Constructing the Lagrangian in the Eulerian coordinate for relativistic hydrodynamics

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(Received 10 May 1993)

The Lorentz-covariant Lagrangian for an ideal relativistic flow is constructed in the Eulerian coordinate. In contrast to the Lagrangian of nonrelativistic flows in the Eulerian formulation, for which the continuity equation is required to be externally imposed as a constraint [Mittag, Stephen, and Yourgrau, in *Variational Principles in Dynamics and Quantum Theory,* edited by W. Yourgrau and X. Mandelstam (Dover, New York, 1968)], this Lorentz-covariant Lagrangian automatically yields the continuity equation as well as the equation of state. In addition, the relativistic generalization of the Bernoulli equation can also be derived from the present formulation.

PACS number(s): 47.10.+g, 03.50.Kk, 03.40.Gc, 95.30.Lz

I. INTRODUCTION

There is a great deal of interest in hydrodynamics [1] using a variational approach, be it in the area of the ordinary fluid (Navior-Stokes equation), the superfluids, or even the X-Y model of the magnetism [2]. The formulation usually involves an identification of the appropriate Lagrangian for the system. Once found, the equation of motion will follow. In addition to the basic interest in theoretical physics for constructing the variational formulation, it can also potentially be quite useful in stability analyses [3]. More importantly, the Lagrangian formulation can be used to construct new theories in composite systems, where the original system is coupled to additional constituents. When all the pieces of the Lagrangian belonging to various components are known, the dynamics of the composite system can then be derived correctly. One example in the context of relativistic fluids is the mixture of fluid and neutrinos in the supernova explosion [4]. A relativistic fluid Lagrangian is also needed to describe the dynamics in the product of heavy-ion collision [5].

For nonrelativistic fluids, the X-Y model has been well studied, and its Lagrangian is derived from an elemental theory by expanding the small misaligned angle of the magnetic moment; the corresponding fluid motion turns out to be potential flows by its very construction. However, the flows may contain vortices, but only in the form of point vortices corresponding to singularities in the field. The Landau theory of the superfluid is more or less similar to the X-Y model in the sense that by constructing it also applies to a potential flow. The primary differences are, first, the fluid has a thermodynamical property, such as pressure; and second, the fluid consists of an invicid superfluid component and a viscous normal fluid component, both of which are coupled together. With the introduction of pressure into the theory, one needs also to consider both the energy transport (or equation of state) as well as the evolution of density. Using the squared-flow velocity to construct the intuitive kinetic energy, and the fluid internal energy as the potential energy, one can construct a primitive Lagrangian. However, such a Lagrangian contains neither information about the thermodynamical property nor the mass conservation. Therefore one has to build either the equation of state or the continuity equation as a necessary constraint into the primitive Lagrangian to yield an effective Lagrangian.

If one ignores the complication arising from two components of the fluid, and retains only one component, a situation relevant to our later discussions, then the effective Lagrangian of a one-component fluid, which reduces to the irrotational Navior-Stokes equation, reads

$$L_{\text{eff}}=\tfrac{1}{2}\rho V^2-\tfrac{3}{2}\rho P-\phi\left[\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{V})\right], \tag{1}$$

where ϕ is the Lagrangian multiplier for the continuity equation [1].

As the aforementioned two theories are derived from relatively simple original models, the way in which the appropriate Lagrangians were found is relatively unambiguous. However, it was not so easy for the ordinary fluid, because the ordinary fluid usually it not a potential flow and has vorticity, a commonly observed property in liquids such as water [6]. While in the X-Y model the vortex is pointlike and its circulation is quantized, the ordinary fluid has extendedly distributed vorticity whose circulation can be of any arbitrary value. This problem was at one time rather bothersome, and has only been solved in the Lagrangian coordinate [7], where the observer moves with the fluid. For the Eulerian-coordinate formulation upon which the field theory is based, the appropriate Lagrangian was not known for almost a decade after the discovery of the Lagrangian-coordinate formulation. Only in the 1960s did people recognize that Kelvin's theorem of invariance of the circulation also should have been implemented as a constraint in the effective Lagrangian [8], similar to the way the continuity equation is built into the effective Lagrangian described above.

