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Avalanches and correlations in driven interface depinning

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We study the critical behavior of a driven interface in a medium with random pinning forces by analyzing spatial and temporal correlations in a lattice model recently proposed by Sneppen [Phys. Rev. Lett. **69**, 3539 (1992)]. The static and dynamic behavior of the model is related to the properties of directed percolation. We show that, due to the interplay of local and global growth rules, the usual method of dynamical scaling has to be modified. We separate the local from the global part of the dynamics by defining a train of causal growth events, or "avalanche," which can be ascribed a well-defined dynamical exponent $z_{\rm loc} = 1 + \zeta_c \simeq 1.63$, where ζ_c is the roughness exponent of the interface.

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I. INTRODUCTION

The behavior of a driven interface subjected to quenched random forces plays an important role in the ordering kinetics of impure magnets and other domain growth phenomena [1]. The driving force F can be realized by a magnetic field, pressure, or chemical potential favoring the growth of one of the coexisting phases. If F is weak, the interface is typically pinned in one of the many locally stable configurations. In this case growth is possible only through thermally activated hopping, which is an extremely slow process at low temperatures. However, when F exceeds some critical value F_c, all metastable states disappear and the interface is then free to move even at zero temperature. The depinning of the interface at F_c can be considered as a critical phenomenon where characteristic quantities show power-law behavior, e.g., the velocity of the interface is expected to scale as $v \sim (F - F_c)^\theta$, where θ is a critical exponent. It is known that charge-density waves pinned by impurities exhibit very similar behavior [2]. The dynamical behavior associated with the disappearance of metastable states (i.e., avalanche) as F is increased towards F_c has been a subject of recent interest in the study of systems far from equilibrium [3–6].

A plausible continuum description of the interface dynamics is given by the following equation with a Kardar-Parisi-Zhang (KPZ) [7] nonlinear term:

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + F - \eta(\mathbf{x}, h), \qquad (1)$$

where $h(\mathbf{x}, t)$ is the height of the interface. Unlike the original KPZ equation for Eden-type growth processes, here $\eta(\mathbf{x}, h)$ is a quenched random force with short-range correlations. In the case $\lambda = 0$, all critical exponents describing the depinning transition have been calculated recently in a functional renormalization group treatment close to four interface dimensions [8]. For growth in an isotropic medium, it is plausible that the λ term is not present when the interface moves with vanishing velocity. However, the term can be present for anisotropic growth. Most solid-on-solid type models are expected to be in the latter category.

In 1+1 dimensions, there is a particularly simple class of lattice models which exhibit a critical depinning transition [9,10]. The mechanism for pinning is the directed percolation of cells with pinning forces η greater than the driving force F. The threshold force F_c needed to depin the interface is then simply related to the critical percolation density ρ_c of such cells by $F_c = 1 - \rho_c$. The roughness exponent of the pinned interface is equal to that of the critical percolation cluster, $\zeta = \zeta_c \simeq 0.63$.

In a separate development, Sneppen introduced a simple model (model B of Ref. [11], hereafter referred to as the Sneppen model) to examine the interplay between local and global rules of growth in determining interface roughening and temporal correlations. He found numerically in 1+1 dimensions that the roughness of the interface also obeys the scaling of a string on a critical directed-percolation cluster.

As we explain below, the same roughness exponent for the two types of models is due to the fact that the same geometrical constraint (the "Kim-Kosterlitz" condition) is invoked in defining a locally stable configuration [12]. There are, however, differences in the way the interface is driven. Although the pinning and elastic forces on a given site in the Sneppen model are *local* as in Eq. (1), each time the site on the interface with minimal $\eta = \eta_{\rm min}$ is selected *globally* and made to grow by one unit in height. Neighboring interface sites then adjust themselves to recover the geometrical constraint. In the language of a uniformly driven interface, such a rule corresponds to increasing F just above $\eta_{\rm min}$ to make one site unstable, and then quickly set F to a much smaller value to prevent an avalanche taking place.

The Sneppen rule allows one to sample a particular sequence of interface configurations, each of them being a metastable configuration at some value of F. Successive configurations in the sequence differ only by an infinitesimal amount, i.e., a few sites (about four) which have moved when one site is made unstable. The situation here resembles that of an interface at a finite temperature and driven by a uniform force F far below F_c: Local irreversible motions are made possible by thermal fluctuations, but the interface stays always close to some metastable state.

Further numerical studies of the model by Sneppen and Jensen [13] revealed interesting spatial correlation of successive growth events. In addition, they have found that, in the saturated regime, the height advance at a given column exhibits complicated scaling with time, with exponents varying with the moment considered.

The aim of the present paper is to analyze the spatial and temporal correlations in the Sneppen model and try to relate the observed scaling behavior to the properties of directed percolation. It turns out that some of these correlations depend on the value of $\eta_{\min}$ (or equivalently F) at a given moment. Since $\eta_{\min}$ fluctuates in time, for these correlations the temporal translational invariance is lost. This is an important feature of the dynamics based on global rules.

Another significant consequence of the global rules is that growth is no longer homogeneous in space. At a given moment, only a small part of the interface is moving. The growth events that follow may be either close by or far away. Due to this property, the usual method to perform dynamical scaling, based on the presumption of a single growing correlation length, should be modified. In this connection we found it useful to distinguish growth events which are close by and hence bear strong correlations, from those which are far apart. We observe that a train of growth events, started with some $\eta^0_{\min}$, propagates laterally with a well-defined dynamical exponent z_{loc}, which can be related to the roughness of the directed-percolation cluster: $z_{\text{loc}} = 1 + \zeta_c \simeq 1.63$. In the context of a driven interface, this motion can be thought of as an avalanche. We found that the distribution of avalanche sizes obeys a power-law decay up to a size related to $\eta^0_{\min}$. The spatial-temporal correlations between successive growth events, on the other hand, involve both local and global motion.

