Conserved energy approximation to wave scattering by a nonlinear interface

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An alternative approach to the investigation of the space-time evolution of an electromagnetic field in a nonlinear medium is presented. An adiabatic theory to obtain the dynamic picture of reflection and penetration of electromagnetic radiation into a nonlinear medium is devised. It is based upon the energy balance equation and uses the nonlinear modes of the stationary problem like trial functions. Filamentation and soliton generation are analyzed and the model easily predicts the giant nonlinear Goos-Hänchen beam shifts that emerge from previously numerically intensive investigations. The method is nonperturbative and is straightforward to apply to linear-nonlinear interfaces.

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I. INTRODUCTION

The problems associated with the interaction of electromagnetic waves with nonlinear interfaces continue to be of very great interest. This interest is driven by the desire to understand bistability in reflection (transmission) and, in particular, to understand how electromagnetic beams behave at linear-nonlinear or nonlinear-linear interfaces. There is, for example, the possibility of soliton generation or substantial nonlinear changes to the magnitudes of the Goos-Hänchen shifts.

This whole field was stimulated by the classic pioneering work of Kaplan [1–7], who in a series of elegant papers laid the foundations of bistability (multistability) theory. The work of Kaplan, reviewed extensively by Gibbs [8], created the first theory of plane-wave scattering at the interface between a semi-infinite linear medium and a semi-infinite nonlinear medium. Both plane homogeneous and inhomogeneous surface results were considered and both positive and negative nonlinearities were investigated. Kaplan was able to calculate [2] amplitude profiles of nonlinear surface waves under conditions of total internal reflection and the total relative power carried by the surface waves, as a function of the input intensity. The question of how a two-dimensional Gaussian beam, incident on a nonlinear medium through a linear dielectric, behaves was addressed subsequently using a numerical simulation [6]. It was concluded that a nonlinear interface cannot exhibit bistable reflectivity for an incident Gaussian beam but that sharp changes occur in the reflectivity at well-defined intensity thresholds. It was also shown, numerically, that giant Goos-Hänchen shifts are possible [6] and that the output beam amplitude varies very rapidly with small intensity changes. This feature led to speculations concerning applications as light-controlled scanning elements. The first detailed experimental investigations of the Kaplan concepts were performed by Smith and co-workers [9–11].

Such applications are also suggested by very recent work that uses a well-known *particle* model [12] to analyze the problem of the oblique incidence of a self-focused channel on an interface separating two or more self-focusing media [13–15]. This method represents the self-focused channel as an equivalent particle and deals with a "matched" interface. In other words, the interface is between nonlinear media which have *almost* identical *linear refractive indices and nonlinear properties.* Solitons approaching such an interface are only weakly perturbed, and it is then possible to represent the physical problem as being that of a "particle" in a potential well and to make stability predictions. The model we present here will address a nonperturbative interface and is not the same as the equivalent particle. It is an adiabatic theory based on an energy balance equation using the known nonlinear stationary states like trial functions. It is worth emphasizing again that it is nonperturbative, unlike the recent *single*-particle theory. It applies to a quite different type of interface. The self-consistent waveguide that constitutes a nonlinear channel is a one-parameter system that can be described by the position of the maximum intensity.

Finally, it is interesting that the way in which nonlinearity, induced by strong fields in the dielectric function of media, changes a reflected field and also any penetration was investigated many years ago, in connection with the generation of electro-acoustic solitons in a plasma whose boundary was exposed to electromagnetic waves [16]. Under certain conditions, electro-acoustic waves escape from a skin layer and, during propagation into the plasma, transform into solitons. The physical picture of such penetration by the incoming radiation is quite simple. The radiation pressure reduces the density of the plasma and, in any region where this occurs, radiation tends to concentrate. It eventually propagates deep into the plasma as an electro-acoustic wave. A similar thing takes place during nonstationary reflection from a weakly inhomogeneous plasma. For instance, if a plasma interacts with p-polarized radiation then radiation capture in "cavitons" occurs. These form in the plasma-resonance region [17,18].

II. PLANE WAVES INCIDENT UPON A NONLINEAR PLANE-WAVE DIELECTRIC

It should be recognized that first plane-wave theoretical results were produced by Kaplan [1–5]. Nevertheless, it is more informative, for the later development in this paper, to generate afresh some of plane-wave results and to present a few illustrative numerical examples in the language we intend to use here. This is not to say that these results are truly original; however, because after some transformation they could be obtained from Kaplan's papers [1–2].

Suppose that a strong, constant-amplitude, *s*-polarized, plane electromagnetic wave $\mathbf{E}=(0,E_y,0)$ and is incident at an angle θ upon an interface between a linear (dielectric permittivity $\epsilon=\epsilon_1$) and a nonlinear (dielectric permittivity $\epsilon=\epsilon_2+\alpha|\mathbf{E}|^2$, $\alpha>0$), nondissipative medium, where α is the nonlinear coefficient. Let the incident angle θ be greater than the critical angle of total internal reflection so that $\theta>\theta_c=\sin^{-1}\sqrt{\epsilon_2/\epsilon_1}$, $\epsilon_1>\epsilon_2$. As shown in Fig. 1, the z axis of the Cartesian coordinate system is perpendicular to the interface, and the x,y axes lie in the plane of interface. Assume that the nonlinear medium occupies the $z>0$ half space and that a stationary nonlinear TE wave has an electric field of the form

$$E_y=\frac{1}{2}\left\{\frac{A}{\sqrt{\alpha}}\exp(-i\omega t+ik_xx)+\text{c.c.}\right\}, \tag{2.1}$$

where A is dimensionless, $1/\sqrt{\alpha}$ has dimensions of the electric field, k_x is a wave number, ω is the angular frequency, c.c. means complex conjugate, and the amplitude A is a solution of the equation

$$\frac{\partial^2 A}{\partial z^2}+k_0^2(\epsilon_2-\gamma^2+|A|^2)A=0\ . \tag{2.2}$$

The quantities γ and k_0 are defined as $\gamma=k_x/k_0=\sqrt{\epsilon_1}\sin\theta$ and $k_0=\omega/c$, where c is the vacuum velocity of light. The amplitude A decreases as $z\to\infty$ and satisfies the usual boundary conditions that the tangential components of the electric and magnetic fields are continuous. Hence, in a nonlinear medium, A has the *stationary-state waveguide channel* [19] form

$$A(z)=\sqrt{2}\kappa_0e^{i\phi_0}\text{sech}[\kappa_0k_0(z\pm z_0)]\ , \tag{2.3}$$

where $\kappa_0=\sqrt{\gamma^2-\epsilon_2}$ and $z=z_0$ is the location of the peak amplitude (the channel position). ϕ_0 is a constant shift which can be determined from the *constant amplitude,* $A_i=A_0$, of an incident stimulant (pump) wave, by the application of the continuity conditions. If $\kappa_1=\sqrt{\epsilon_1-\gamma^2}$, $\xi_0=\tanh(k_0\kappa_0z_0)$ and the phase of the incoming pump wave is zero, then, under the total internal reflection conditions,

$$A_0=\frac{1}{\sqrt{2}}\kappa_0\left[(1-\xi_0^2)\left[1+\frac{\kappa_0^2}{\kappa_1^2}\xi_0^2\right]\right]^{1/2}, \tag{2.4}$$

$$\phi_0=\tan^{-1}\left[\frac{\kappa_0}{\kappa_1}\xi_0\right]. \tag{2.5}$$

These are the stationary-state conditions. Note that, since z_0 is the position of the peak intensity of the waveguide channel (2.3), ξ_0 will now be called the *channel parameter.*