Concerning the relativistic flows, a Lorentz covariant Lagrangian in the Lagrangian coordinate has indeed been constructed [9]. As in the nonrelativistic case, the Lagrangian coordinate has the advantage of not having to impose mass conservation explicitly, as its formulation uses mass conservation as the starting principle for defining a fluid element. As for the Eulerian-coordinate Lagrangian, the simplest way to work it out is by analogy to Eq. (1), explicitly building the constraints into the primitive Lagrangian [10], which is simply the Lagrangian of a noninteracting gas. However, both formulations can be rather cumbersome. The latter involves introducing additional fields, the Lagrangian multipliers, into the Lagrangian and thus in the equations of motion. The former has to deal with integration of the deformation matrix along fluid trajectories; moreover, since the field theories are mostly based on the Eulerian coordinate, to be useful in a wide range of situations, such as coupling to other new fields [4], a Eulerian Lagrangian is certainly desired.

In constructing the Eulerian Lagrangian, the energy-momentum tensor alone does not contain enough information, since information about the mass conservation and the equation of state is missing in the energy-momentum tensor. A correct Lagrangian must contain all pieces of information that the energy-momentum tensor does not.

In this report, we will derive such a Lagrangian. Amazingly, this Lagrangian has no externally built-in constraints, and the continuity equation, the conservation of circulation, and the equation of state can automatically be derived from the Euler-Lagrange equations by variating appropriate field variables. In addition, the relativistic generalization of the Bernoulli equation can also be derived.

II. CONSTRUCTION OF THE LORENTZ COVARIANT LAGRANGIAN FOR A RELATIVISTIC FLOW

We begin by examining the energy-momentum tensor

$$T_\mu^\nu=(E+P)U_\mu U^\nu-P\delta_\mu^\nu \ , \tag{2}$$

which contains only the field variables; the internal energy E, the pressure P, and the four-velocity U_μ. We thus assume the kinetic energy to be $[K(E,P)U_\mu U^\mu]/2$. But, as the components of the energy-momentum tensor all contain a factor $(E+P)$ associated with the four-velocity, we thus assume that $K(E,P)=a(E+P)/c^2$, where a is a numerical constant. On the other hand, the potential energy should contain the variables describing the thermal property, and we let it be $W(E,P)/2$, which is to be determined. Note that the variables E and P are functions of the proper density ρ through the appropriate equation of state to be derived later. The Lagrangian now has the form

$$L=\frac{1}{2}\left[a\frac{E+P}{c^2}U_\mu U^\mu-W(E,P)\right] \ . \tag{3}$$

Next, we shall proceed by examining the nonrelativistic limit. In nonrelativistic flows, the flow velocity can be described by an irrotational component and a rotational component:

$$\mathbf{V}=\nabla\phi+\alpha\nabla\beta \ . \tag{4}$$

The variables α and β are the Clebsch variables, related to the vorticity by $\nabla\times\mathbf{V}=\nabla\alpha\times\nabla\beta$. The intersection of the two surfaces $\alpha(\mathbf{x},t)=\text{const}$ and $\beta(\mathbf{x},t)=\text{const}$ represents a vortex line, which is tied to the fluid element throughout the evolution in an invicid fluid. There are infinitely many possibilities to choose the two surfaces that possess this property [11]. However, it is convenient to choose a particular set of the two surfaces that satisfies $d\alpha/dt=0$ and $d\beta/dt=0$, where d/dt consists of an explicit time derivative and a convective derivative. In other words, the particular surfaces are frozen into the fluid elements [11].

Due to Eq. (4) in the nonrelativistic flows, we therefore expect that in relativistic flows, $U_\mu\propto\partial_\mu\phi+\alpha\partial_\mu\beta$. The next guidance for choosing the proportional factor is the continuity equation

$$\partial_\nu(\rho U^\nu)=0 \ . \tag{5}$$

To be consistent with the dimension of the nonrelativistic velocity, we choose $U_\mu=\partial_\mu\phi+\alpha\partial_\mu\beta$. By variating with respect to ϕ, we obtain that $\partial_\mu[(E+P)U^\mu]=0$. This is not a correct equation for either the internal energy or the pressure. However, as $E+P\to\rho c^2$ in the nonrelativistic limit, we can also choose

$$U_\mu=\frac{\rho c^2}{E+P}(\partial_\mu\phi+\alpha\partial_\mu\beta) \ . \tag{6}$$