The paper is organized as follows. In Sec. II we recall the definition of the Sneppen model and present a theorem which relates the stable (static) configurations of the Sneppen model to directed percolating strings. In Sec. III we define the avalanches (causal events) and determine their dynamical behavior as well as their size distribution. In Sec. IV the spatial-temporal correlations are investigated by considering the distribution of lateral distances between successive growth events. Section V contains conclusions and a summary.

II. DISTRIBUTION OF PINNING FORCES AND ROUGHNESS

We first review the definition of the Sneppen model [11]. Each cell (i, h) on a square lattice is assigned a random pinning force $\eta(i, h)$ uniformly distributed in the interval $[0, 1)$. The interface is specified by a set of integer column heights h_i ($i = 1, ..., L$) with the local slope constraint $|h_i - h_{i-1}| \leq 1$ for all h_i ("Kim-Kosterlitz" condition). Growth $h_j \to h_j + 1$ proceeds at the site j where the pinning force $\eta(j, h_j) = \eta_{\min}$ is the minimum among all interface sites, followed by necessary adjustments at neighboring sites until the slope constraint is recovered. The growth rules are illustrated in Fig. 1.

Since we want to relate the behavior of the interface in the Sneppen model to directed percolation, we first recall some of the properties of directed-percolation clusters [14]. When the density ρ of occupied sites is less than some threshold value ρ_c, a typical cluster of occupied sites connected horizontally or diagonally extends over a distance of the order of $\xi_\parallel$ in the parallel direction and a distance of the order of $\xi_\perp$ in the perpendicular direction. For $\rho > \rho_c$, there appears a directed percolating cluster which extends over the whole system. This cluster has a network structure of nodes and compartments, where each compartment has an anisotropic shape similar to the connected clusters below ρ_c, characterized by $\xi_\parallel$ and $\xi_\perp$. On both sides of the percolation transition, the two lengths have the power-law behavior

$$\xi_\parallel \sim |\rho - \rho_c|^{-\nu_\parallel}, \qquad \xi_\perp \sim |\rho - \rho_c|^{-\nu_\perp}. \qquad (2)$$

Series calculations give $\nu_\parallel = 1.733 \pm 0.001$, $\nu_\perp = 1.097 \pm 0.001$ [15], and $\rho_c = 0.5387 \pm 0.003$ [16]. The roughness of a percolating string scales as $\xi_\perp \sim \xi_\parallel^{\nu_\perp/\nu_\parallel}$, i.e., the roughness exponent $\zeta_c = \nu_\perp/\nu_\parallel \simeq 0.63$.

In Ref. [12] we proposed to study the distribution $P_p(\eta)$ of pinning forces $\eta(i, h_i)$ at the interface and the probability distribution $P_m(\eta_{\min})$ during growth. When starting at $t = 0$ with a flat interface $h_i \equiv 0$, the forces $\eta(i, h_i)$ are uniformly distributed. During the transient regime the interface becomes rough and since the smallest pinning force $\eta_{\min}$ is always updated, the sites with small $\eta(i, h_i)$ become rare. This in turn implies that the typical value of the selected $\eta_{\min}$ increases with time. The

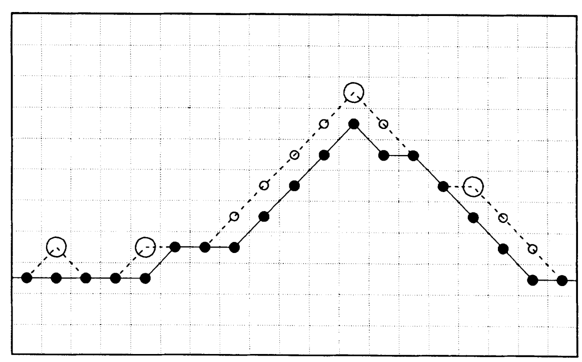

FIG. 1. An interface configuration (filled circles and full lines) is updated (big circles) at the site which has the smallest pinning force $\eta_{\min}$. Four possible growth events are shown including local adjustments (small circles and dashed lines) due to the slope constraint $|h_i - h_{i-1}| \leq 1$. Time is measured in units of such growth events. Each interface configuration may be considered as a directed percolating path.

distributions $P_p(\eta)$ and $P_m(\eta_{\min})$ shown in Fig. 2(a) were recorded in the transient regime in a time interval around $t = L/2$ for a system of size $L = 8192$. The peak of $P_m(\eta_{\min})$ moves to the right with increasing time and thereby "eats up the store" of small $\eta(i, h_i)$ which were present at $t = 0$ for the flat interface. As long as there were no $\eta_{\min}$ larger than a value $\eta_u(t)$, the distribution $P_p(\eta)$ is still a constant for $\eta > \eta_u(t)$. [In Fig. 2(a) $\eta_u(t) \approx 0.3$.] In the next paragraph we show that in the thermodynamic limit, there will never be an $\eta_{\min}$ larger than a critical value $F_c = 1 - \rho_c \simeq 0.461$ and therefore the transient regime ends when $\eta_{\min}$ first comes close to F_c. When the peak of $P_m(\eta_{\min})$ approaches F_c, its height vanishes and the distribution becomes stationary, which we show in Fig. 2(b) together with $P_p(\eta)$ in the saturated regime. Since $\eta_{\min} \leq F_c$, the stationary distribution $P_p(\eta)$ is flat for $\eta > F_c$ in the limit $L \to \infty$ [17].