A typical dependence of the channel parameter ξ_0 upon the amplitude A_0 of the pump wave is plotted in Fig. 2, where Fig. 2(a) corresponds to angles of incidence that satisfy the condition $\kappa_1>\kappa_0$:$\epsilon_2<\gamma^2<\frac{1}{2}(\epsilon_2+\epsilon_1)$ and Fig. 2(b) corresponds to $\kappa_1<\kappa_0$:$\frac{1}{2}(\epsilon_1+\epsilon_2)<\gamma^2<\epsilon_1$. Figure 2 shows that threshold amplitudes $A_0^*=\kappa_0/\sqrt{2}$ and $A_0^m=A_0^*[(\kappa_0^2+\kappa_1^2)/2\kappa_0\kappa_1]$ exist. If the pump-wave amplitude exceeds these values then stationary solutions that vanish as $z\to\infty$ cannot be created. As stated earlier, it is more appropriate to show how to generate Fig. 2 from the present model than to try and quote it directly. It should be emphasized, however, that Fig. 2 could be obtained from the Fig. 6 contained in the fundamental Kaplan paper [2]. The energy density of the electromagnetic field, calculated per unit area of the (x,y) plane in the nonlinear nonmagnetic medium, is

$$W=\tfrac{1}{2}\int_0^\infty\langle\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}\rangle dz$$
$$=\frac{\epsilon_0}{2}\int_0^\infty\langle\epsilon_2E_y^2+c^2\mu_0^2H_x^2+c^2\mu_0^2H_z^2\rangle dz\ , \tag{2.6}$$

where $\langle\ \rangle$ means the time average, μ_0 is the permeability of free space, $\mathbf{D}$ is the displacement vector, $\mathbf{H}$ is the

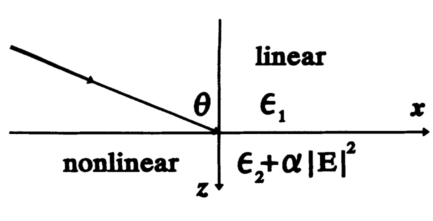

FIG. 1. Coordinate system at a linear-nonlinear interface.

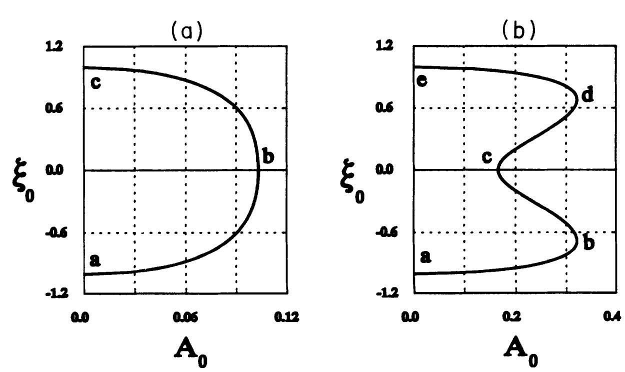

FIG. 2. The stationary dependence of the channel axis coordinate on the incident wave amplitude. Nondissipative nonlinear medium. (a) $\kappa_0=0.146$, $\kappa_1=0.196$; (b) $\kappa_0=0.235$, $\kappa_1=0.065$.

magnetic-field vector, and **B** is the magnetic-flux density vector.

Hence

$$W=\frac{\kappa_0\epsilon_0}{2k_0\alpha}(1+\xi_0)\left\{\epsilon_2+\gamma^2+\frac{5}{3}\kappa_0^2+\frac{\kappa_0^2}{3}\xi_0(1-\xi_0)\right\}$$

$$\simeq\frac{\kappa_0\epsilon_0}{2k_0\alpha}(\epsilon_2+\gamma^2)(1+\xi_0)\ , \qquad (2.7)$$

because the terms $\frac{5}{3}\kappa_0^2$, and $(\kappa_0^2/3)\xi_0(1-\xi_0)$ are always much smaller than the first two terms. The dependence of the energy density W on A_0, the amplitude of the incident pump wave, is determined by Eqs. (2.4) and (2.7). The maximum value of W is $W^{\max}=\epsilon_0\kappa_0(\epsilon_2+\gamma^2)/k_0\alpha)$ and occurs for $\xi_0=1$ when the nonlinear channel axis moves out to infinity (i.e., $z_0\rightarrow\infty$). Hence

$$W=\tfrac{1}{2}W^{\max}(1+\xi_0) \qquad (2.8)$$

and the energy flux density along the x direction is

$$P_x=\int_0^\infty S_x dz=-\tfrac{1}{2}\mathrm{Re}\int_0^\infty E_y H_z^* dz=\tfrac{1}{2}P_x^{\max}(1+\xi_0)\ , \qquad (2.9)$$

where $P_x^{\max}=2\kappa_0\gamma\epsilon_0 c/(\alpha k_0)$.

III. BASIC DYNAMICAL EQUATION

The investigation of the space-time evolution of the electromagnetic field in a nonlinear medium can proceed from the energy conservation law

$$\frac{\partial w}{\partial t}+\mathrm{div}S=\frac{\partial w}{\partial t}+\frac{\partial S_x}{\partial x}+\frac{\partial S_z}{\partial z}=0\ , \qquad (3.1)$$

where w is the *energy per unit volume* ($W=\int_0^\infty w dz$) and **S** is the Poynting vector. The integration of Eq. (3.1) over the range $0\le z\le\infty$ yields

$$\frac{\partial W}{\partial t}+\frac{\partial P_x}{\partial x}=S_i(z=0,t,x)-S_r(z=0,t,x)\ , \qquad (3.2)$$

where S_i,S_r are the z components of the flux of the incident and reflected energy at the interface, respectively, $S_z(z=\infty,t,x)=0$ and $S_z(z=0,t,x)=S_i(z=0,t,x)-S_r(z=0,t,x)$.