Variation with respect to ϕ yields the continuity equation (5). Variation with respect to α yields that

$$U^\nu\partial_\nu\beta=0 \ , \tag{7}$$

describing free convection of the field variable β by the fluid element. Variation with respect to β yields that

$$\partial_\nu(\alpha\rho U^\nu)=0 \ . \tag{8}$$

By virtue of the continuity equation (4), this equation can be rewritten as

$$U^\nu\partial_\nu\alpha=0 \ , \tag{8'}$$

also describing the free convection of the field variable α. Thus the variables α and β recover their physical meanings in the nonrelativistic flow.

So far, we have implicitly made a serious assumption in Eq. (6). We know that the four components of U_μ are not independent, and they are related by the fact that $U_\mu U^\mu=c^2$. By assuming Eq. (6), we have imposed a dynamical relation for the evolution of the field variables ϕ, α, and β. In contrast, this problem does not exist for the nonrelativistic flow, for there the flow velocity itself only involves the spatial derivatives, and therefore Eq. (4) is simply a kinematic choice for the classification of the flow, having nothing to do with the dynamics. We need to examine this choice of U^0.

We can check against the nonrelativistic limit of U^0 in Eq. (6) by a careful expansion of the small parameters, the mechanical kinetic energy, and the thermal energy relative to the rest mass energy. The definitions of the internal energy and the pressure are

$$E=\rho c^2\int d^3u\left[1+\frac{u^2}{c^2}\right]^{1/2} f(u)\to\rho c^2\left[1+\frac{\langle u^2\rangle}{2c^2}\right] \quad (9)$$

and

$$P=\frac{\rho}{3}\int d^3u\frac{u^2}{(1+u^2/c^2)^{-1/2}}f(u)\to\frac{\rho\langle u^2\rangle}{3}\ , \quad (10)$$

where u is the spatial component of the particle four-velocity, $f(u)$ is the proper distribution function, and the limits are taken at $u/c\to 0$. We thus have

$$\partial_t\phi=\frac{E+P}{\rho^c}(1-V^2/c^2)^{-1/2}\to\frac{V^2}{2}+\frac{5P}{2\rho}+c^2 \quad (11)$$

for a potential flow. This is just the Bernoulli equation. Indeed, our choice of $U^0/c\propto\partial_t\phi$ recovers correct nonrelativistic dynamics, so will it be in the relativistic flows.

Thus the relation $U_\mu U^\mu=c^2$ indeed contains useful dynamical information:

$$(\partial_\mu\phi+\alpha\partial_\mu\beta)(\partial^\mu\phi+\alpha\partial^\mu\beta)=\left[\frac{E+P}{\rho c}\right]^2 . \quad (12)$$

For a potential flow, i.e., $\alpha=0$, Eq. (12) becomes the relativistic generalization of the Bernoulli equation [12].

Thus far, we have not addressed the potential energy $W(E,P)$ in Eq. (2). The guidance for its choice is contained in the energy-momentum tensor, which can be derived from the Lagrangian by Noether's theorem:

$$T_\mu^\nu=\sum_i\frac{\partial L}{\partial\partial_\nu q^{(i)}}\partial_\mu q^{(i)}-L\delta_\mu^\nu\ , \quad (13)$$

where the summation is over all fields $q^{(i)}$. But we also know that the energy-momentum tensor has been derived from a completely different formulation, as shown in Eq. (2). From Eqs. (2), (3), and (13), we find that $a=1$ and $L=\frac{1}{2}(E+P-W)=P$, where the second equality is the only choice determined by a comparison between the δ tensor terms in Eqs. (2) and (13). It follows that $W=E-P$, and the desired Lagrangian reads

$$L=\frac{1}{2}\left[\frac{\rho^2c^2}{E+P}(\partial_\mu\phi+\alpha\partial_\mu\beta)(\partial^\mu\phi+\alpha\partial^\mu\beta)+P-E\right] . \quad (14)$$

A straightforward inspection of this expression can be done by taking the derivatives of L with respect to $\partial_\mu\phi$ and $\partial_\mu\beta$ in accordance with Eq. (13), and it yields the energy-momentum tensor given by Eq. (2). The equation of motion, i.e., the relativistic Euler equation, then follows immediately as a result of the conservation law $\partial_\nu T_\mu^\nu=0$.