To see that the growing interface always has $\eta_{\min} \leq F_c$ in the thermodynamic limit we first note that every interface configuration satisfying the slope constraint is a path on a directed-percolation cluster of sites with η

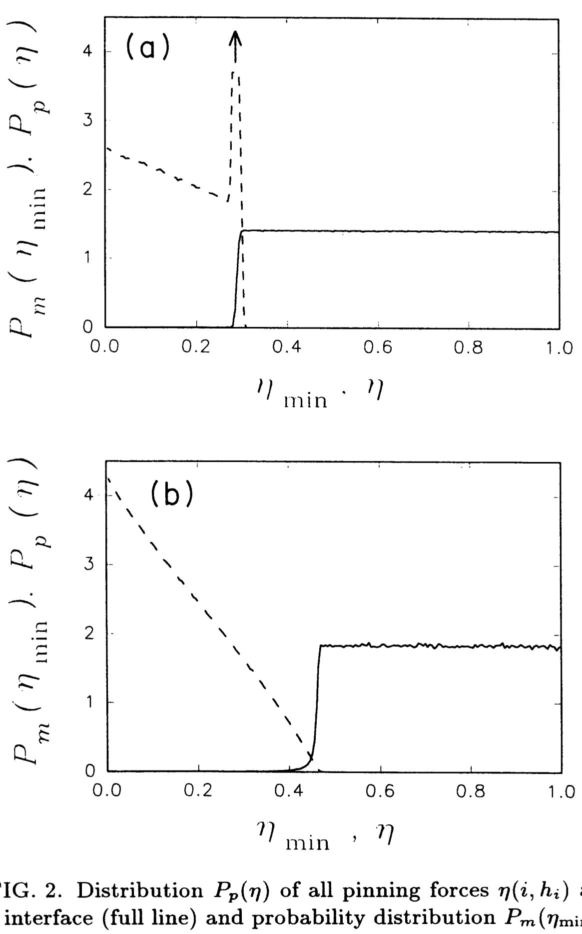

FIG. 2. Distribution $P_p(\eta)$ of all pinning forces $\eta(i, h_i)$ at the interface (full line) and probability distribution $P_m(\eta_{\min})$ (dashed line) for a system of size $L = 8192$. (a) The distributions in the transient regime are averaged over the time interval $L/2 \leq t \leq \frac{33}{64}L$ and over 25 000 independent runs. (b) $P_p(\eta)$ and $P_m(\eta_{\min})$ averaged over time in the saturated regime.

greater than or equal to $\eta_{\min}$. Such a path only exists if the density $1 - \eta_{\min}$ of these cells on the lattice is greater than the critical percolation density ρ_c, i.e., $\eta_{\min} \leq 1 - \rho_c$ for all interfaces. Paths on the infinite *critical* percolating cluster have the *largest* $\eta_{\min} = F_c = 1 - \rho_c \simeq 0.461$.

In a numerical simulation with a finite system, however, we see only a part of an infinite critical percolating cluster. Thus an interface which traces out a critical path can have a value $\eta_{\min}$ slightly larger than F_c and the distribution $P_m(\eta_{\min})$ in the saturated regime in Fig. 2(b) is not exactly zero for $\eta_{\min} > F_c$. The motion of the flat interface at $t = 0$ to the first critical path corresponds to the transient regime [see Fig. 2(a)].

In the following we show that the interface motion is quite restricted by a set of strings which make only minimal reference to the initial interface position. All configurations referred to below are assumed to satisfy the Kim-Kosterlitz condition. Denoting by $A = \{h_i^A\}$ and $B = \{h_i^B\}$ two arbitrary interface configurations. We say $A > B$ if $h_i^A \geq h_i^B$ for all i and that the inequality holds at least for one i, $A = B$ if $h_i^A = h_i^B$ for all i, and $A < B$ if $B > A$. Of course, not all pairs of configurations obey one of the three relations. However, in a given realization of the Sneppen model, we have $H(t) \equiv \{h_i(t)\} > H(t')$ for $t > t'$.

In a given realization of the random forces η, each configuration $H = \{h_i\}$ has a minimum random force $\eta_{\min}(H)$ on the interface. Given a number c $(0 \leq c < 1)$, we define a *closest* configuration $E(c; H)$ to the configuration H as a configuration in the set $\mathcal{S} = \{H' | H' \geq H \text{ and } \eta_{\min}(H') > c\}$ such that any other $H' \in \mathcal{S}$ satisfies $H' > E(c; H)$. The existence of $E(c; H)$ can be shown by noting that, for any two configurations A and B in the set $\mathcal{S}$, $C = \{h_i^C = \min[h_i^A, h_i^B]\} \leq A, B$ is also in $\mathcal{S}$, even though A and B may not be assigned any relation. The components of $E(c; H)$ are in fact given by $h_i^E = \min_{H' \in \mathcal{S}}\{h_i'\}$.

Let us now consider an interface configuration at t_0, $H_0 = H(t_0)$. For $c < \eta_{\min}(H_0)$, $E(c; H_0) = H_0$. In the thermodynamic limit, $E(c; H_0)$ does not exist for $c > F_c$. The interesting case occurs at $\eta_{\min}(H_0) \leq c < F_c$. The following argument shows that, for c in this range, $H(t_1) = E(c; H_0)$ holds at a later time $t_1 > t_0$. Since $h_i(t)$ is an ever increasing function of t for all i, there is a time $t_1 \geq t_0$ such that $H(t_1 - 1) < E(c; H_0)$ but no inequality for $H(t_1)$. Let $\eta_{\min}(t) \equiv \eta_{\min}[H(t)]$. Obviously $\eta_{\min}(t_1 - 1) \leq c$ as otherwise we contradict the definition of $E(c; H_0)$. This implies that the site $i_{\min}$ at which $\eta_{\min}(t_1 - 1)$ is realized is not on $E(c; H_0)$, i.e., $h_{i_{\min}}(t_1 - 1) < h_{i_{\min}}^E$. According to the growth rules, each column may advance at most one unit in height in one time step. Therefore $h_{i_{\min}}(t_1) \leq h_{i_{\min}}^E$. Simple geometrical consideration shows that the neighboring columns which are made to grow by growth at $i_{\min}$ due to the Kim-Kosterlitz condition must also all lie below $E(c; H_0)$ at $t_1 - 1$. Hence $h_i(t_1) \leq h_i^E$ for all i. The only possibility for $H(t_1) < E(c; H_0)$ not to hold is $H(t_1) = E(c; H_0)$.

