Transverse particle motion in radio-frequency linear accelerators

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The transverse motion of a relativistic charged particle in a radio-frequency linear accelerator (rf linac) is examined. The spatially averaged equations of motion are derived for a particle in a periodic accelerating cavity system, and solved exactly in the ultrarelativistic limit. These solutions, along with an impulse treatment of the transients at the entrance and exit of the linac cavities, allow derivation of a linear transport matrix through the cavity. This generalized matrix is improved over previously derived results in that it is applicable to both traveling- and standing-wave structures, allows for arbitrary injection phase and spatial-harmonic content of the rf fields, and is more accurate in approximating the exact charged-particle motion.

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There has been much recent interest in the focusing effects of radio-frequency (rf) fields in linear accelerators [1–3]. This focusing is important in understanding the design and performance of high-gradient rf electron guns [1–3] and linacs [2], and can contribute to multibunch beam breakup instability in linear electron-positron colliders [4]. The derivation of the transverse particle motion in high-frequency rf fields has been approached in a variety of ways [1,5,6] which are all specific to certain physical situations. The purpose of this paper is to present a general approach to solving for this motion which is broadly applicable, covering standing- and traveling-waves linacs in a single formalism. The solutions to the equations of particle motion, which are derived by averaging over the motion in a periodic cell of the linac structure, allow for arbitrary injection phase, and include the effects of higher spatial harmonics in the rf fields. A treatment of the transient kicks that the particles experience while traversing the fringing field regions at the ends of the linac cavities, which is suitable for matching into the periodic focusing of the interior linac cells, is also given. Finally, a generalized matrix description of the motion through a full cavity is obtained.

We begin the discussion by writing the radial electromagnetic forces on an ultrarelativistic ($v \equiv \beta c \cong c$) paraxial charged particle due to the transverse rf fields in a cylindrically symmetric, spatially periodic, rf cavity. In terms of the longitudinal (accelerating) field profile, we have

$$F_r \cong -\frac{qr}{2}\frac{d}{dz}E_z\ , \tag{1}$$

where q is the charge of the particle and the total derivative with respect to z, the distance along the beam axis that the particle propagates [7].

To maintain the discussion in most general terms, we write the accelerating field profile in Floquet form [6],

$$E_z \equiv E_0 \mathrm{Re}\left[\sum_{n=-\infty}^{\infty} b_n e^{i(\omega t - k_n z)}\right]\ , \tag{2}$$

where E_0 is defined as the average accelerating field experienced by a particle injected at the phase which gives maximal acceleration, $k_n \equiv (\psi + 2\pi n)/d$ is the wave number of the traveling wave associated with the nth space harmonic, and $\psi \equiv l\pi/m$ is the phase advance per cavity cell (l,m integers, $l \le m$) of length d, the periodicity length of the structure. While this sum is explicitly written in terms of traveling waves, the special case of a π mode standing-wave structure is obtained by requiring that the field coefficients obey the relation $b_{-(n+1)} = b_n^*$. For ultrarelativistic particles only the $n=0$ component of the field contributes to the average acceleration; in the present normalization this implies that $b_0 = 1$.

In order to take advantage of the form of Eq. (1), we substitute $\omega t = k_0 z + \Delta\phi$, where $\Delta\phi$ is the phase of the particle with respect to the maximum acceleration phase, and $k_0 = \psi/d = \omega/c$ for an ultrarelativistic particle, to obtain

$$E_z = E_0 \mathrm{Re}\left[\sum_{n=-\infty}^{\infty} b_n e^{i[2k_0(m/l)nz + \Delta\phi]}\right]\ . \tag{3}$$