The last field variables to be variated is ρ. Variation with respect to ρ leads to

$$\frac{\partial E}{\partial\rho}=\frac{E+P}{\rho}\ , \quad (15)$$

the equation of state for an isentropic flow. This can be verified by considering the ultrarelativistic limit, where $E=3P$. It correctly yields the adiabatic index of the equation of state, $\Gamma=\frac{4}{3}$. This equation can also be checked against the nonrelativistic limit, where $E=\rho c^2+3P/2$. We indeed recover the adiabatic index $\Gamma=\frac{5}{3}$.

III. DISCUSSION

In spite of the success, this Lagrangian does have a limitation, in that is applies only to the isentropic flows, as does the existing nonrelativistic Lagrangian in the Eulerian formulation [1]. In other words, the variables ρ, P, and E must be functions of each other, and not explicitly of space and time. This cannot be the most general case in nature. We can imagine a given initial condition where the entropy varies arbitrarily. Initially Eq. (15) fails to satisfy, and so will the subsequent ideal hydrodynamic evolution. In such a situation, the present Lagrangian will not be valid. Nevertheless, one may argue that arbitrary entropy variations must be created by local dissipation, and, since our theory only applies to the ideal fluids, this situation should be excluded.

Note that this Lagrangian density is effectively derived from an empirical formulation, since during the derivation we have used the energy-momentum tensor which is derived on an empirical basis [10]. One may instead attempt to derive a Lagrangian from the first principle. Guided by the impression that an ideal gas also behaves like an ideal fluid, one may want to construct the Lagrangian by summing up the Lagrangians of individual free particles. The action is

$$S_{\text{free}}=\Sigma mc^2\int ds\int d^3x\ \delta(\mathbf{x}-\mathbf{x}_i(t))$$

$$=\int dt\int d^3x\left[\rho c^2\int d^3u\frac{f(u)}{\gamma}\right] , \quad (16)$$

where the terms in the bracket of the second equality are the free Lagrangian. Using the identity $\gamma^{-1}=\gamma-\gamma v^2/c^2$, we find, with the aid of Eqs. (9) and (10), that $L_{\text{free}}=E-3P$, which also equals the trace of the energy-momentum tensor. It differs significantly from our Lagrangian $L=P$. Hence we conclude that the free Lagrangian cannot be the Lagrangian of the fluid. The reason is that an ideal fluid must undergo infinitely many collisions in the hydrodynamical time scale, and it cannot be a collisionless gas. When collisions become so frequent, the potential energy, although short ranged, can be as important as the kinetic energy, and hence the free Lagrangian can strictly apply only to collisionless gases. As the potential energy of frequent collisions is very difficult to model, an empirical approach must be adopted. The empirical approach is guided by preserving the known symmetries in the system. In the usual formulation of the Lagrangian in the Eulerian coordinate, the symmetries are the mass and circulation conservation, manifested as added constraints in the Lagrangian. The effects caused by the Lagrange multipliers are to exert additional forces to the freely streaming particles, mimicking the particle collision. In our approach, these additional forces are not as apparent as the constraint formulation. However, we did blend the use of the mass and circulation conservation to choose the appropriate fields, and the use of energy-momentum tensor, derived from the symmetry of Lorentz invariance [13], to choose the potential energy. These considerations are exactly the rationale behind the present approach, and this is also the rationale frequently adopted in particle physics in deriving an effective Lagrangian [14].

A natural extension of the present formulation is the construction of the Lagrangian for magnetofluids. Here, a nontrivial task is to incorporate the frozen-in condition of the magnetic-field lines into the Lagrangian. Regarding the above discussions on the particle collisions, there is another twist for magnetofluids. That is, a collection of collisionless charge particles can also behave as a fluid when magnetic fields are present [15–17]. Will the Lagrangian be the same for the collisional magnetofluids as for the collisionless one, and if not, then in what limit will they be?

ACKNOWLEDGMENTS

This work is supported by Grants Nos. NSC 81-0208-M-513 and NSC 82-F-sp-008-02 of the National Science Council of Taiwan. I also thank Y. C. Lin for useful conversations.

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