The motion with $c = \eta_{\min}^0$ will be called an "avalanche" below (see Sec. III). If we choose, however, $c = F_c$, it follows that paths on *critical* percolation clusters which are (at one time) closest act as "checking points" where the interface has to go through. There, a "snapshot" of the distribution $P_p(\eta)$ is exactly a step function. It can be seen from Fig. 2(b) that the intermediate interface configurations between two checking points are also close to critical, in the sense that only a small percentage (which we expect to vanish in the thermodynamic limit) of sites on the interface have $\eta < F_c$. Hence the interface in the Sneppen model has approximately the roughness exponent ζ of the critical directed percolating path with $\zeta \simeq \zeta_c \simeq 0.63$.

The roughness exponent ζ can be measured by the equal time height-height correlation function $C(r) = \langle\overline{[h(r+r',t) - h(r',t)]^2}\rangle \sim r^{2\zeta}$ where the overbar and the angular brackets denote the spatial and the configurational average, respectively. For a system of size $L = 8192$ we found a roughness exponent $\zeta = 0.655 \pm 0.005$ which is somewhat larger than the critical value 0.63. However, the measured exponent ζ varies systematically with the system size: For $L = 900$ we get $\zeta = 0.665 \pm 0.005$, whereas for $L = 65\,536$ we found $\zeta = 0.648 \pm 0.005$. An explanation is that in a finite system the distribution $P_p(\eta)$ is not exactly a step function and the motion between two critical clusters yields an effective exponent larger than 0.63 similar to a moving interface of Ref. [9]. In our simulations, $P_p(\eta)$ approaches the step function with increasing system size. Thus we expect that the measured discrepancy to the exponent of the percolating cluster $\zeta_c \simeq 0.63$ vanishes in the thermodynamic limit. This is also supported by a simulation where we measured $C(r)$ only when $\eta_{\min}$ is close to F_c. In this case $\zeta = 0.64 \pm 0.01$, which is consistent with the expected critical value.

III. CAUSAL EVENTS: AVALANCHES

After a transient regime, when $\eta_{\min}$ first comes close to F_c, the interface in the Sneppen model exhibits a steady-state critical behavior (saturated regime), which allows a convenient study of the dynamics at criticality. However, the behavior is complicated, caused by the interplay between the local adjustments due to the slope constraint and the rule that the growth site with $\eta = \eta_{\min}$ is chosen among all interface sites. Successive growth events can be far apart and the motion is therefore inhomogeneous in space. Hence a single growing correlation length does not exist and the usual dynamical scaling has to be considered with care. The realized sequence of growth sites after a time $t = t_0$ depends on the globally chosen value $\eta_{\min}(t = t_0)$, which is responsible for the growth inhomogeneity in space. Thus we will separate the local part from the global part of the dynamics by defining an "avalanche," which has the property that the sequence of growth sites *inside* the avalanche does *not* depend on $\eta_{\min}(t = t_0)$.

An avalanche is defined by a sequence of growth events (including the necessary adjustments due to the slope constraint) started at any integer time $t = t_0$ where $\eta_{\min}(t = t_0)$ is denoted by $\eta^0_{\min}$. This avalanche is terminated at the first time τ when $\eta_{\min}(\tau)$ is larger than $\eta^0_{\min}$, i.e., for all times t with $t_0 < t < \tau$ the growth events have $\eta_{\min}(t) < \eta^0_{\min}$. We call this causal events because the avalanche consists of a train of growth events which are all induced by local adjustments in h_i due to the slope constraint after $t = t_0$.

To see in what sense the sequence of growth sites inside an avalanche is independent of $\eta^0_{\min}$, we consider at $t = t_0$ two identical interface configurations A and B with the same random forces $\eta(i,h)$ above the interface but with different $\eta(i,h_i)$ at the interface such that $\eta^0_{\min}[A] < \eta^0_{\min}[B]$ at the same site j. For the configuration A there may exist forces $\eta(i,h_i)$ with $\eta^0_{\min}[A] < \eta(i,h_i) < \eta^0_{\min}[B]$ but for interface B all $\eta(i,h_i) > \eta^0_{\min}[B]$. Since the random forces above the interface are assumed to be the same, all $\eta_{\min}$ of the growing interfaces A and B inside both avalanches are identical because they are all induced by identical local adjustments after $t = t_0$. When, however, at time τ^A, $\eta^0_{\min}[A] < \eta_{\min}(\tau^A) < \eta^0_{\min}[B]$, i.e., when avalanche A is terminated, this new growth site of interface A can be far away, but for the interface B it has to be still induced by the local adjustments of avalanche B because there were no $\eta(i,h_i) < \eta^0_{\min}[B]$ at $t = t_0$. We have seen that although the random environment at (and below) the two interfaces A and B is different, the motion is identical until one of the two avalanches terminates.