If the characteristic scale of the incident wave is much larger than $(1/k_0\kappa_0)$, the characteristic scale of the electromagnetic field in the nonlinear medium, then the rate of change of the electromagnetic field structure is adiabatically slow. This physical property permits the dynamics of space-time distribution to be determined in terms of the stationary states. The basic step is to use the quantities P_x and W, in Eq. (3.2), to form the evolution equation of the channel parameter. This is done by introducing $\xi_0=\xi(x,t)$ as a slowly varying function of x and t. The substitution of W and P_x from Eqs. (2.8) and (2.9), using $\xi_0=\xi(x,t)$, gives

$$\frac{\partial\xi}{\partial t}+v_g\frac{\partial\xi}{\partial x}=\frac{2}{W_s^{\max}}\{S_i(z=0,t,x)-S_r(z=0,t,x)\}\ , \qquad (3.3)$$

where $v_g=P_x^{\max}/W^{\max}=2\gamma c/(\epsilon_2+\gamma^2)$ is a group velocity. The latter quantifies the transport of the electromagnetic energy along the axis of the solitary self-consistent waveguides that are formed. The left-hand side of Eq. (3.3), therefore, describes the space-time evolution of the channel axis parameter ξ, while the right-hand side of Eq. (3.3) is the net energy flow determined by the value of the incident-field amplitude at the interface. The energy-transfer method is similar to that used in astrophysics or space physics, where we imagine a source pumping energy down a "light tube." In the nonstationary state, the pump wave has an amplitude $A_i(x,t)$ which is not now a constant, as it was in the derivation of Eq. (2.4). The incident energy flow is

$$S_i=\frac{c}{2}\kappa_1|A_i(t,x)|^2\frac{\epsilon_0}{\alpha}$$

and, for a field distribution *close to the stationary state* in the nonlinear medium, the interface radiates

$$S_r\simeq\frac{c}{2}\kappa_1|A_0|^2\frac{\epsilon_0}{\alpha}\ ,$$

where A_0 is given by Eq. (2.4). The substitution of S_i and S_r into Eq. (3.3) yields

$$\frac{\partial\xi}{\partial t}+v_g\frac{\partial\xi}{\partial x}=\frac{\omega}{\epsilon_2+\gamma^2}\left\{\frac{\kappa_1}{\kappa_0}A_i^2(t,x)-\frac{\kappa_0\kappa_1}{2}(1-\xi^2)\left[1+\frac{\kappa_0^2}{\kappa_1^2}\xi^2\right]\right\}\ , \qquad (3.4)$$

which will now be used as the basic evolution equation. The validity of this approach depends upon whether any background radiation, which is ignored here, is significant. The theory will be applied to soliton generation, with energy provided by the arrival of a pump wave, or beam, at an interface between a linear medium and a nonlinear medium. In the framework of the theory, the generated solitons must be sufficiently far apart for the soliton interaction not to involve any significant background energy. In the ideal theory developed here the solitons move far away from the interface before the next soliton is generated, which is why the background is absent. In a real situation the solitons do not, of course, reach infinity before the next soliton is generated. Hence the theory here is only valid for a sparse lattice of solitons. In any case, for this problem, "infinity" is only the order of the skin depth. As will be made clear, later on, the period of soliton generation is restricted here to be large enough for them to be far from each other, and the calculation assumes the form of the channels without any background. The conditions we derive will show that the input energy is consistent with the original assumption of a sech channel. The numerical simulations will also show that most of the radiation is in the reflected field (because of the total reflection condition in the linear limit), which is also consistent with ignoring the background.

The introduction of the dimensionless variables $\eta=\frac{1}{2}(\omega/c)x$ and $\tau=\frac{1}{2}c\omega t/\gamma v_f$, where $v_f=(\epsilon_2+\gamma^2)c/$

(2γ) is the phase velocity along the interface and $v_f v_g = c^2$, transforms Eq. (3.4) into

$$\frac{\partial \xi}{\partial \tau}+\gamma\frac{\partial \xi}{\partial \eta}=\frac{\kappa_1}{\kappa_0}\left[A_i^2(\tau,\eta)-\frac{\kappa_0^2}{2}(1-\xi^2)\left[1+\frac{\kappa_0^2}{\kappa_1^2}\xi^2\right]\right] . \tag{3.5}$$

IV. INTERACTION OF PLANE HOMOGENEOUS WAVES OR BEAMS WITH A NONLINEAR INTERFACE

A. Plane waves

This case is included here as the first illustration of the use of the new method. A plane wave, of constant amplitude, is represented here by $A_i(\tau,\eta)=A_0=\text{const}$. First, the stability of the stationary solution, with respect to small perturbations $\delta\xi$ to the steady-state nonlinear channel axis, is investigated. The basic equation is, from Eq. (3.5),

$$\begin{aligned}\frac{\partial}{\partial \tau}(\delta\xi)+\gamma\frac{\partial}{\partial \eta}(\delta\xi)&=\delta\left[\frac{\partial \xi}{\partial \tau}+\gamma\frac{\partial \xi}{\partial \eta}\right]\\&=\delta\left\{\frac{\kappa_1}{\kappa_0}\left[A_i^2-\frac{\kappa_0^2}{2}(1-\xi^2)\left[1+\frac{\kappa_0^2}{\kappa_1^2}\xi^2\right]\right]\right\}\\&=\rho(\delta\xi) ,\end{aligned} \tag{4.1}$$

where

$$\rho=\kappa_0\kappa_1\xi_0\left[\left[1-\frac{\kappa_0^2}{\kappa_1^2}\right]+2\frac{\kappa_0^2}{\kappa_1^2}\xi_0^2\right] ,$$

and $\xi_0(A_0^2)$ determines the dependence of the steady-state channel axis coordinate position upon the amplitude of the incident pump field. For a given $\delta\xi>0$ (or <0), $\rho>0$ leads to $\partial/\partial\tau(\delta\xi)+\gamma(\partial/\partial\eta)(\delta\xi)>0$ (or <0), which means that $|\delta\xi|$ will increase exponentially. Such a system is unstable and the state, for which $\rho<0$, is stable. From Eq. (4.1) it can be appreciated that the *b*-*c* part of the curve shown in Fig. 2(a), and the *b*-*c*, *d*-*e* parts of the curve shown in Fig. 2(b), cover the convective unstable state regime [20]. Indeed, these parts are characterized by $\partial W/\partial A_0<0$. The equilibrium states in Fig. 2(a) correspond, physically, to a nonlinear skin effect and are labeled (*a*-*b*). If the physical conditions correspond to Fig. 2(b), then a stable channel-like state (*c*-*d*) occurs in the nonlinear medium. It is important to note that transitions from unstable to stable states are accompanied by radiation or absorption of electromagnetic energy, equal in magnitude to the energy difference between the corresponding steady states. These energies are monotonically increasing functions of ξ. The stable steady state of the waveguide channel type (*c*-*d*) can be achieved by first increasing the amplitude of the incident pump field to a value A_0^m, thus leading to a displacement of the nonlinear channel to the region $\xi>0$ [in accordance with Eq. (3.5)]. The amplitude is then decreased to a value that lies in the interval $A_0^* < A_i < A_0^m$.

