under propagation. Using Eq. (95) we have

$$\begin{aligned} 1_{n,m}(T) &= G(n, m; 0, 0) \\ &= e^{-2\pi imn\alpha} \int_0^1 d\nu \int_0^1 d\nu' e^{-i\pi\xi[(\nu+\alpha n)\bmod 1]^2/\alpha} \\ &\times e^{i\pi\xi\nu^2/\alpha} e^{-2\pi im\nu} e^{-2\pi i\nu\nu'} \\ &= \delta_{n,0} \int_0^1 d\nu e^{-2\pi im\nu} = \delta_{n,0} \delta_{m,0}. \end{aligned} \quad (101)$$

Thus the uniform distribution is an eigenfunction with unit eigenvalue. [This result also implies that the dynamics is measure (i.e., area) preserving.]

Second, note that the Schrödinger propagator is an eigenfunction of the Liouville propagator since  $\hat{U}^\dagger(\hat{U})\hat{U} = \hat{U}^{-1}(\hat{U})\hat{U} = \hat{U}$ . It remains to demonstrate that the Wigner function associated with  $\hat{U}$  is  $L^2$  on a unit cell. To see this note the form of  $\hat{U}$  in the Wigner-Weyl representation:

$$U^{(w)}(P, Q) = \sum_{n, m=-\infty}^{\infty} g_{-n}(\eta) g_m^*(\xi) e^{-i\pi\alpha n m} e^{2\pi i(nP+mQ)}. \quad (102)$$

Then

$$\int_0^1 dP \int_0^1 dQ |U^{(w)}(P, Q)|^2 = \int_0^1 dP \int_0^1 dQ \sum_{n, m, k, l=-\infty}^{\infty} g_{-k}^*(\eta) g_l(\xi) g_{-n}(\eta) g_m^*(\xi) e^{2\pi i[(n-k)P+(m-l)Q]} e^{-i\pi(nm-kl)\alpha} \quad (103)$$

$$= \sum_{k=-\infty}^{\infty} |g_k(\eta)|^2 \sum_{l=-\infty}^{\infty} |g_l(\xi)|^2 \quad (104)$$

$$= 1^2 = 1 \quad (105)$$

since

$$\sum_{k=-\infty}^{\infty} g_{j+k}(z) g_k^*(z) = \delta_{j,0}. \quad (106)$$

The latter identity is proven in Appendix B.

Thus  $U^{(w)}$  is an  $L^2$  eigenfunction of the mapping with eigenvalue 1, as is the uniform distribution and so [18] the mappings are not ergodic. Note, furthermore, the  $\alpha \rightarrow 0$  limit of  $U^{(w)}$  displays an essential singularity. Hence this eigenfunction does not exist classically, a result which is essential to achieve the proper ergodic classical limit for classically ergodic cases like the cat map.

The extended quantization procedure therefore provides a straightforward treatment of the cat map, a system which has been the subject of considerable controversy.

## VI. WHY NO HAMILTONIAN EIGENFUNCTIONS?

In addition to the explicit results on maps, this work has also introduced a quantization procedure leading to results which differ significantly from the traditional Schrödinger equation approach. This difference arises primarily from the fact that periodic boundary conditions applied to the density matrix [ $T^{-1}\hat{\rho}T = \hat{\rho}$  for  $T = T_q(a)$  and  $T = T_p(b)$ ] allow for solutions when similar boundary conditions, applied to the wave function, preclude the possibility of a solution. Formally this is so because periodic boundary conditions on the wave functions (corresponding to the condition  $T\hat{\rho} = \hat{\rho}$ ,  $\hat{\rho}T = \hat{\rho}$ , on the density matrix) are a far more restrictive requirement than the condition  $T^{-1}\hat{\rho}T = \hat{\rho}$ . Nonetheless it is of interest to examine the character of the mapping dynamics and to expose the physics underlying the lack of wave functions.