A size s of the avalanche can be defined by the number of growth events including the necessary adjustments, $s = \sum_i [h_i(\tau) - h_i(t_0)]$, and a width by $\max\{i$, such that $h_i(\tau) > h_i(t_0)\}$ $-$ $\min\{i$, such that $h_i(\tau) > h_i(t_0)\}$.

From Sec. II we know that an interface with $\eta^0_{\min}$ is driven to configurations with $\eta^c_{\min} > \eta^0_{\min}$. At $t = t_0$ with $\eta_{\min}(t = t_0) = \eta^0_{\min}$, a part of the interface starts to move which was "pinned" just before t_0 by a path with $\eta(i,h_i) \geq \eta^0_{\min}$. This part of the interface moves through a compartment of the percolation cluster to the next path which again pins the interface with $\eta(i,h_i) > c = \eta^0_{\min}$ (see Sec. II). Since the compartment has a height of the order of $\xi_\perp(\eta^0_{\min})$ and a width of the order of $\xi_\parallel(\eta^0_{\min})$, it is a natural conjecture that the width of the avalanche scales with $\xi_\parallel(\eta^0_{\min}) \sim [F_c - \eta^0_{\min}]^{-\nu_\parallel}$ and the size is at most $\xi_\parallel(\eta^0_{\min})\xi_\perp(\eta^0_{\min}) \sim [F_c - \eta^0_{\min}]^{-(\nu_\perp+\nu_\parallel)}$.

Note that at every (integer) time an avalanche is started and big avalanches can contain smaller ones. Thus successive growth events inside a big avalanche can be quite far apart (jumps between small avalanches), but all events are inside the correlation length $\xi_\parallel(\eta^0_{\min}) \sim [F_c - \eta^0_{\min}]^{-\nu_\parallel}$. In this sense an avalanche is "localized" and we call the corresponding motion "local dynamics." The presumption of dynamical scaling, that there is only a single growing correlation length, is reestablished for the local dynamics and we can ascribe a well-defined local dynamical exponent $z_{\rm loc}$ to the lateral propagation of growth inside the avalanche: $\tau \sim \xi_\parallel^{z_{\rm loc}}$. Since the size of an avalanche is proportional to the elapsed time τ, one has from $\xi_\parallel^{z_{\rm loc}} \sim \tau \sim \xi_\perp \xi_\parallel$

$$z_{\rm loc} \simeq 1 + \nu_\perp/\nu_\parallel = 1 + \zeta_c \simeq 1.63. \tag{3}$$

In a simulation, this local dynamical exponent can be detected by considering the infinite moment of the height-height time correlation function in the saturated regime

$$C_q(t) = \left\langle \overline{[h_i(t+t') - h_i(t') - \overline{h_i(t+t') - h_i(t')}]^q}^{1/q} \right\rangle, \quad (4)$$

which becomes for $q \to \infty$

$$C_\infty(t) \simeq \langle \max_i \{h_i(t+t') - h_i(t')\} \rangle. \quad (5)$$

Since only the column with maximum height advance $\Delta h_{\max}$ contributes to the infinite moment $C_\infty(t)$, most growth events during the time t are involved in the motion through a compartment of a percolation cluster of height $\Delta h_{\max}$, i.e., $\Delta h_{\max} \sim \xi_\perp \sim t/\xi_\parallel \sim t^{1-1/z_{\rm loc}} = t^{\zeta/z_{\rm loc}}$. Thus $C_\infty(t) \sim t^{\beta_\infty}$ with $\beta_\infty = \zeta/z_{\rm loc} = \beta_{\rm loc}$. Sneppen and Jensen observed a scaling of $C_\infty(t)$ with $\beta_\infty = 0.41 \pm 0.02$ which is close to the exponent $\beta_{\rm loc} = \zeta/z_{\rm loc} = \nu_\perp/(\nu_\perp + \nu_\parallel) \simeq 0.39$. Our own simulations give $\beta_\infty = 0.40 \pm 0.01$ which is in perfect agreement with $\beta_{\rm loc}$ if we insert the measured $\zeta \simeq 0.655$.

The distribution $P_h(\Delta h, t)$ of height advances $\Delta h(t) = h_i(t'+t) - h_i(t')$ is shown in Fig. 3(a) for $2^6 \le t \le 2^{15}$ and $\Delta h > 0$. P_h has a large peak at $\Delta h = 0$ which is not shown, i.e., most of the columns i have not grown (over 99% for $t = 2^6$ and 70% for $t = 2^{15}$ with $L = 8192$). Next we show that the distribution $P_h(\Delta h, t)$ for the moving columns can be roughly brought to a "local" scaling form when Δh is scaled by $t^{\beta_{\rm loc}}$. To normalize $P_h(\Delta h > 0, t)$ we note that the portion $n_{\rm mov}$ of columns which have moved scales as the correlation length divided by the system size, $n_{\rm mov} \sim t^{1/z_{\rm loc}}/L$. Thus one has

$$P_h(\Delta h > 0, t) = \frac{t^{1/z_{\rm loc}}}{L} \frac{1}{t^{\beta_{\rm loc}}} \Gamma\left(\frac{\Delta h}{t^{\beta_{\rm loc}}}\right)$$
$$\sim t^{1-2\beta_{\rm loc}} \Gamma\left(\frac{\Delta h}{t^{\beta_{\rm loc}}}\right), \quad (6)$$

where $\Gamma(y)$ is a scaling function [see Fig. 3(b)]. For fast growing columns with large $\Delta h/t^{\beta_{\rm loc}}$ the above argument for $\Delta h_{\max}$ applies and the data collapse in Fig. 3(b) is perfect. For smaller $\Delta h/t^{\beta_{\rm loc}}$, however, there are significant deviations from the "local" scaling form.

