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Mutually destructive fluctuations in globally coupled arrays

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The phenomenon of mutually destructive fluctuations is observed in numerical simulations of a Josephson-junction array. As an instability is approached, each array element exhibits increasingly wild voltage fluctuations but the total voltage remains relatively steady. Such behavior is expected in any globally coupled oscillator array near the onset of a symmetry-breaking bifurcation; the same phenomenon is demonstrated for systems of coupled iterative maps.

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Populations of coupled nonlinear oscillators are an increasingly studied class of many degree-of-freedom dynamical systems. Particularly good progress has been made for arrays with *global* coupling, i.e., where each oscillator is coupled to all others with equal strength. Global coupling arises, for example, in the study of multimode lasers [1–4], solid-state laser arrays [5], and electrical circuits [6–8] such as one-dimensional series [9–15] and parallel [16,17] Josephson-junction arrays.

In addition to problems where the dynamics is governed by differential equations, a great deal of attention has been devoted to discrete-time globally coupled-map lattices [18]. While the connection between discrete-time arrays and continuous-time systems is not always clear, coupled-map lattices serve as an interesting paradigm for investigating complex nonlinear dynamical systems [19].

Past work on globally coupled arrays has uncovered a number of interesting phenomena, including chaotic itineracy [20–22], attractor crowding [23,24], selective targeting of splay phase states [4], and nongeneric neutral stability [5,12–15,25–27]. In these studies, two themes are recurrent. First, globally coupled arrays tend to exhibit subtle and persistent correlations between the various degrees of freedom, even when operated in the chaotic regime [28–31]. Second, it is often the case that the dynamics is sensitive to even small amounts of external noise, due to either peculiarly weak (or even neutral) dynamics, or the presence of a very large number of coexisting attractors.

It was recently suggested that any globally coupled array might, under the right circumstances, demonstrate a curious kin of robustness to random noise [32,33]. Specifically, as a bifurcation point is approached with fixed input noise, each element in the array exhibits increasingly wild fluctuations, but the total output across the entire array remains relatively steady. This effect was deduced from an analytic study of linearized differential equations describing arrays of identical elements. In this paper we show direct evidence of this effect in simulations of a Josephson-junction array, using the complete nonlinear equations. Our simulations show that this phenomenon persists even if the array elements are not identical. We then consider a model of globally coupled iterative maps and show that the onset of mutually destructive fluctuations is a generic phenomenon which occurs near *symmetry-breaking* bifurcations of the in-phase state. The map model allows us to make quantitative predictions for the magnitude of the effect, and shows that no such effect occurs near *symmetry-preserving* bifurcations. This last prediction is also borne out by our Josephson-junction simulations.

Consider first the Josephson-junction series array depicted in the inset of Fig. 1. Assuming the junctions are identical, the governing dynamical equations can be written in dimensionless form as

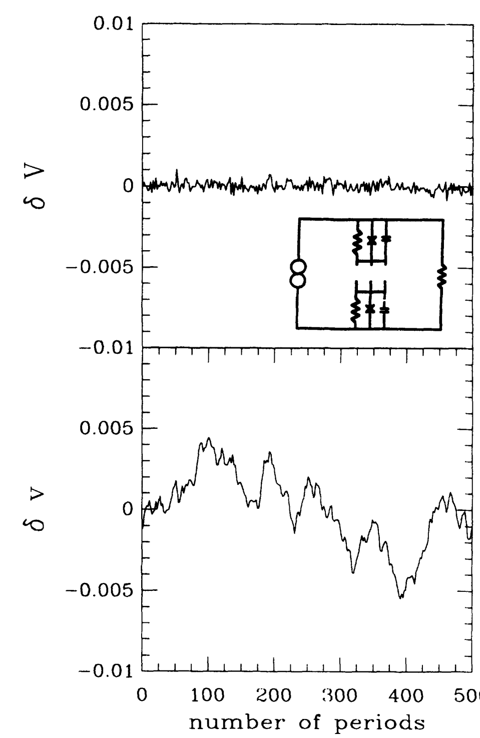

FIG. 1. Circuit schematic of the Josephson-junction array (inset); output fluctuations δV for the entire array (top) and δv for a single junction (bottom). $N=10$, $R=8$, $I_{ac}=2.269$, $I_{dc}=0.25$, $\omega=1$, and $\kappa=10^{-4}$. β's chosen with a 2% spread about a mean of 64.