Consider the Arnold cat map [18] as a specific example. Written explicitly, the cat map [ $\eta = \xi = 1$  in Eq. (54)] reads

$$Q_{n+1} \bmod 1 - Q_n \bmod 1 = P_n \bmod 1 + V(Q_n \bmod 1, P_n \bmod 1), \quad (107)$$

$$P_{n+1} \bmod 1 - P_n \bmod 1 = P_n \bmod 1 + Q_n \bmod 1 + F(Q_n \bmod 1, P_n \bmod 1), \quad (108)$$

where

$$V(Q \bmod 1, P \bmod 1) = -1\chi_{A_3 \cup A_4}(Q \bmod 1, P \bmod 1), \quad (109)$$

$$F(Q \bmod 1, P \bmod 1) = -1\chi_{A_2 \cup A_3}(Q \bmod 1, P \bmod 1) - 2\chi_{A_4}(Q \bmod 1, P \bmod 1). \quad (110)$$

Here  $\chi_B$  is the characteristic function on the set  $B$  and the phase space regions  $A_i$  are defined by

$$A_2 = \{\frac{1}{2} \leq Q \bmod 1 < 1\} \times \{1 - 2Q \bmod 1 \leq P \bmod 1 < 1 - Q \bmod 1\}, \quad (111)$$

$$A_3 = \{0 \leq Q \bmod 1 < \frac{1}{2}\} \times \{1 - Q \bmod 1 \leq P \bmod 1 < 2 - 2Q \bmod 1\}, \quad (112)$$

$$A_4 = \{\frac{1}{2} \leq Q \bmod 1 < 1\} \times \{2 - 2Q \bmod 1 \leq P \bmod 1 < 1\}. \quad (113)$$

Equations (107) and (108) show that the changes in  $q$  and  $p$  under the mapping depend upon an external boost  $V$  and external force  $F$  which are functions of position and momentum. Thus the cat map equations are a discretization of a Langevin type process

$$\begin{aligned} \frac{dq}{dt} &= p/\mu + V(q, p), \\ \frac{dp}{dt} &= -\zeta p - \frac{\partial \Phi}{\partial q} + F(p, q), \end{aligned} \quad (114)$$

with a negative coefficient of friction  $\zeta = -1$  and potential  $\Phi = -q^2/2$ . Since this set of equations describes the Hamiltonian dynamics of a particle moving in the vicinity of a single hyperbolic point if  $V = \zeta = F = 0$ , Eqs. (44) and (45) constitute a set of perturbed dynamics around an unstable point. Thus the classical cat map can be regarded as a system subject to external boosts and forces, rather than a stroboscopic viewing of a true Hamiltonian system. Since the system is one which is subjected to external boosts and forces, it does not admit a pure-state description.

## VII. SUMMARY

In summary, we have introduced a von Neumann based quantization procedure in concert with Bloch type boundary conditions in order to achieve a physically complete quantization of mappings on a torus [17(b)]. The more traditional approach [1,2] of quantizing the mappings via the Schrödinger equation with associated periodic boundary conditions yields solutions only for tori satisfying  $ab = hN$ , and excessively restricts the mapping dynamics to the dynamics on a set of rational points in  $(q, p)$  space, associated with a discrete Liouville spectrum. Alternative approaches such as applying Bloch type boundary conditions on the wave function or periodic boundary conditions on the density matrix encounter similar difficulties. From the perspective of the density matrix periodic boundary conditions imply  $T\hat{\rho} = \hat{\rho}$ ,  $\hat{\rho}T = \hat{\rho}$  for  $T = T_q(a)$  and  $T = T_p(b)$ . Adopting Bloch boundary conditions on the density matrix ( $T^{-1}\hat{\rho}T = \hat{\rho}$ ), the only remaining possibility, allows for a broader class of solutions obtainable even when wave functions do not exist. The fact that pure states do not in general exist is a consequence of the fact that boundary conditions here correspond to the application of external forces to the system.

The resultant quantum system has a continuous Liouville spectrum, and the von Neumann quantized cat map dynamics is not ergodic until the proper classical limit is reached. This result demonstrates a straightforward limiting procedure which completely recovers the full chaotic classical dynamics in the  $\hbar \rightarrow 0$  limit. Further studies are in progress to ascertain the character of the quantum dynamics and its dependence on  $\hbar$  as the system approaches the classical limit. Further, this study suggests the possibility of quantization via a generalized von Neumann (or continuity) equation for systems described as a Langevin

type process, i.e., Eqs. (114), a fertile general area for further study.