Consider now solutions spatially homogeneous in the x direction $(\partial/\partial\eta=0)$, for the case in which the amplitude of the pump wave $A_i(\tau,\eta)=A_i=\text{const}$ is greater than the steady-state threshold amplitudes (A_0^*, A_0^m). Equation (3.5) then becomes

$$\frac{d\xi}{d\tau}=\frac{\kappa_1}{\kappa_0}\left[A_i^2-\frac{\kappa_0^2}{2}(1-\xi^2)\left[1+\frac{\kappa_0^2}{\kappa_1^2}\xi^2\right]\right] . \tag{4.2}$$

In this case the interaction results in the periodic generation of solitons (waveguides) that penetrate deeply into the nonlinear medium. According to Eq. (4.2), the right-hand part of which is always positive $(A_i > A_0^*, A_0^m)$, the displacement of the channel or waveguide is $\xi(\tau)$, and this tends to infinity, as time increases. It should be recalled, however, that $|\xi|\le 1$, so that, when $\xi=1$, the process of soliton radiation ceases and the field in the medium returns to its initial state where $\xi=-1$. In practice, the connection of the nonlinear channel to the pump wave is effectively cutoff when the coordinate value of the channel axis reaches the skin-layer thickness $(L_{sk}\simeq 1/k_0\kappa_0)$. After leaving the skin-layer region, the soliton speed remains approximately constant and equal to $L_{\rm sk}/T_s$, where $T_{s)}$ is the period of soliton generation, i.e., the time interval between two successively radiated solitons. T_s can be calculated from Eq. (4.2). For angles of incidence close to the angle of total internal reflection $(\kappa_0\ll\kappa_1)$, Eq. (4.2) reduces to

$$\frac{d\xi}{d\tau}=\frac{\kappa_1}{\kappa_0}A_i^2-\frac{\kappa_0\kappa_1}{2}(1-\xi^2) . \tag{4.3}$$

Equation (4.3) integrates to

$$\xi(\tau)=\sqrt{\mu^2-1}\tan\left[\frac{\kappa_0\kappa_1}{2}\sqrt{\mu^2-1}\tau-\tan^{-1}\left[\frac{1}{\sqrt{\mu^2-1}}\right]\right] , \tag{4.4a}$$

$$\tau=\frac{2}{\kappa_0\kappa_1}\frac{1}{\sqrt{\mu^2-1}}\left[\tan^{-1}\left[\frac{\xi}{\sqrt{\mu^2-1}}\right]+\tan^{-1}\left[\frac{1}{\sqrt{\mu^2-1}}\right]\right] , \tag{4.4b}$$

where $\mu=\sqrt{2}(A_i/\kappa_0)=A_i/A_0^*$ is going to be called the *superthreshold,* since $\mu>1$ corresponds to energy above the threshold needed for soliton generation.

The dimensionless soliton generation period, T_s, the time during which ξ goes from -1 to $+1$, is obtained by substituting $\xi=1$ into Eq. (4.4), i.e.,

$$T_s=\frac{4}{\kappa_0\kappa_1}\frac{1}{\sqrt{\mu^2-1}}\tan^{-1}\left[\frac{1}{\sqrt{\mu^2-1}}\right] . \tag{4.5}$$

The dependence of $T_s(\mu^2)$ upon μ^2 is shown in Fig. 3. As the angle of incidence of the wave increases, the soliton generation threshold increases from A_0^*, when $\kappa_0<\kappa_1$ to A_0^m for which $\kappa_0>\kappa_1$.

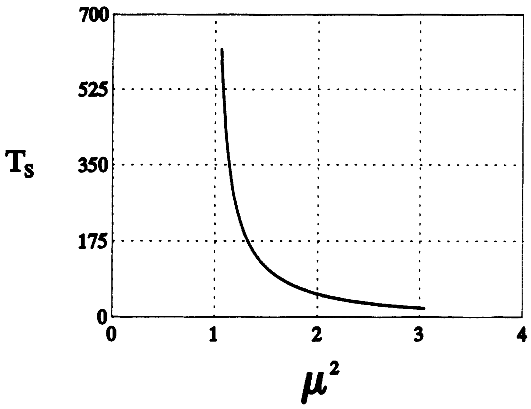

FIG. 3. Dependence of the soliton generation period on the superthreshold rate (μ^2).

The nonstationary reflection coefficient, $R(\tau)$, is defined as the ratio of the instantaneous energy flux leaking from the medium,

$$S_r=\frac{c}{2}\kappa_1|A_0(\tau)|^2\frac{\epsilon_0}{\alpha}\ ,$$

to the incident energy flux

$$S_i=\frac{c}{2}\kappa_1|A_i|^2\frac{\epsilon_0}{\alpha}$$

and is given by

$$R(\tau)=\frac{[1-\xi^2(\tau)]\left[1+\frac{\kappa_0^2}{\kappa_1^2}\xi^2(\tau)\right]}{\mu^2}\ , \tag{4.6}$$

where $\xi(\tau)$ is given by Eq. (4.4a) and $R(\tau)=1/\mu^2$ at its maximum. Maximum values of $1/\mu_1^2$, $1/\mu_2^2$ are shown in Fig. 4(a). For angles of incidence close to the critical angle ($\kappa_0 \ll \kappa_1$), $R(\tau)$ reduces to $(1-\xi^2)/\mu^2$.

As μ^2, the superthreshold, increases, T_s and the peak value of $R(\tau)$ decrease. The dependence of $R(\tau)$ on the time τ is illustrated in Fig. 4(a), where $\mu_1>\mu_2$. The minima in the reflection coefficient correspond to the moments when solitons are emitted. The average of the reflection coefficient $R(\tau)$, over the period T_s, is

$$\langle R\rangle=1-\frac{\sqrt{\mu^2-1}}{\mu^2\tan^{-1}\left[\frac{1}{\sqrt{\mu^2-1}}\right]}\ . \tag{4.7}$$

Its dependence upon μ^2 is shown in Fig. 4(b). For $\mu^2<1$, no soliton will be generated.

Stationary solutions also exist in the form of nonlinear plane waves that can penetrate deeply into the nonlinear medium. Such waves, as is well known, are unstable and break down into self-consistent waveguide channels (filaments) in which the energy of the electromagnetic field is concentrated. The steady nonlinear state, as a result of the development of this instability, may also be investigated with Eq. (3.5) after setting $\partial/\partial\tau=0$, i.e.,

$$\gamma\frac{d\xi}{d\eta}=\frac{\kappa_1}{\kappa_0}\left[A_i^2-\frac{\kappa_0^2}{2}(1-\xi^2)\left[1+\frac{\kappa_0^2}{\kappa_1^2}\xi^2\right]\right]\ . \tag{4.8}$$

The solutions of Eq. (4.8) are similar to the solutions of Eq. (4.2) but with the substitution $\tau\to\eta/\gamma$. Hence, by analogy with Eq. (4.5), the *spatial* period of the filament "lattice" formed is, in dimensionless form,

$$L_s=\frac{4\gamma}{\kappa_0\kappa_1}\frac{1}{\sqrt{\mu^2-1}}\tan^{-1}\left[\frac{1}{\sqrt{\mu^2-1}}\right]\ . \tag{4.9}$$