For the second moment $C_2(t)$ [18] Sneppen and Jensen observed a scaling with an exponent $\beta_2 = 0.69 \pm 0.02$ [13]. However, due to the inhomogeneity in growth, the application of dynamical scaling is questionable, as mentioned above. The deviation of the effective exponent β_2 from the scaling of the local dynamics is caused by the fact that only a small part of the interface ($n_{\rm mov} \sim t^{1/z_{\rm loc}}/L$) has moved for short times t. The observed value for β_2 can be explained by using that Δh scales *roughly* with $t^{\beta_{\rm loc}}$ for $\Delta h > 0$ [see Fig. 3(b)]. For general integer q we write

$$C_q(t) = \left\langle \overline{|\Delta h - \overline{\Delta h}|^q}^{1/q} \right\rangle$$
$$= \left\langle |\overline{\Delta h^q} - q\overline{\Delta h^{q-1}}\;\overline{\Delta h} + \cdots + \overline{\Delta h}^q|^{1/q} \right\rangle$$
$$\simeq \left\langle \left| \frac{t^{1/z_{\rm loc}}}{L} t^{q\beta_{\rm loc}} - q\frac{t^{1/z_{\rm loc}}}{L} t^{(q-1)\beta_{\rm loc}} \frac{t}{L} + \cdots + \frac{t^q}{L^q} \right|^{\frac{1}{q}} \right\rangle$$

$[\overline{\Delta h(t)} = t/L]$. In the observed scaling regime $t \ll L^{z_{\rm loc}}$ and therefore the main contribution comes from the first term. Thus we have

$$C_q(t) \sim t^{\beta_{\rm loc} + 1/qz_{\rm loc}} \sim t^{1+(1-q)/qz_{\rm loc}}, \quad (7)$$

i.e., $\beta_q \simeq 1 + (1-q)/q(1+\zeta)$ and $\beta_2 \simeq 0.70$, which agrees with the simulations, while for $q \to \infty$, $\beta_q \to \beta_{\rm loc}$.

Next the avalanche size is investigated. We observe that the distribution $P_{\rm av}(s, \eta^0_{\min})$ of the avalanche size s for a given $\eta^0_{\min}$ shows a power-law decay with an exponent $\kappa \simeq 1.25 \pm 0.05$ up to a size $s_0 \sim \xi_\parallel(\eta^0_{\min})\xi_\perp(\eta^0_{\min})$ and then drops to zero for $s > s_0$ [see Fig. 4(a)]. Thus the avalanche size distribution obeys the scaling form

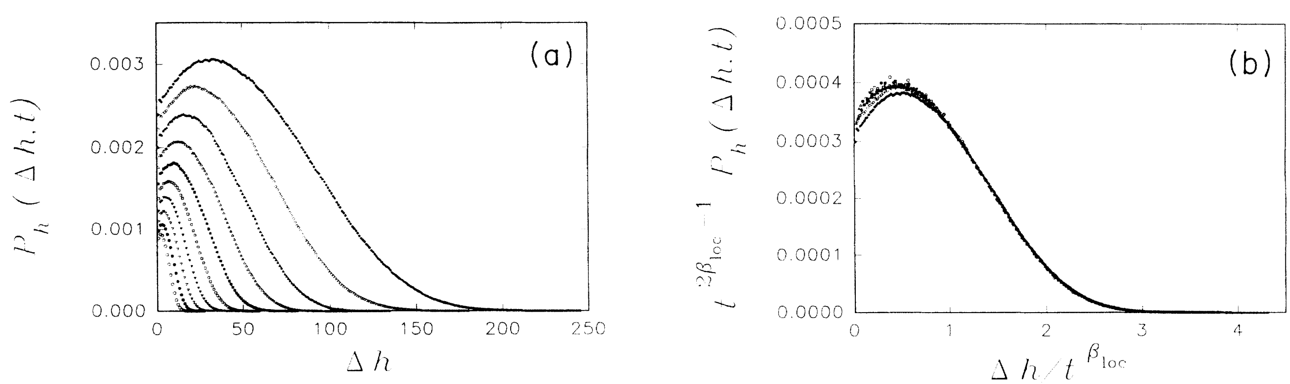

FIG. 3. (a) Distribution $P_h(\Delta h, t)$ of height advances $\Delta h > 0$ for time differences $2^6 \le t \le 2^{15}$ ($L = 8192$). Higher curves correspond to larger t. (The same plotting symbol is used for data at a given time t.) In addition, there is a large peak of P_h at $\Delta h = 0$ which is not shown. The curves are normalized if the peak at $\Delta h = 0$ is included. (b) Scaling plot, Eq. (6), with the local exponent $\beta_{\rm loc} \simeq 0.40$.