$$\ddot{\phi}_k+(1/\sqrt{\beta})\dot{\phi}_k+\sin\phi_k+\sum\dot{\phi}_j/R=I+\xi_k\ ,\qquad k=1,\ldots,N\ ,$$

where the overdot denotes differentiation with respect to time, β is the McCumber parameter, R is the resistance of the parallel load, $I=I_{\rm dc}+I_{\rm ac}\cos\omega t$ is the supplied bias current, and ξ_k models the Johnson noise in the junction shunt resistors, taken to be independent white-noise sources $\langle\xi_k(t)\rangle=0;\langle\xi_j(t)\xi_k(s)\rangle=\kappa\delta(t-s)\delta_{jk}$. (For simplicity, we have neglected the noise due to the load resistor.) In these units, the voltage drop across the kth junction is $v_k=\dot{\phi}_k$, while the voltage drop across the entire array is $V=\sum v_k$. We should mention that in applications, it is the total voltage V that is of dominant interest, thus the fluctuations in V are of special concern.

In the absence of noise, we can identify the in-phase state in which all of the junctions oscillate identically, $\phi_k(t)=\phi_0(t)$ for all k. As the parameters are varied, an in-phase attractor can lose stability in a variety of ways. One finds in this particular array that the in-phase attractor can suffer a period-doubling bifurcation as $I_{\rm ac}$ is increased past some critical value, all other parameters being held fixed. Near the bifurcation point we expect even small amounts of noise will generate large output fluctuations. The bottom of Fig. 1 shows the output fluctuations $\delta v=v_1-\langle v_1\rangle$ of one of the junctions. (For clarity only the envelope of the time series is plotted, strobing the voltage once every two drive periods.) Naively, we would expect that the total voltage output would have fluctuations *larger* by a factor of $\sqrt{N}$, but in fact the fluctuations $\delta V=V-\langle V\rangle$ are *smaller,* as seen at the top of Fig. 1. This effect grows more pronounced as the system is tuned closer to the bifurcation point: v_1 fluctuates ever more wildly, while V remains relatively steady.

We emphasize that this effect persists even if the array elements are not identical: in fact, in generating Fig. 1 we introduced a spread in the junction parameter β of 2%.

What we see in Fig. 1 is not simply a case of uncorrelated fluctuations. Though the input fluctuations are uncorrelated, the global coupling generates mutually destructive correlations which result in large-scale cancellations in the total output. This can be seen quite clearly in the simplest "array" case with just two Josephson junctions; however, for larger arrays the cancellations cannot be discerned simply by looking at the time series.

To understand the origin of this effect, consider the system of globally coupled one-dimensional iterative maps:

$$x_k(n+1)=F(x_k(n))+G(X)+\xi_k(n),\quad k=1,\ldots,N\ ,$$

where F and G are any function of their arguments x_k and $X=\sum x_j$, respectively, and the ξ_k's are independent white-noise sources $\langle\xi_k(n)\rangle=0,\langle\xi_j(m)\xi_k(n)\rangle=\kappa\delta_{jk}\delta_{mn}$. Supposing that the noise-free system has an in-phase period-one attractor, $x_k(n)=x^*$ for all k, we can examine the effect of small perturbations by setting $x_k=x^*+h_k$ and linearize the maps about x^*, with the result

$$h_k(n+1)=F'h_k+G'\sum_{j=1}^{N}h_j+\xi_k(n)\ , \tag{1}$$

where the derivatives F' and G' are evaluated on the period-one attractor. The sum of these N equations give the "bulk" response analogous to the total voltage fluctuations in the Josephson-junction array example:

$$H(n+1)=(F'+NG')H(n)+\sum_{k=1}^{N}\xi_k(n)\ .$$

This is a linear equation involving the single variable H, so it is a straightforward matter to calculate the rms fluctuations, with the result

$$\langle H^2\rangle=\frac{N\kappa}{1-(F'+NG')^2}\ . \tag{2}$$

On the other hand, the stability of the fixed point x^* is determined by the N eigenvalues μ_k of the noise-free map Eq. (1), namely $\mu_1=F'+NG'$, and $\mu_2=\mu_3=\cdots=\mu_N=F'$. Thus x^* can go unstable in two fundamentally different ways, depending on whether μ_1 or μ_2 exits the unit interval. These two cases are directly tied to the symmetry type of the instability: The former case corresponds to a symmetry-preserving bifurcation, while the latter case corresponds to a symmetry-breaking bifurcation. From Eq. (2), we see that the system may undergo a symmetry-breaking bifurcation while the bulk fluctuations remain relatively small.

In fact, Eq. (1) can also be solved to give the rms fluctuations for a single element. One finds

$$\langle h_k^2\rangle=\frac{N-1}{N}\frac{\kappa}{1-\mu_2^2}+\frac{\kappa}{N(1-\mu_1^2)}\ . \tag{3}$$

This shows clearly that the fluctuations of a single element grow dramatically regardless of the bifurcation type.

This analysis suggests how to obtain an extremely pronounced effect, namely by choosing parameter values that make μ_1 as small as possible when μ_2 is close to -1 (for a period-doubling bifurcation). Figure 2 shows the results of simulations of linearly coupled logistic maps: $F(x)=ax(1-x), G=\beta\sum x_j/N$, with $N=100$, $\beta=1$, and $\kappa=10^{-8}$ as the system approaches the symmetry-breaking period-doubling bifurcation at $\alpha=1$. Shown are the rms fluctuations in x_1 and X, as well as the analytic results Eqs. (2) and (3). Far from the bifurcation point ($\alpha=0.7$, not shown), the bulk fluctuations are about ten times larger than the single-element fluctuations, which is what one expects for the sum of 100 uncorrelated random variables. As the bifurcation point is approached, the fluctuations in x_1 grow while those in X remain about the same. For $\alpha>0.995$ the fluctuations in x_1 are larger than those in X. Very close to the bifurcation point the linearized analysis is no longer adequate: the observed bulk fluctuations are larger than predicted by Eq. (2), while the single-element fluctuations are smaller than predicted by Eq. (3), though still larger than those in X. The largest value of α in our simulations was $1-\alpha=1.0\times10^{-6}$, for which the rms fluctuation.

We can draw one other conclusion from this analysis,

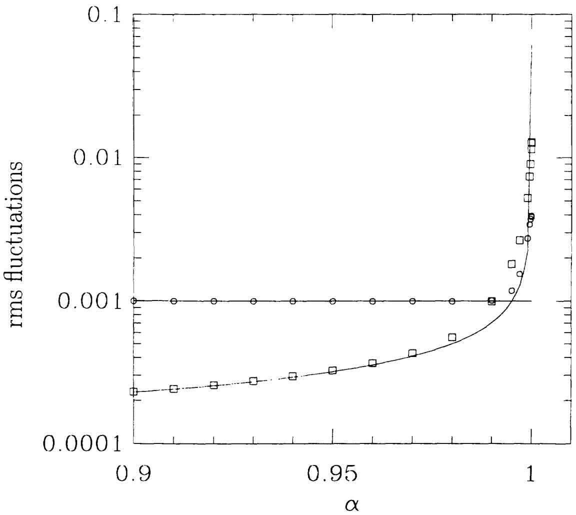

FIG. 2. rms fluctuations vs control parameter for a single element (squares), and the sum of all elements (circles) for the coupled logistic maps. Solid lines represents the predictions of Eqs. (2) and (3).

namely that the single-element fluctuations should have a significantly longer correlation time τ_c than the bulk fluctuations. In particular, one can show

$$\tau_c \sim (\ln|\mu_2|)^{-1} \sim (1-|\mu_2|)^{-1} .$$

This effect is clearly evident in the Josephson-junction simulations shown in Fig. 1.