The general (space-time) solution of Eq. (3.5), for the case when the angles of incidence are close to the angle of total internal reflection, is

$$\xi(\tau,\eta)=\sqrt{\mu^2-1}\left\{\frac{\tan\left[\frac{\kappa_0\kappa_1}{2}\sqrt{\mu^2-1}\,\tau\right]+\frac{F_0(\eta-\gamma\tau)}{\sqrt{\mu^2-1}}}{1-\frac{F_0(\eta-\gamma\tau)}{\sqrt{\mu^2-1}}\tan\left[\frac{\kappa_0\kappa_1}{2}\sqrt{\mu^2-1}\,\tau\right]}\right\}, \tag{4.10}$$

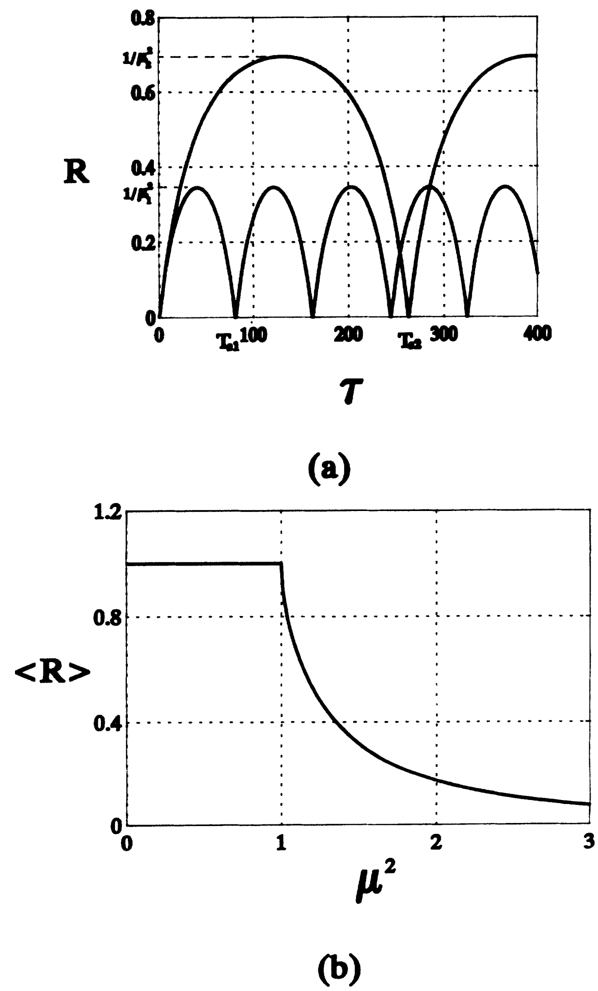

FIG. 4. (a) Time dependence of the reflection coefficient when the incident field is a monochromatic plane wave. (b) Time-averaged dependence of the reflection coefficient on the superthreshold rate (μ^2).

where $F_0(\eta)=\xi(0,\eta)$, the initial displacement of the channel, is determined by the pump wave. In this case a two-dimensional and nonstationary "lattice" of solitons makes up the spatial structure of the wave field. Equation (4.10) returns to Eq. (4.4) if $F_0(\eta-\gamma\tau)\equiv-1$.

The character of this solution of the nonlinear problem is corroborated qualitatively by numerical calculations of interaction of a plane wave with an interface [20]. These calculations demonstrate the penetration of an incoming electromagnetic field into a nonlinear medium in the form of generated stationary channels. As previously discussed, the theory developed above does not take into account the interaction between the modes and, therefore, the distance between the channels (L_s) must exceed the characteristic scale ($1/k_0\kappa_0$). Also the period of the soliton excitation (T_s) must be much greater than the steady-state relaxation time ($\gamma/\omega\kappa_0$). These requirements lead to an upper limit on value of the superthreshold rate, i.e.,

$$\mu^2 \ll \mu_{c1}^2=\frac{8}{\kappa_1}\frac{\gamma^2+\epsilon_2}{2\gamma}$$

(in the case of soliton generation) and

$$\mu^2 \ll \mu_{c2}^2=\frac{8}{\kappa_1}\gamma$$

(in the case of filamentation) [21]. In the limit of total reflection, $\gamma\simeq\epsilon_2^{1/2}$, the values of μ_{c1} and μ_{c2} are close to each other.

B. Electromagnetic beams

In this section we consider the interaction of time-continuous but space-limited waves (beams) with a nonlinear medium. The analysis is based on Eq. (3.5), and uses an assumption that $\partial\xi/\partial\tau=0$. Of particular interest here are the changes in the form of beam after reflection, any space shift that occurs, and, finally, soliton generation.

Equation (3.5) becomes, in this case,

$$\gamma\frac{d\xi}{d\eta}=\frac{\kappa_1}{\kappa_0}\left[A_i^2(\eta)-\frac{\kappa_0^2}{2}(1-\xi^2)\left[1+\frac{\kappa_0^2}{\kappa_1^2}\xi^2\right]\right], \tag{4.11}$$

where $A_i(\eta)$ is a slowly space-varying function.

Beams are localized in space and have two possible ways of interacting with a nonlinear medium, as can be seen from the constant wave amplitude case already considered. First of all, there is the possibility that the peak intensity of the electromagnetic beam is less than the threshold values A_0^*, A_0^m. For this condition there is only a nonlinear modification of the skin layer. This occurs in a manner similar to a linear interaction of a beam with the medium. If the peak amplitude of the beam is above threshold ($A_i>A_0^*$, if $\kappa_0<\kappa_1$, or $A_i>A_0^m$ if $\kappa_0>\kappa_1$) it is possible to excite a nonlinear channel which, through its dependence upon value of A_i, can radiate completely back to the linear medium or can break its connection to the pump wave and penetrate deeply into the nonlinear medium. For both cases the nonlinear interaction can be usefully characterized by a reflected-beam space shift, the Goos-Hänchen shift, defined as

$$\Delta=\frac{\int_{-\infty}^{\infty}\eta[A_r^2(\eta)-A_i^2(\eta)]d\eta}{\int_{-\infty}^{\infty}A_i^2(\eta)d\eta}, \tag{4.12}$$

where A_r is the reflected amplitude, and there is a spreading factor

$$K=\frac{\int_{-\infty}^{\infty}\eta^2A_i^2(\eta)d\eta\int_{-\infty}^{\infty}A_r^2(\eta)d\eta}{\int_{-\infty}^{\infty}A_i^2(\eta)d\eta\int_{-\infty}^{\infty}\eta^2A_r^2(\eta)d\eta}, \tag{4.13}$$

i.e., K is the ratio of the widths of the incident and reflected beams.