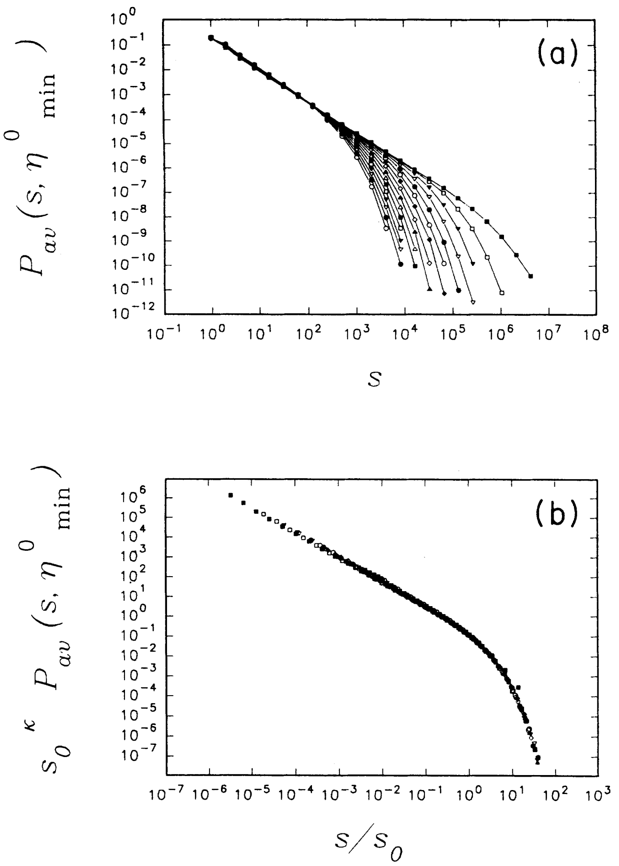

FIG. 4. (a) Logarithmically binned distribution of avalanche sizes s, where $P_{av}(s, \eta_{min})$ is the density of events in the range $[s, 2s)$. Lower curves correspond to smaller η_{min}. ($0.30 \le \eta_{min} < 0.46$, $L = 8192$). $P_{av}(s, \eta_{min})$ has a power-law decay with an exponent $\kappa \simeq 1.25 \pm 0.05$ up to a size $s_0 \sim \xi_{\parallel}(\eta^0_{min})\xi_{\perp}(\eta^0_{min})$ and then drops to zero for $s > s_0$. (b) Scaling plot of the avalanche size distribution [Eq. 8].

$$P_{av}(s) = s^{-\kappa}\, \Phi\left(\frac{s}{s_0}\right), \tag{8}$$

with $\Phi(y)$ =const for $y < 1$ and a rapid decay for $y > 1$ [19]. The good data collapse onto the scaling form Eq. (8) in Fig. 4(b) supports our picture that an avalanche corresponds to the motion through a compartment of a percolation cluster, which is characterized by the lengths $\xi_{\perp}$ and $\xi_{\parallel}$.

IV. SPATIAL-TEMPORAL CORRELATIONS

In this section we try to understand the spatial-temporal correlations between successive growth events. To this end we investigate the probability distribution $P_{co}(x, \Delta t)$, where x is the distance parallel to the interface between growth events which occur after a time Δt. Sneppen and Jensen [13] observed that $P_{co}(x, \Delta t)$ is constant for sufficiently small x and has a power-law decay with an exponent $\gamma = 2.25 \pm 0.05$ above a value x_c which increases with Δt [Fig. 5(a)].

We next explain that the behavior $P_{co}(x, \Delta t)$ =const corresponds to the dynamics of causal growth events inside an avalanche. The local adjustments due to the slope constraint after the avalanche has started induce randomly distributed $\eta(i, h_i)$. During the avalanche all $\eta_{min}(t) < \eta^0_{min}$ are taken from these newly appeared $\eta(i, h_i)$. Thus the $\eta_{min}(t)$ are randomly distributed in space, i.e., the distance between successive growth events is also equally distributed as long as Δt is smaller than the duration of the avalanche τ, i.e., as long as $x < \xi_{\parallel}(\eta^0_{min})$. Therefore we can cast $P_{co}(x, \Delta t)$ into the scaling form

$$P_{co}(x, \Delta t) = \frac{1}{\Delta t^{1/z_{loc}}}\, \Psi\left(\frac{x}{\Delta t^{1/z_{loc}}}\right), \tag{9}$$

with $z_{loc} = 1 + \zeta$ and $\Psi(y)$ =const for $y < 1$ and $\Psi \sim y^{-\gamma}$ for $y > 1$. The scaling form Eq. (9) with a satisfactory data collapse is shown in Fig. 5(b).

We see that the spatial-temporal correlations depend

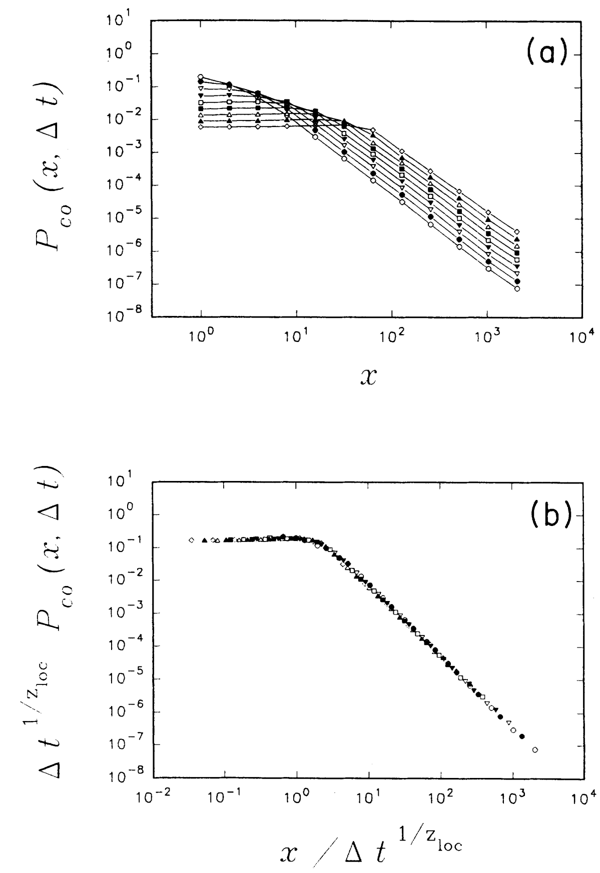

FIG. 5. (a) Logarithmically binned distribution of distances x between two growth events, where $P_{co}(x, \Delta t)$ is the density of x in the range $[x, 2x)$. The two growth events occur after a time Δt ($1 \le \Delta t \le 256$, $L = 8192$). Curves with a larger Δt have a wider plateau. The power-law decay has an exponent $\gamma = 2.20 \pm 0.05$. (b) Scaling plot of the distribution $P_{co}(x, \Delta t)$ Eq. (9), where x is scaled by $\Delta t^{1/(1+\zeta)}$ using the measured value $\zeta = 0.655$.

on the value $\eta_{\min}$. For $\eta_{\min}$ close to F_c, x is equally distributed even for large x. For small $\eta_{\min}$, on the other hand, $P_{co}(x,\Delta t)$ obeys a power-law decay also for small x, i.e., it is more probable that successive growth events are close by. Thus for these correlations the temporal translational invariance is destroyed.