Following Ref. [1], we average the periodic force derived from Eqs. (1) and (3) to yield

$$\begin{aligned}\bar{F}_r &= \frac{(qE_0)^2}{8\gamma m_0 c^2} r \sum_{n=1}^{\infty} b_n^2 + b_{-n}^2 + 2b_n b_{-n}\cos(2\Delta\phi) \\ &\equiv \eta(\Delta\phi)\frac{(qE_0)^2}{8\gamma m_0 c^2} r\ ,\end{aligned} \tag{4}$$

where $\gamma m_0 c^2$ is the total particle energy and the b_n's are assumed to be real. The knowledge of the field components, and therefore $\eta(\Delta\phi)$, is straightforwardly obtained by using electromagnetic design computer codes, as well as bench measurements. This second-order ponderomotive focusing force can, because of its cylindrical symmetry, be written as a focusing strength parallel to either transverse Cartesian axis,

$$K_r \equiv -\frac{\bar{F}_r}{\gamma m_0 c^2 r} = K_x = K_y = \frac{\eta(\Delta\phi)}{8}\left[\frac{qE_0}{\gamma m_0 c^2}\right]^2 = \frac{\eta(\Delta\phi)}{8\cos^2(\Delta\phi)}\left[\frac{\gamma'}{\gamma}\right]^2 . \tag{5}$$

Here the prime indicates the derivative with respect to the longitudinal position ($\equiv d/dz$), and $\gamma' = qE_0\cos(\Delta\phi)/m_0c^2$ is the gradient in the particle energy averaged over a period of the structure. It should be noted that this expression is equivalent to the compact form previously found by Helm and Miller [6],

$$K_r \equiv \frac{1}{4\gamma m_0 c^2}\left[\langle E_z^2\rangle - \langle E_z\rangle^2\right] , \tag{6}$$

written explicitly in terms of the spatial harmonics of the accelerating field.

For ultrarelativistic particles, the averaged equation of motion for paraxial trajectories is thus [8], for either transverse Cartesian coordinate, of the form

$$x'' + \left[\frac{\gamma'}{\gamma}\right]x' + \frac{\eta(\Delta\phi)}{8\cos^2(\Delta\phi)}\left[\frac{\gamma'}{\gamma}\right]^2 x = 0 . \tag{7}$$

The solution to Eq. (7) can be written in terms of initial conditions on the position and angle (x_i, x_i') as

$$x = x_i\cos(\alpha) + x_i'\left[\frac{8}{\eta(\Delta\phi)}\right]^{1/2}\frac{\gamma_i}{\gamma'}\cos(\Delta\phi)\sin(\alpha) , \tag{8}$$

where

$$\alpha = \left[\frac{\sqrt{\eta(\Delta\phi)/8}}{\cos(\Delta\phi)}\right]\ln\left[\frac{\gamma_f}{\gamma_i}\right] ,$$

and $\gamma_{i(f)}$ is the initial (final) normalized energy of the particle. The transport matrix form of the solution to the initial-value problem is thus

$$\begin{bmatrix} x \\ x' \end{bmatrix}_f = \begin{bmatrix} \cos(\alpha) & \left[\frac{8}{\eta(\Delta\phi)}\right]^{1/2}\frac{\gamma_i}{\gamma'}\cos(\Delta\phi)\sin(\alpha) \\ -\left[\frac{\eta(\Delta\phi)}{8}\right]^{1/2}\frac{\gamma'}{\gamma_f\cos(\Delta\phi)}\sin(\alpha) & \frac{\gamma_i}{\gamma_f}\cos(\alpha) \end{bmatrix}\begin{bmatrix} x \\ x' \end{bmatrix}_i . \tag{9}$$

The complete traversal of an rf cavity requires that the particle experience a first-order transient force in the fringe field regions at both the entrance and exit of the cavity. Ignoring the variation of the both the particle energy and transverse position in the transient region, we can integrate Eq. (1) to give the change in particle angle at the entrance (exit) of the cavity [6],

$$\Delta x' = \mp\frac{qE_m\cos(\Delta\phi)}{2\gamma_{i(f)}mc^2}x \equiv \mp\frac{q\gamma'}{2\gamma_{i(f)}}\left[\sum_{n=-\infty}^{\infty} b_n\right]x = \mp\frac{q\gamma'}{2\gamma_{i(f)}}gx , \tag{10}$$

where g is the ratio of the maximum accelerating field in the end cell to the average accelerating field experienced by a synchronous ($\Delta\phi = 0$) particle in the structure. In writing the series form for the longitudinal field in the end cell, we assume that the last half of the end cell is identical in field profile to the periodic interior cells. Let us note that Eq. (10) can be derived by simple application of the Panofsky-Wenzel theorem [9], as shown in Ref. [3].