For states close to their linear values, $\xi=-1+\alpha(\eta)$ ($\alpha\ll1$), so that

$$\frac{d\alpha}{d\eta_b}+2\alpha=\frac{2}{\kappa_0^2\left[1+\frac{\kappa_0^2}{\kappa_1^2}\right]}A_i^2(\eta_b), \tag{4.14}$$

where

$$\eta_b=\frac{\kappa_0\kappa_1}{2\gamma}\left[1+\frac{\kappa_0^2}{\kappa_1^2}\right]\eta .$$

Equation (4.14) is a first-order, linear, ordinary differential equation, which, after applying the condition $\alpha(\eta_b=-\infty)=0$, has the solution

$$\alpha(\eta_b)=2e^{-2\eta_b}\int_{-\infty}^{\eta_b}e^{2t'}A_i^2(t')\frac{dt'}{\kappa_0^2\left[1+\frac{\kappa_0^2}{\kappa_1^2}\right]} . \tag{4.15}$$

Since

$$A_r^2(\eta_b)\simeq A_0^2=\frac{\kappa_0^2}{2}(1-\xi^2)\left[1+\frac{\kappa_0^2}{\kappa_1^2}\xi^2\right]$$
$$\simeq\kappa_0^2\left[1+\frac{\kappa_0^2}{\kappa_1^2}\right]\alpha ,$$

then

$$A_r^2(\eta_b)=2e^{-2\eta_b}\int_{-\infty}^{\eta_b}e^{2t'}A_i^2(t')dt' . \tag{4.16}$$

The various integrations $\int A_r^2d\eta$, $\int\eta A_r^2d\eta$, $\int\eta^2A_r^2d\eta$ can now be calculated to give the following *linear* ($\alpha\ll1$) results:

$$\Delta_L=\frac{2\gamma}{\kappa_0\kappa_1\left[1+\frac{\kappa_0^2}{\kappa_1^2}\right]}\simeq\frac{2\gamma}{\kappa_0\kappa_1}, \tag{4.17}$$

$$K_L=\frac{2\eta_0^2}{1+2\eta_1+2\eta_0^2}, \tag{4.18}$$

where $\eta_0^2=\int_{-\infty}^{\infty}\eta_b^2A_i^2(\eta_b)d\eta_b/\int_{-\infty}^{\infty}A_i^2(\eta_b)d\eta_b$ is the square of the mean width of the incident beam, and $\eta_1=\int_{-\infty}^{\infty}\eta_bA_i^2(\eta_b)d\eta_b/\int_{-\infty}^{\infty}A_i^2(\eta_b)d\eta_b$ measures the distortion of the input beam from a symmetrical one. η_1 is zero for a perfectly symmetrical input beam.

In the region where the incident angles are close to the total internal reflection angle ($\kappa_0^2\ll\kappa_1^2$), ϕ_0, the phase shift of the field given by Eq. (2.5), is a rapidly varying function and the Goos-Hänchen shift of the beam can be obtained from the theory given by Brekhovskikh [22], i.e.,

$$\Delta_L = -2\frac{\partial\phi_0}{\partial\gamma} = \frac{2\gamma}{\kappa_0\kappa_1} . \tag{4.19}$$

This agrees with Eq. (4.17).

In the nonlinear regime, there is also a space shift of the reflected beam. This nonlinear shift occurs for values of α that are no longer small. In this case, the square of the reflected field amplitude is, from $A_r^2 \simeq A_0^2$, Eqs. (4.11) and (2.4),

$$A_r^2(\eta) = A_i^2(\eta) - \gamma\frac{\kappa_0}{\kappa_1}\frac{d\alpha}{d\eta} .$$

The substitution of this into Eq. (4.12), and integrating by parts, give the nonlinear shift

$$\Delta = \Delta_L \int_{-\infty}^{\infty} \alpha(\eta)d\eta \bigg/ \int_{-\infty}^{\infty} \mu^2(\eta)d\eta , \tag{4.20}$$

where $\mu^2(\eta) = 2A_i^2(\eta)/\kappa_0^2$.

For the angles of incidence close to the critical angle ($\kappa_0 \ll \kappa_1$), α satisfies the Riccati equation

$$\frac{d\alpha}{d\eta_b} = \mu^2 - 2\alpha + \alpha^2 , \tag{4.21}$$

where $\eta_b = \eta/\Delta_L$, and Eq. (4.11) has been used. The Riccati equation has an exact solution [23] for certain dependences of $\mu^2(\eta_b)$ upon η_b. A simple example that illustrates the principles of the theory is the square profile beam

$$\mu^2(\eta_b) = \begin{cases} \mu_0^2, & |\eta_b| \le \eta_0 \\ 0, & |\eta_b| > \eta_0 , \end{cases} \tag{4.22}$$

for which the solutions of Eq. (4.21) are the following.

(1) For $\mu_0^2 < 1$,

$$\alpha = \begin{cases} 0, & \eta_b < -\eta_0 \\ 1 - \sqrt{1-\mu_0^2}/\tanh[\sqrt{1-\mu_0^2}(\eta_b+\eta_0) + \tanh^{-1}(\sqrt{1-\mu_0^2})], & -\eta_0 \le \eta_b < \eta_0 \\ \alpha_0 e^{-2(\eta_b-\eta_0)}, & \eta_0 \le \eta_b \end{cases} \tag{4.23}$$

where $\alpha_0 = 1 - \sqrt{1-\mu_0^2}/\tanh[2\eta_0\sqrt{1-\mu_0^2} + \tanh^{-1}(\sqrt{1-\mu_0^2})]$.

(2) For $\mu_0^2 > 1$,

$$\alpha = \begin{cases} 0, & \eta_b < -\eta_0 \\ 1 + \sqrt{\mu_0^2-1}\tan\left[\sqrt{\mu_0^2-1}(\eta_b+\eta_0) - \tan^{-1}\left[\frac{1}{\sqrt{\mu_0^2-1}}\right]\right], & -\eta_0 \le \eta_b < \eta_0 \\ \alpha_0 e^{-2(\eta_b-\eta_0)}, & \eta_0 \le \eta_b \end{cases} \tag{4.24}$$

where

$$\alpha_0 = 1 + \sqrt{\mu_0^2-1}\tan\left[2\eta_0\sqrt{\mu_0^2-1} - \tan^{-1}\left[\frac{1}{\sqrt{\mu_0^2-1}}\right]\right] .$$

The shifts are the following.

(1) For $\mu_0^2 \le 1$,

$$\frac{\Delta}{\Delta_L} = \frac{1}{\eta_0\mu_0^2}\ln\left\{\frac{\sqrt{1-\mu_0^2}[1-\tanh^2(\chi)]e^{2\eta_0}}{\sqrt{1-\mu_0^2}[1+\tanh^2(\chi)] + (2-\mu_0^2)\tanh(\chi)}\right\} , \tag{4.25}$$

where $\chi = \eta_0\sqrt{1-\mu_0^2}$.