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## APPENDIX A

Here we discuss briefly the algebra generated by the operators  $\hat{f}_{n,m}$ . It is trivial to show that these operators satisfy the following conditions: (1) closure  $\hat{f}_{n,m}\hat{f}_{k,l} = e^{i\pi(nl-mk)\alpha}\hat{f}_{n+k,m+l}$ , (2) associativity  $(\hat{f}_{n,m}\hat{f}_{k,l})\hat{f}_{i,j} = \hat{f}_{n,m}(\hat{f}_{k,l}\hat{f}_{i,j})$ , (3) the existence of an identity  $\hat{f}_{0,0} = \hat{1}$ , (4) the existence of inverses  $\hat{f}_{n,m}\hat{f}_{-n,-m} = \hat{1} = \hat{f}_{-n,-m}\hat{f}_{n,m}$ , and (5)  $\hat{f}_{n,m}^\dagger = \hat{f}_{-n,-m} = (\hat{f}_{n,m})^{-1}$  which implies that they are unitary operators. Furthermore, (6) every element of this algebra can be generated from the set  $\{\hat{f}_{1,0}, \hat{f}_{-1,0}, \hat{f}_{0,1}, \hat{f}_{0,-1}\}$ .

## APPENDIX B

Here we will prove some of the simple formulas used in this paper. Underlying all of these formulas is the identity in Eq. (17). We have defined integrals  $g_k(z) = \int_0^\infty d\nu e^{-i\pi z\nu^2/\alpha} e^{2\pi i k\nu}$  and here we will prove two simple identities and one formula specific to the quantum map problem involving these integrals.

The first identity is that of Eq. (88). Using identity (17)

$$\begin{aligned} e^{-i\pi z(x \bmod 1)^2/\alpha} &= \int_0^1 d\nu \delta(\nu - x) e^{-i\pi z\nu^2/\alpha} \\ &= \int_0^1 d\nu \sum_{j=-\infty}^{\infty} e^{2\pi i(\nu-x)j} e^{-i\pi z\nu^2/\alpha} \\ &= \sum_{j=-\infty}^{\infty} g_j(z) e^{-2\pi i jx}. \end{aligned} \quad (\text{B1})$$

The second identity is that of Eq. (106). Again using identity (17)

$$\begin{aligned} \sum_{j=-\infty}^{\infty} g_{j+k}(z) g_j^*(z) &= \int_0^1 d\nu \int_0^1 d\nu' e^{-i\pi z\nu^2/\alpha} e^{i\pi z\nu'^2/\alpha} e^{2\pi i k\nu} \sum_{j=-\infty}^{\infty} e^{2\pi i j(\nu-\nu')} \\ &= \int_0^1 d\nu \int_0^1 d\nu' e^{-i\pi z\nu^2/\alpha} e^{i\pi z\nu'^2/\alpha} e^{2\pi i k\nu} \delta(\nu - \nu') \\ &= \int_0^1 d\nu e^{2\pi i k\nu} = \delta_{k,0}. \end{aligned} \quad (\text{B2})$$

The formula specific to the quantum map problem is Eq. (74). The proof is as follows:

$$\begin{aligned}
\langle x|\hat{K}\hat{U}_0(T)|x'\rangle &= e^{i\pi\xi(X\text{mod }1)^2/\alpha} \int_{-\infty}^{\infty} dp \langle x|p\rangle \langle p|x'\rangle e^{-i\pi\eta(P\text{mod }1)^2/\alpha} \\
&= e^{i\pi\xi(X\text{mod }1)^2/\alpha} \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} e^{ip(x-x')/\hbar} e^{-i\pi\eta(P\text{mod }1)^2/\alpha} \\
&= e^{i\pi\xi(X\text{mod }1)^2/\alpha} \sum_{k=-\infty}^{\infty} g_k(\eta) \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} e^{ip(x-x'-\alpha ak)/\hbar} \\
&= e^{i\pi\xi(X\text{mod }1)^2/\alpha} \sum_{k=-\infty}^{\infty} g_k(\eta) \delta(x-x'-\alpha ak),
\end{aligned} \tag{B3}$$

where we have used identity (88) and the closure relation for momentum.