In order to construct transport matrices for these impulsive kicks for the *secular* (averaged) particle motion, one must be careful to subtract the angle of the oscillating orbit corresponding to the homogeneous periodic solutions to the zeroth-order (in transverse position) equations [1] assumed in deriving Eq. (4). This angle is, at the maximum of the accelerating field in the entrance (exit) cell,

$$\theta_{\mathrm{eq}} = \mp\frac{x}{2}\frac{\gamma'}{\gamma}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} b_n \equiv \mp\frac{x}{2}\frac{\gamma'}{\gamma}(g-1) , \tag{11}$$

where we have made the same assumption concerning the field profile as was done in writing Eq. (10). The transport matrices corresponding to traversal of the entrance (exit) of the cavity are therefore

$$\begin{bmatrix} 1 & 0 \\ \mp\frac{\gamma'}{2\gamma_{i(f)}} & 1 \end{bmatrix} . \tag{12}$$

It should be emphasized that this result applies only to the *secular* portion of the trajectory. If one is concerned with the actual angular deflection of the particle as it enters or leaves a cavity (for example, the phase-dependent, emittance diluting deflection of particles leaving an rf gun [8,9]), then the results of Eq. (10) should be applied. On the other hand, the secular motion is of more importance when the beam envelope inside of the rf cavity is of concern, such as when one is designing an emittance compensation system [10].

The full transport matrix for the secular motion through the cavity is now obtained by multiplying the component matrices in sequence,

$$\begin{bmatrix} \cos(\alpha)-\left[\dfrac{2}{\eta(\Delta\phi)}\right]^{1/2}\cos(\Delta\phi)\sin(\alpha) & \left[\dfrac{8}{\eta(\Delta\phi)}\right]^{1/2}\dfrac{\gamma_i}{\gamma'}\cos(\Delta\phi)\sin(\alpha) \\ -\dfrac{\gamma'}{\gamma_i}\left[\dfrac{\cos(\Delta\phi)}{\sqrt{2\eta(\Delta\phi)}}+\left[\dfrac{\eta(\Delta\phi)}{8}\right]^{1/2}\dfrac{1}{\cos(\Delta\phi)}\right]\sin(\alpha) & \dfrac{\gamma_i}{\gamma_f}\left[\cos(\alpha)+\left[\dfrac{2}{\eta(\Delta\phi)}\right]^{1/2}\cos(\Delta\phi)\sin(\alpha)\right] \end{bmatrix} . \quad (13)$$

For a particle injected into a pure π-mode standing-wave accelerating cavity at $\Delta\phi=0$, we have $b_0=b_{-1}=1$ (with all other field components vanishing, $\eta=1$), and the transport matrix reduces to

$$\begin{bmatrix} \cos(\alpha)-\sqrt{2}\sin(\alpha) & \sqrt{8}\dfrac{\gamma_i}{\gamma'}\sin(\alpha) \\ -\dfrac{3\gamma'}{\sqrt{8}\gamma_f}\sin(\alpha) & \dfrac{\gamma_i}{\gamma_f}[\cos(\alpha)+\sqrt{2}\sin(\alpha)] \end{bmatrix} . \quad (14)$$

This is identical to the result obtained previously by Chambers using a different method for the special case of a pure (no higher spatial harmonics) π-mode standing-wave linac [5].