Returning to the Josephson-junction array equations, we expect that the mutual destructive interference of fluctuations will not occur near a symmetry-preserving bifurcation. However, we find that this particular array does not undergo a symmetry-preserving bifurcation for any positive value of load resistance R. One way to obtain the desired instability is to allow $R<0$, physically; one imagines using a negative impedance converter in place of the conventional resistor [34]. Figure 3 shows the same information as Fig. 1, except that now the system is poised near the onset of a symmetry-preserving period-doubling bifurcation. The fluctuations are now mutually constructive, so that the total voltage fluctuations are substantially larger than those of an individual junction.

In summary, we expect mutually destructive fluctuations to be a generic phenomenon in globally coupled arrays. The essential ingredients are the existence of an in-phase attractor which can suffer a symmetry-breaking bifurcation, and independent noise sources acting on each element. (Typically, one expects an in-phase attractor to suffer both symmetry-preserving and symmetry-breaking bifurcations depending on the parameter regime.) The global-coupling dynamics introduces correlations which greatly suppress the total output fluctuations. Besides the Josephson-junction array, a good candidate to observe this effect is the Nd:YAG (yttrium aluminum garnet) multimode laser with intracavity doubling crystal

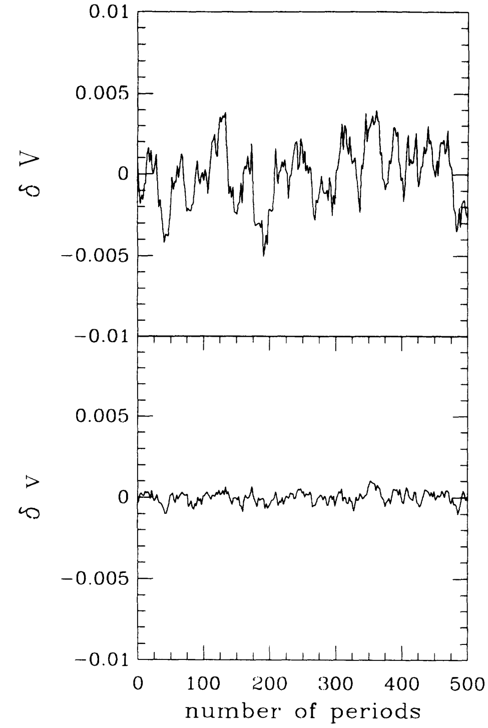

FIG. 3. Same as Fig. 1, but near a symmetry-preserving period-doubling bifurcation; $N=10$, $R=-80$, $\kappa=10^{-4}$, $I_{ac}=1.1754$, $I_{dc}=0.25$, and $\omega=1$. β's chosen with a 2% spread about a mean of 2.56.

[2]. The dynamics of this system is globally coupled, exhibiting an in-phase attractor for sufficiently low pump levels which is known to undergo a symmetry-breaking bifurcation [35,36]. Other electrical circuits [6–8] with globally coupled dynamics are also natural candidates where mutually destructive fluctuations might be observed.

Finally, one is led naturally to the question of whether there is a deterministic counterpart to the phenomenon studied here. That is, are there situations where (in the absence of noise) each element evolves chaotically while the bulk output remains relatively steady? If so, this would have to involve phase-space considerations of a global nature, not just in the vicinity of a simple in-phase orbit. Consequently, if such behavior exists at all it is probably less typical than noisy destructive fluctuations. Nevertheless, an effect like this may be behind recent experimental observations in multimode laser experiments [37–39] wherein the chaotic fluctuations of the individual mode intensities display decidedly different statistics than do the total intensity.

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