### APPENDIX C

Here we will show that Eq. (81) can be put in the form of Eq. (85). To begin, we substitute expression (78) for  $\kappa_{n,m}(P, Q)$  into Eq. (81). This gives

$$\begin{aligned}
\kappa(P, Q, T; P_0, Q_0, 0) &= \sum_{n,m=-\infty}^{\infty} e^{2\pi i[n(P-P_0)+m(Q-Q_0)]} \sum_{k,l=-\infty}^{\infty} g_k(\eta) g_l^*(\eta) e^{2\pi i(l-k)P} \\
&\quad \times e^{-i\pi\alpha(k+l)m} e^{i\pi\xi\{[Q-\alpha(n-k+l)/2]\text{mod }1\}^2/\alpha} e^{-i\pi\xi\{[Q+\alpha(n-k+l)/2]\text{mod }1\}^2/\alpha}.
\end{aligned} \tag{C1}$$

Using identity (17) for the sum over  $m$  then gives

$$\begin{aligned}
\kappa(P, Q, T; P_0, Q_0, 0) &= \sum_{n,m,k,l=-\infty}^{\infty} e^{2\pi i n(P-P_0)} \delta(Q-Q_0-(k+l)\alpha/2-m) g_k(\eta) g_l^*(\eta) e^{2\pi i(l-k)P} \\
&\quad \times e^{i\pi\xi\{[Q-\alpha(n-k+l)/2]\text{mod }1\}^2/\alpha} e^{-i\pi\xi\{[Q+\alpha(n-k+l)/2]\text{mod }1\}^2/\alpha}
\end{aligned} \tag{C2}$$

$$\begin{aligned}
&= \sum_{n,m,k,l=-\infty}^{\infty} e^{2\pi i n(P-P_0)} \delta(Q-Q_0-(k+l)\alpha/2-m) g_k(\eta) g_l^*(\eta) e^{2\pi i(l-k)P} \\
&\quad \times e^{i\pi\xi\{[Q_0-\alpha(n-2k)/2]\text{mod }1\}^2/\alpha} e^{-i\pi\xi\{[Q_0+\alpha(n+2l)/2]\text{mod }1\}^2/\alpha}.
\end{aligned} \tag{C3}$$

Now using identity (17) in reverse we obtain

$$\begin{aligned}
\kappa(P, Q, T; P_0, Q_0, 0) &= \sum_{n,m,k,l=-\infty}^{\infty} e^{2\pi i[n(P-P_0)+m(Q-Q_0)]} g_k(\eta) g_l^*(\eta) e^{2\pi i(l-k)P} e^{-i\pi\alpha(k+l)m} \\
&\quad \times e^{i\pi\xi\{[Q_0-\alpha(n-2k)/2]\text{mod }1\}^2/\alpha} e^{-i\pi\xi\{[Q_0+\alpha(n+2l)/2]\text{mod }1\}^2/\alpha}.
\end{aligned} \tag{C4}$$

Making the change in the integer of summation  $n' = n + l - k$  we then obtain

$$\begin{aligned}
\kappa(P, Q, T; P_0, Q_0, 0) &= \sum_{n',m,k,l=-\infty}^{\infty} e^{2\pi i(n'P+mQ)} g_k(\eta) g_l^*(\eta) e^{-2\pi i[(n'+k-l)P_0+mQ_0]} e^{-i\pi\alpha(k+l)m} \\
&\quad \times e^{i\pi\xi\{[Q_0-\alpha(n'-k-l)/2]\text{mod }1\}^2/\alpha} e^{-i\pi\xi\{[Q_0+\alpha(n'+k+l)/2]\text{mod }1\}^2/\alpha}.
\end{aligned} \tag{C5}$$

Now setting  $n' = n$  and changing the order of the factors we obtain Eq. (85).