It is useful to compare the results of an exact solution of the equations of motion to the generalized matrix solutions developed above, as well as to Chambers' matrix solution. Figure 1 shows a π-mode standing-wave case where there is a higher spatial harmonic ($b_1=b_{-2}=-0.2$) which flattens out the spatial field profile near its extrema, as is the case with many high gradient linac structures. To test the accuracy of the top left and right matrix elements, the solutions with initial

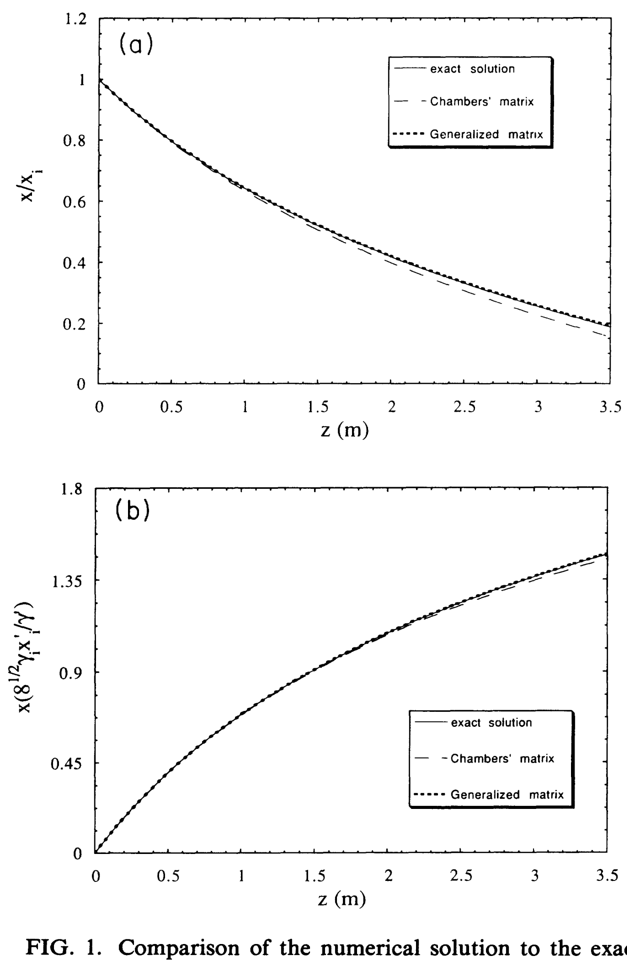

FIG. 1. Comparison of the numerical solution to the exact equations of motion in a π-mode standing-wave cavity [$\gamma_i=100$, $E_0=50$ MV/m, $k_0=59.8$ m^{-1} ($f=\omega/2\pi=2856$ MHz)] containing a higher spatial harmonic ($b_1=b_{-2}=-0.2$), with the predictions of Chambers' matrix and our generalized matrix. Initial conditions are (a) $(x_i,x_i')=(1,0)$ and (b) $(x_i,x_i')=(0,1)$.