(2) For $\mu_0^2 > 1$,

$$\frac{\Delta}{\Delta_L} = \frac{1}{\eta_0\mu_0^2}\ln\left\{\frac{\sqrt{\mu_0^2-1}[1+\tan^2(\psi)]e^{2\eta_0}}{\sqrt{\mu_0^2-1}[1-\tan^2(\psi)] + (2-\mu_0^2)\tan(\psi)}\right\} , \tag{4.26}$$

where $\psi=\eta_0\sqrt{\mu_0^2-1}$. Note that $\Delta\to\Delta_L=2\gamma/\kappa_0\kappa_1$ as $\mu_0\to 0$, which corresponds to the linear limit of the theory. Equation (4.26) shows that the value of the shift goes logarithmically to infinity at the threshold of soliton generation. The logarithmic breakup of Eq. (4.26) can be understood from simple physical considerations. At the threshold for soliton formation the field forms a solitary channel which travels to infinity. This solitary channel will not return to the linear medium again. The dependences Δ/Δ_L for different values of the parameter η_0 are shown in Fig. 5.

If the peak intensity is large enough, it is possible to generate more than one waveguide (soliton) channel. In order to calculate the number of solitons to be formed, it is necessary to find a general solution of Eq. (4.11) for a range of angles of incidence. This is possible when the condition $\kappa_0 \ll \kappa_1$ holds. Equation (4.21), which is derived from Eq. (4.11) under this condition, reduces to the linear stationary Schrödinger equation after the substitution $\alpha=1-dU/d\eta_b/U$, i.e.,

$$\frac{d^2U}{d\eta_b^2}+[\mu^2(\eta_b)-1]U=0 . \tag{4.27}$$

This equation describes the motion of a "particle" in a potential field [24]. The number of coupled states that are solutions of Eq. (4.27) determines the number of radiated solitons. The limiting values $\xi=\pm 1, 0$ correspond to the states of $\mu=0$, 1, which further correspond to the asymptotic values of the eigenfunctions, which are $U(\eta_b)\sim\exp(\mp\eta_b)$ or $U(\eta_b)=C$, where C is a constant. As a further example, we can find the number of the solitons, generated by beam with the form

$$\mu^2(\eta_b)=\frac{\mu_0^2}{\cosh^2(\eta_b/\eta_0)} , \tag{4.28}$$

for which Eq. (4.27) has an exact solution. The number of solitons, generated by the beam (4.28), with width η_0, and superthreshold rate μ_0, is equal to

$$N_g=\left[1+\left[\frac{1}{2^2}+\eta_0^2\mu_0^2\right]^{1/2}-\left[\frac{1}{2}+\eta_0\right]\right] , \tag{4.29}$$

where the square brackets denote the integral part. The dependence of the number of radiated solitons on the values μ_0^2 and η_0 is shown in Fig. 6.

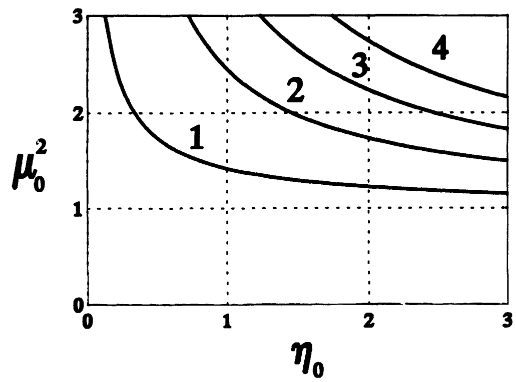

FIG. 6. Number of solitons generated shown on the (μ_0^2, η_0) parameter plane.

The electromagnetic energy that is transmitted into the nonlinear medium is a discrete value, determined by the number of radiated solitons. This is illustrated in Fig. 7, where the dependence of the (total) reflection coefficient upon μ^2, namely,

$$R_T=\int_{-\infty}^{\infty}A_r^2(\eta)d\eta\Big/\int_{-\infty}^{\infty}A_i^2(\eta)d\eta ,$$

is shown. The same discrete characteristic is exhibited by the coefficient $K(\mu^2)$ and the shift $\Delta(\mu^2)$ of the reflected beam. The minima in the functions $R_T(\mu^2)$, $K(\mu^2)$, and $\Delta(\mu^2)$ correspond to values of the superthreshold rate when a coupled state is formed. The behavior of R_T, as shown in Fig. 7, is already established in the literature.

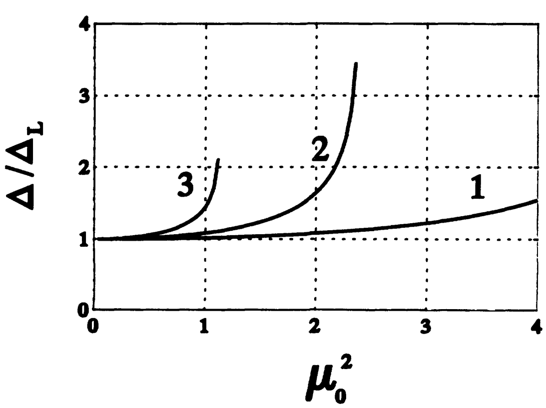

FIG. 5. Dependences of the nonlinear Goos-Hänchen shift Δ/Δ_L on the superthreshold rate μ_0^2 for (1) $\eta_0=0.04$, (2) $\eta_0=0.2$, (3) $\eta_0=2.0$.

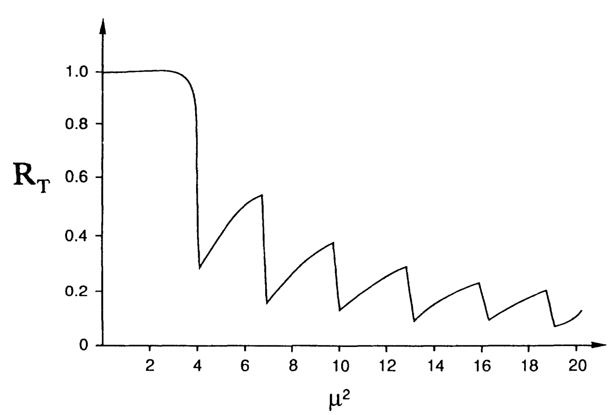

FIG. 7. The dependence of the total reflection coefficient R_T upon μ^2 for $\gamma^2=2.67$.

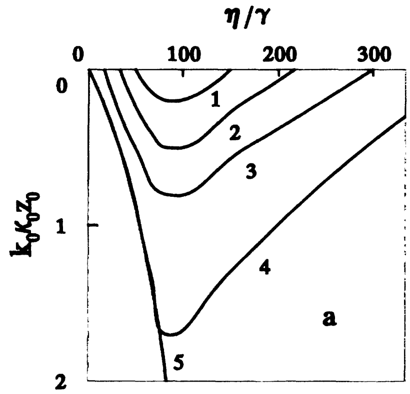

FIG. 8. Positions of peak intensity of beams in a nonlinear medium for $\gamma^2=2.65$ and (1) $\mu^2=2.8$, (2) $\mu^2=3.0$, (3) $\mu^2=3.5$, (4) $\mu^2=3.77$, (5) $\mu^2=3.78$.