The value of the exponent γ can be understood by considering the conditional distribution P_{co} for fixed $\eta_{\min}$ of the first event, which we denote by $\tilde{P}_{co}(\eta_{\min},x,\Delta t)$. We express $P_{co}(x,\Delta t)$ by

$$P_{co}(x,\Delta t) = \int d\eta_{\min}\, \tilde{P}_{co}(\eta_{\min},x,\Delta t)\, P_\eta(\eta_{\min}). \quad (10)$$

From our simulation [see Fig. 2(b)] we see that the probability distribution $P_\eta(\eta_{\min}) \sim (F_c - \eta_{\min})$ close to F_c. From the above discussion we know that $\tilde{P}_{co}(\eta_{\min},x,\Delta t) \simeq 1/\xi_\parallel$ if $\xi_\parallel > x$. Thus the integral Eq. (10) takes the form

$$P_{co}(x,\Delta t) = \int_{\xi_\parallel > x} d\eta_{\min}\, \frac{1}{\xi_\parallel}\, (F_c - \eta_{\min})$$

from which one obtains $P_{co}(x,\Delta t) \sim x^{-\gamma}$ with

$$\gamma = 1 + (2/\nu_\parallel) \simeq 2.16. \quad (11)$$

This is quite close to our numerical result $\gamma = 2.20 \pm 0.05$. We have also directly measured the distribution $\tilde{P}_{co}(\eta_{\min},x,\Delta t)$ to check our assumptions. We found that $\tilde{P}_{co}(\eta_{\min},x,\Delta t = 1)$ first decreases for small x due to a high probability for choosing the next $\eta_{\min}$ from the newly appeared η from the local adjustments. However, to explain the exponent γ we are interested in large x (and thus in large $\xi_\parallel$), for which we indeed observe a constant $\tilde{P}_{co}(\eta_{\min},x,\Delta t)$ for $x < \xi_\parallel(\eta_{\min})$.

V. CONCLUSIONS AND SUMMARY

We have analyzed a number of spatial and temporal correlations in the Sneppen model [11,13]. The imposed slope constraint $|h_i - h_{i-1}| \leq 1$ is the reason for the relation of the static and dynamic behavior to the properties of directed percolation. The roughness exponent of the interface in the Sneppen model and of the pinned interface in the model of Ref. [9] is equal to that of a percolating string, $\zeta_c \simeq 0.63$.

The difference between the two models is, however, that in the model of Ref. [9] the interface is driven by a uniform force whereas in the Sneppen model there is a self-tuned driving force which keeps the interface at the onset of steady-state motion, giving rise to "self-organized" critical behavior. This is achieved by the rule that the site which has the weakest pinning force $\eta_{\min}$ among all sites of the interface grows. This induces a nonlocal part in the dynamics. As a consequence, the motion of the interface is inhomogeneous in space and the methods of dynamical scaling are not applicable in a direct way because there is no single growing correlation length. Thus we have separated the local from the global part of the motion by introducing an "avalanche," and assigned a well-defined dynamical exponent $z_{\rm loc} = 1 + \zeta_c$ to the lateral propagation of the growth inside an avalanche. We found that the size distribution of the avalanches started with $\eta^0_{\min}$ has a power-law decay with an exponent $\kappa \simeq 1.25$ up to a size $\xi_\parallel(\eta^0_{\min})\xi_\perp(\eta^0_{\min})$.

The spatial-temporal correlations were investigated by the probability distribution $P_{co}(x,\Delta t)$ which shows a crossover from a behavior determined by causal growth events [$P_{co}(x,\Delta t)$=const] to a power-law decay with an exponent $\gamma \simeq 2.2$ which can also be related to exponents of directed percolation. We have seen that for the distribution P_{co} the temporal translational invariance is lost, which is due to the global part of the dynamics.

Upon completion of the paper we became aware of an independent work by Z. Olami, I. Procaccia, and R. Zeitak where ideas similar to ours have been developed.

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[17] The upward shift of $P_p(\eta)$ in the transient regime is similar to the initial buildup of local slopes in sandpile models, and in this sense it may be viewed as a self-organization process towards a critical state. The same feature appears in the model proposed by Bak and Sneppen [Phys. Rev. Lett. **71**, 4083 (1993)] for biological evolution.

[18] The definition of $C_q(t)$ in Ref. [13] is slightly different from the one we used in Eq. (4) which yields the expression in Eq. (5) in the limit $q \to \infty$. Since in Eq. (4) we first take the power $1/q$ and then average over time, we still average in Eq. (5) over time (or over the disorder).

[19] We have not been able to explain the observed exponent $\kappa = 1.25 \pm 0.05$. O. Narayan and D. S. Fisher (Ref. [8]) derived a scaling relation between the avalanche exponent and the correlation length exponent. Translated into our case, this result implies $\kappa = 1 + (1 - 1/\nu_\parallel)/(1+\zeta) \simeq 1.26$, in excellent agreement with the simulation result.