### APPENDIX D

Here we will show that taking the complex conjugate of propagator (95) and interchanging  $n$  and  $k$ , and  $m$  and  $l$  gives the inverse of propagator (95). This inverse propagator propagates distributions backward in time. The inverse propagator according to this prescription is

$$\begin{aligned}
G^{-1}(n, m; k, l) &= e^{i\pi\alpha(kl-nm)} \int_0^1 d\nu \int_0^1 d\nu' e^{i\pi\xi[(\nu+\alpha k)\text{mod }1]^2/\alpha} e^{-i\pi\xi\nu^2/\alpha} e^{2\pi i(l-m)\nu} \\
&\quad \times e^{i\pi\eta[(\nu'+\alpha m)\text{mod }1]^2/\alpha} e^{-i\pi\eta\nu'^2/\alpha} e^{2\pi i(k-n)\nu'}.
\end{aligned} \tag{D1}$$

To verify that  $G^{-1}$  is indeed the inverse of  $G$  we need only show that

$$\sum_{r,s=-\infty}^{\infty} G(n, m; r, s) G^{-1}(r, s; k, l) = \delta_{n,k} \delta_{m,l} = \sum_{r,s=-\infty}^{\infty} G^{-1}(n, m; r, s) G(r, s; k, l). \quad (\text{D2})$$

We begin with the first equality;

$$\begin{aligned} \sum_{r,s=-\infty}^{\infty} G(n, m; r, s) G^{-1}(r, s; k, l) &= e^{i\pi(kl-nm)\alpha} \sum_{r,s=-\infty}^{\infty} \int_0^1 d\nu \int_0^1 d\nu' \int_0^1 d\nu'' \int_0^1 d\nu''' e^{-i\pi\xi[(\nu+\alpha n)\bmod 1]^2/\alpha} \\ &\quad \times e^{i\pi\xi\nu^2/\alpha} e^{2\pi i(s-m)\nu} e^{-i\pi\eta[(\nu'+\alpha s)\bmod 1]^2/\alpha} e^{i\pi\eta\nu'^2/\alpha} e^{2\pi i(r-n)\nu'} e^{i\pi\xi[(\nu''+\alpha k)\bmod 1]^2/\alpha} \\ &\quad \times e^{-i\pi\xi\nu''^2/\alpha} e^{2\pi i(l-s)\nu''} e^{i\pi\eta[(\nu'''+\alpha s)\bmod 1]^2/\alpha} e^{-i\pi\eta\nu'''^2/\alpha} e^{2\pi i(k-r)\nu''}. \end{aligned} \quad (\text{D3})$$

Using identity (17) to do the sum over  $r$  gives the delta function  $\delta(\nu''' - \nu')$  and then doing the integral over  $\nu'''$  we obtain

$$\begin{aligned} \sum_{r,s=-\infty}^{\infty} G(n, m; r, s) G^{-1}(r, s; k, l) &= e^{i\pi(kl-nm)\alpha} \sum_{r,s=-\infty}^{\infty} \int_0^1 d\nu \int_0^1 d\nu' \int_0^1 d\nu'' e^{-i\pi\xi[(\nu+\alpha n)\bmod 1]^2/\alpha} \\ &\quad \times e^{i\pi\xi\nu^2/\alpha} e^{2\pi i(s-m)\nu} e^{-i\pi\eta[(\nu'+\alpha s)\bmod 1]^2/\alpha} e^{i\pi\eta\nu'^2/\alpha} e^{2\pi i(r-n)\nu'} \\ &\quad \times e^{i\pi\xi[(\nu''+\alpha k)\bmod 1]^2/\alpha} e^{-i\pi\xi\nu''^2/\alpha} e^{2\pi i(l-s)\nu''} e^{i\pi\eta[(\nu'+\alpha s)\bmod 1]^2/\alpha} \\ &\quad \times e^{-i\pi\eta\nu'^2/\alpha} e^{2\pi i(k-r)\nu'} \end{aligned} \quad (\text{D4})$$

$$\begin{aligned} &= e^{i\pi(kl-nm)\alpha} \sum_{r,s=-\infty}^{\infty} \int_0^1 d\nu \int_0^1 d\nu' \int_0^1 d\nu'' e^{-i\pi\xi[(\nu+\alpha n)\bmod 1]^2/\alpha} \\ &\quad \times e^{i\pi\xi\nu^2/\alpha} e^{2\pi i(s-m)\nu} e^{2\pi i(k-n)\nu'} e^{i\pi\xi[(\nu''+\alpha k)\bmod 1]^2/\alpha} e^{-i\pi\xi\nu''^2/\alpha} e^{2\pi i(l-s)\nu''}. \end{aligned} \quad (\text{D5})$$