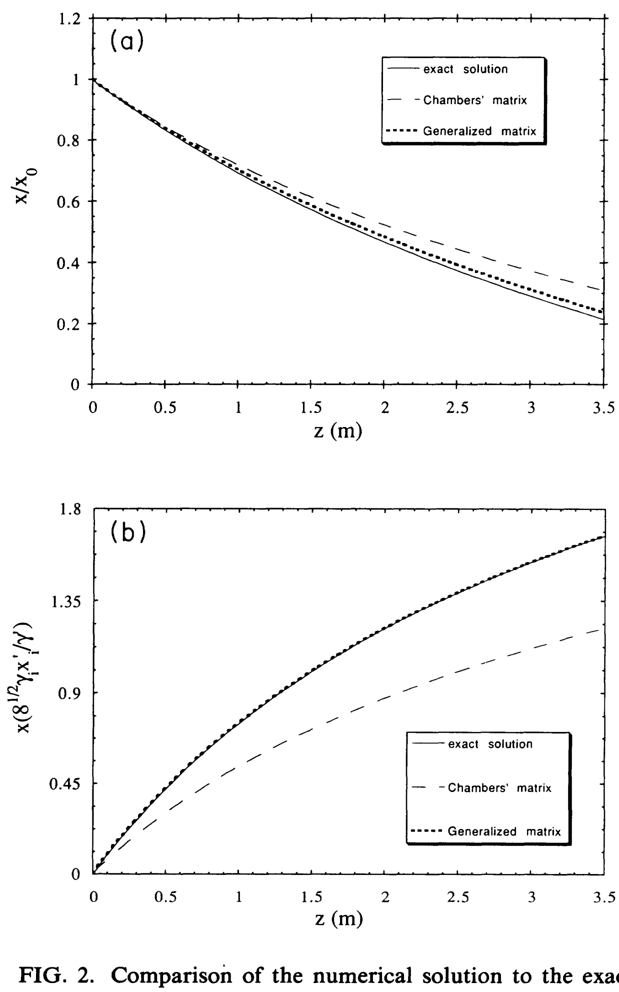

FIG. 2. Comparison of the numerical solution to the exact equations of motion in a pure π-mode standing-wave cavity [$\gamma_i=100$, $E_0=50$ MV/m, $k_0=59.8$ m^{-1} ($f=\omega/2\pi=2856$ MHz)] of a particle injected at $\Delta\phi=\pi/4$, with the predictions of Chambers' matrix and our generalized matrix. Initial conditions are (a) $(x_i,x_i')=(1,0)$ and (b) $(x_i,x_i')=(0,1)$.

conditions $(x_i,x_i')=(1,0)$ and $(x_i,x_i')=(0,1)$, respectively, are displayed in Figs. 1(a) and 1(b). It can be seen that the solutions corresponding to Eq. (13) are more accurate than that obtained from Chambers' matrix [Eq. (14)].

In practice, in order to compensate for the phase-dependent wake-field energy losses in high intensity linacs, particle bunches are injected "off crest," that is $\Delta\phi\neq 0$. Figure 2 illustrates the effects on the rf focusing of altering the injection phase of the particle. In this case we have taken $\Delta\phi=\pi/4$, and used a pure π-mode standing-wave field. Again, the agreement of our generalized matrix solutions with the exact solution is much improved over the previous results.

While the accuracy of this method is quite good, some discussion concerning the range of applicability is necessary. The averaging technique for the second-order focusing, as well as the derivation of the kick matrices at the ends of the cavities, require that the relative change in the energy over a cell be small. Likewise, the fast transverse oscillations of the particles in the rf field must be small compared to the slow secular part of the transverse offset. For further clarification of these requirements, see the discussions in Refs. [1] and [5].

Even though the results given above are considerably more general than those produced before, being applicable for traveling- and standing-wave cavities with arbitrary spatial harmonics and injection phase, the assumptions made in deriving them need a bit more comment. In particular, the kick matrices were derived assuming that the transition (fringe) field region can be modeled by requiring only that the fields at the first and last accelerating field maxima match those of the periodic interior cells. This assumption is fairly general; it allows cavities which begin on the "half-cell," such as the plane-wave transformer linac currently under development at UCLA [11], which have a very short fringing region near the accelerating field maximum, yielding a short transient region consisting of a mainly electric transverse kick. It also allows for cavities which begin on the "full cell" (the case considered by Chambers, which is typical of superconducting rf cavities), where the accelerating electric field is near zero as the particles enter the structure, and the transverse kick is produced by nearly equal components of electric and magnetic deflection. Unlike the half-cell case, in the full cell start there is non-negligible acceleration in the fringe field region. This is not taken into account in the kick matrices, but instead is included in the interior second-order matrix, Eq. (9). The approximation involved in this procedure is valid to second order in the field strength.

If the above considerations are kept in mind, the linear transport of particle beams can be calculated quite accurately and quickly using the matrix given in Eq. (13). This may be of particular applicability when modeling large multibunch systems, such as one encounters in calculating beam breakup in linear colliders [3]. If the transport is nonlinear, e.g., in the presence of space-charge forces, the results of Eqs. (5) and (12) may be used to integrate an envelope equation [1,12].

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