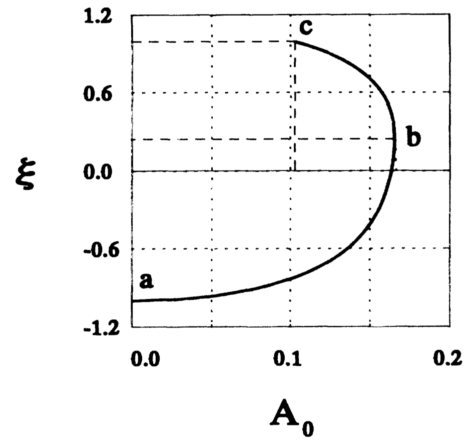

FIG. 9. Stationary dependence of the channel axis coordinate upon the incident wave amplitude in a dissipative medium. $\kappa_0<\kappa_1$. $\epsilon_i=0.001$.

The interesting thing here is that we have obtained it from quite a simple and elegant formalism. We have also been able to compute the positions of the peak intensity in the nonlinear medium for various values of power, from Eq. (4.11) directly. These are shown in Fig. 8 and are precisely of the form obtained from purely numerical simulations [7].

V. INFLUENCE OF WEAK DISSIPATION IN THE NONLINEAR MEDIUM

In the previous sections, dissipation in the nonlinear medium has been neglected, but the influence of weak dissipation can be calculated readily, however, within the framework of the energy method. The equation for the energy balance (3.1), when the dissipation is taken into account, is

$$\frac{\partial w}{\partial t}+\mathrm{div}\mathbf{S}=-q_{\mathrm{diss}}\ , \tag{5.1}$$

where $q_{\mathrm{diss}}=2\epsilon_i w$ is the time-averaged energy dissipating per unit volume of the nonlinear medium, and ε_i is the imaginary part of its dielectric permittivity. Under conditions when the imaginary part of the dielectric permittivity is small enough in comparison with its real part, the calculation of the energy loss can obviously be made using a field structure that is close to that of the solution of the nondissipative problem. As an example, we investigate the influence of energy dissipation in a spatially homogeneous medium such as the one discussed in Sec. IV. In this case,

$$\int_0^\infty q_{\mathrm{diss}}dz=2\epsilon_i W=\frac{\kappa_0\epsilon_0\epsilon_i}{k_0\alpha}(\epsilon_2+\gamma^2)(1+\xi)\ . \tag{5.2}$$

Equation (5.1) then becomes

$$\frac{\partial\xi}{\partial\tau}+\gamma\frac{\partial\xi}{\partial\eta}=\frac{\kappa_1}{\kappa_0}\left[A_i^2(\tau,\eta)-\frac{\kappa_0^2}{2}(1-\xi^2)\left[1+\frac{\kappa_0^2}{\kappa_1^2}\xi^2\right]\right]-2\epsilon_i(1+\xi)\ , \tag{5.3}$$

which has an extra dissipative term on the right-hand side, compared with Eq. (3.5). If an incident wave with constant amplitude $A_i(\tau,\ \eta)=A_0$ is used, then, for the stationary case, $\partial\xi/\partial\tau=0$, $\partial\xi/\partial\eta=0$, and Eq. (5.3) gives

$$A_0^2=\frac{\kappa_0^2}{2}(1-\xi^2)\left[1+\frac{\kappa_0^2}{\kappa_1^2}\xi^2\right]+\frac{\kappa_0^2}{2}\delta(1+\xi)\ , \tag{5.4}$$

where $\delta=4\epsilon_i/\kappa_0\kappa_1$. For simplicity, attention is restricted

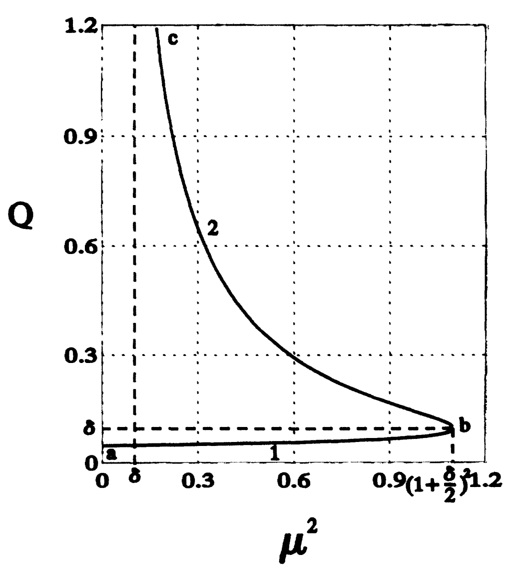

FIG. 10. Absorption coefficient Q dependence upon superthreshold μ^2.

here to the case of $\kappa_0 < \kappa_1$. The dependence of $\xi(A_0^2)$ for weak dissipation ($\delta < 1$) is shown in Fig. 9. The (ab) part of the curve covers the stable regime and (bc) covers the unstable regime. The absorption coefficient Q is defined as the ratio of the dissipated power to the energy density flux of the incident radiation, i.e.,

$$Q = \frac{\omega}{\epsilon_2 + \gamma^2} \frac{\int q_{\text{diss}} dz}{S_i} = \frac{\delta(1+\xi)}{\mu^2} . \tag{5.5}$$

For the case when the incident angle is close to the angle of total internal reflection ($\kappa_0 \ll \kappa_1$), the coefficient Q has the form, on the basis of Eq. (5.4),

$$Q_{1,2} = \frac{\delta}{\mu^2} \left\{ 1 + \frac{\delta}{2} \pm \left[\left[\frac{\delta}{2} + 1 \right]^2 - \mu^2 \right]^{1/2} \right\} , \tag{5.6}$$

where $\mu^2 = 2A_0^2/\kappa_0^2$. Q, with its branches 1 and 2, is illustrated in Fig. 10. The sign $(-)$ in Eq. (5.6) gives the absorption along the stable branch of the solution (part ab in Fig. 10) and the sign $(+)$ gives the absorption along the unstable branch (part bc in Fig. 10). Thus it is seen that, in the stationary problem, strong absorption takes place only along the unstable branch, which is characterized by a higher field intensity in the nonlinear medium.

As in the nondissipative case, if the amplitude of the pump wave is greater than threshold value A_0^*, periodic generation of solitons is possible. The period, in this case, is

$$T_s = \frac{\dfrac{2}{\kappa_0 \kappa_1}}{\left[\mu^2 - \left[1 + \dfrac{\delta}{2} \right]^2 \right]^{1/2}} \left\{ \tan^{-1} \left[\frac{1 - \dfrac{\delta}{2}}{\left[\mu^2 - \left[1 + \dfrac{\delta}{2} \right]^2 \right]^{1/2}} \right] + \tan^{-1} \left[\frac{1 + \dfrac{\delta}{2}}{\left[\mu^2 - \left[1 + \dfrac{\delta}{2} \right]^2 \right]^{1/2}} \right] \right\} ,$$

which shows that T_s increases with increasing dissipation.

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