Now performing the sum over  $s$ , again through the use of identity (17), gives the delta function  $\delta(\nu'' - \nu)$  and doing the integral over  $\nu''$  we obtain

$$\begin{aligned} \sum_{r,s=-\infty}^{\infty} G(n, m; r, s) G^{-1}(r, s; k, l) &= e^{i\pi(kl-nm)\alpha} \int_0^1 d\nu \int_0^1 d\nu' e^{-i\pi\xi[(\nu+\alpha n)\bmod 1]^2/\alpha} e^{i\pi\xi\nu^2/\alpha} \\ &\quad \times e^{2\pi i(l-m)\nu} e^{i\pi\xi[(\nu+\alpha k)\bmod 1]^2/\alpha} e^{-i\pi\xi\nu'^2/\alpha} e^{2\pi i(k-n)\nu'}. \end{aligned} \quad (\text{D6})$$

Doing the integral over  $\nu'$  gives a factor  $\delta_{n,k}$  and we obtain

$$\begin{aligned} \sum_{r,s=-\infty}^{\infty} G(n, m; r, s) G^{-1}(r, s; k, l) &= e^{i\pi(kl-nm)\alpha} \int_0^1 d\nu e^{-i\pi\xi[(\nu+\alpha n)\bmod 1]^2/\alpha} e^{i\pi\xi\nu^2/\alpha} e^{2\pi i(l-m)\nu} \\ &\quad \times e^{i\pi\xi[(\nu+\alpha k)\bmod 1]^2/\alpha} e^{-i\pi\xi\nu^2/\alpha} \delta_{n,k} \end{aligned} \quad (\text{D7})$$

$$= e^{i\pi(kl-nm)\alpha} \int_0^1 d\nu e^{2\pi i(l-m)\nu} \delta_{n,k}. \quad (\text{D8})$$

Finally, doing the integral over  $\nu$  we obtain a factor  $\delta_{m,l}$  and so

$$\sum_{r,s=-\infty}^{\infty} G(n, m; r, s) G^{-1}(r, s; k, l) = \delta_{n,k} \delta_{m,l} \quad (\text{D9})$$

as we claimed. The identity in Eq. (D2) can be proven in the same fashion.

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- [16] These operators were also introduced by S. Knabe, *J. Phys. A* **23**, 2013 (1990) and subsequently used in [9].
- [17] (a) The usual commutation relation, like the classical Poisson bracket relation  $\{q \bmod a, p \bmod b\} = 1$ , is satisfied on the interior of the torus. (b) The allowed states obey the uncertainty principle if the phase space is large enough to hold at least one quantum, i.e.,  $ab \geq h$ .
- [18] V.I. Arnold and A. Avez, *Ergodic Problems of Classical Mechanics* (Addison-Wesley, New York, 1989).
- [19] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I.A. Stegun (Dover, New York, 1965).
- [20] R.S. Dumont and P. Brumer, *J. Chem. Phys.* **87**, 6437 (1987).
- [21] It can be simply shown that the rule for the evaluation of the expectation of any observable  $\hat{A}(MT)$  is  $\langle \hat{A}(MT) \rangle = \int_0^b dp \int_0^a dq [\hat{A}\hat{\rho}(MT)]^{(w)}(P, Q) = \sum_{n,m} A_{n,m} \rho_{-n,-m}(MT)$ . This is the same as the rule for evaluating averages classically.
- [22] A second class of mappings with  $\phi = \begin{pmatrix} 1 + \eta\xi & \eta \\ \xi & 1 \end{pmatrix}$  can be quantized by integrating from just before the  $M$ th kick to just before the  $(M+1)$ st kick. The resulting quantum propagator for the Fourier coefficients is

$$G(n, m; k, l)$$

$$= e^{i\pi\alpha(kl-nm)} \int_0^1 d\nu \int_0^1 d\nu' e^{-i\pi\xi[(\nu+\alpha k) \bmod 1]^2/\alpha} \\ \times e^{i\pi\xi\nu^2/\alpha} e^{2\pi i(l-m)\nu} e^{-i\pi\eta[(\nu'+\alpha m) \bmod 1]^2/\alpha} \\ \times e^{i\pi\eta\nu'^2/\alpha} e^{2\pi i(k-n)\nu'}$$