Border-collision bifurcations: An explanation for observed bifurcation phenomena

Helena E. Nusse,[1,2] Edward Ott,[3] and James A. Yorke[1,4]
[1]Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742
[2]Rijksuniversiteit Groningen, Vakgroep Econometrie, Postbus 800, NL-9700 AV Groningen, The Netherlands
[3]Laboratory for Plasma Research, Department of Physics, and Department of Electric Engineering, University of Maryland, College Park, Maryland 20742
[4]Department of Mathematics, University of Maryland, College Park, Maryland 20742

(Received 19 March 1993)

Recently physical and computer experiments involving systems describable by continuous maps that are nondifferentiable on some surface in phase space have revealed novel bifurcation phenomena. These phenomena are part of a rich new class of bifurcations which we call *border-collision bifurcations.* A general criterion for the occurrence of border-collision bifurcations is given. Illustrative numerical results, including transitions to chaotic attractors, are presented. These border-collision bifurcations are found in a variety of physical experiments.

PACS number(s): 05.45.+b

In the literature dealing with bifurcation theory, it is usually assumed that the dynamical system arises from a differentiable process. On the other hand, systems that are not differentiable on a surface in phase space occur in a variety of physical situations, and this circumstance leads to a rich class of new bifurcation phenomena. Indeed several experimental and numerical studies [1–6] in systems of this type have reported seeing a new kind of bifurcation. For example, in experimental and numerical studies of self-synchronization of digital phase-locked loops and the chaotic synchronization of digital phase-locked loops [1,2], it was observed that there is a transition from a periodic orbit to chaos, and this transition is different from those of maps which are everywhere differentiable. A similar phenomenon was observed in a He-Ne laser [3,4] and in studies of grazing impact in mechanical oscillators [5], to mention a few examples. In this paper we give an explanation of these observed bifurcation phenomena.

The purpose of this paper is to study the occurrence of such a new bifurcation phenomenon for a variety of physical models, two of which will be emphasized. In [7] two-dimensional piecewise smooth maps are examined, and a variety of examples are presented. At that time, no physical examples were known. In this paper the maps we investigate are more general than those in [7]. In particular, we treat the physical important case of a square-root singularity. We obtain a general result on bifurcations of maps that are not differentiable on some surface in phase space. We illustrate this result in numerical experiments.

As an example involving a two-dimensional map, assume that a particle undergoes forced damped harmonic motion:

$$\ddot{x}+2a\gamma\dot{x}+\gamma^2x=F(t) \quad \text{for } x>x_c\ , \tag{1}$$

with a hard wall at $x=x_c$ at which the particle undergoes elastic reflection; here F is a periodic function of the time t with period 2π, $a>0$ is the damping parameter, and $\gamma^{-1}>0$ is the ratio of the driving to the natural frequency for the system. Forced impact oscillators have been intensively studied for several years as examples of nonlinear systems exhibiting complicated dynamics; see [5,8–11] and references therein. In this case, a curve Γ in the two-dimensional time-2π surface of section is determined by the condition of grazing incidence of the hard wall, and the derivative of the time-2π map has a square-root singularity on Γ. As a special case, the impact oscillator (1) is considered in [5] with elastic impact at $x=x_c=-1$ and

$$F(t)=(\mu+1)[(\gamma^2-1)\cos(t)-2a\gamma\sin(t)]\ .$$

Here μ is regarded as the bifurcation parameter. Note that $x(t)=(\mu+1)\cos(t)$ is the attracting solution as long as $-2<\mu<0$. As μ is increased through 0, the stable periodic solution $x(t)=(\mu+1)\cos(t)$ will be grazing the wall at $\mu=0$. Our results for this system are summarized later in this paper.

In the bifurcation theory for maps, attention has been focused on differentiable maps when one or more eigenvalues of a fixed point (or periodic point) cross the unit circle. When this occurs, the nature of the fixed point changes. For example, a fixed-point attractor becomes a saddle (possibly a flip saddle) or a repellor. We say a map is *smooth* if the map has a continuous derivative. We assume that there is a smooth surface Γ which separates the k-dimensional phase space into two regions denoted by R_A and R_B. Let $F(\cdot,\mu)=F_\mu$ be a one-parameter family of continuous maps from the k-dimensional phase space to itself, being smooth on the regions R_A and R_B and depending smoothly on the parameter μ. In particular, physical situations arise in which F_μ has a square-root singularity on Γ. Let E_μ denote a fixed point of F_μ defined on $-\varepsilon<\mu<\varepsilon$ and depending continuously on μ, for some $\varepsilon>0$. We say that E_μ is a *border-crossing fixed point* if it crosses the border Γ between the two regions

R_A and R_B. We will assume that the crossing occurs at $\mu=0$. We say that a border-crossing fixed point E_μ is an *isolated border-crossing fixed point* if the point E_μ is isolated in phase space when $\mu=0$, that is, in the phase space, there exists a neighborhood U of the point E_0 such that E_0 is the only periodic point in U when $\mu=0$. The case where E_μ is a periodic orbit can be dealt with similarly, but for simplicity we restrict our attention to the case where E_μ is a fixed point.

For border-crossing fixed points, the Jacobian matrix of the fixed point generally changes discontinuously, and the fixed point can, for example, change from being strongly repelling to strongly attracting as μ crosses zero. To analyze these bifurcations that occur while the fixed point crosses the border, we need the concept of the "orbit index" of a typical periodic orbit [12]. This approach was used in [7] to investigate the bifurcations that occur in the case that the crossing fixed point for two-dimensional piecewise smooth maps changes from being a repellor to a saddle as μ crosses zero. We say that an orbit of period p is *typical* if the Jacobian matrix of the pth iterate of the map at a point of the orbit exists and no eigenvalue of the Jacobian matrix is on the unit circle. Let P denote a typical fixed point of F_μ. Let A be the Jacobian matrix of F_μ evaluated at P, let m be the number of real eigenvalues of A smaller than -1, and let n be the number of real eigenvalues of A greater than $+1$. The *orbit index* I of the fixed point P is defined by [12]

$$I=\begin{cases}0 & \text{if } m \text{ is odd}\\ -1 & \text{if } m \text{ is even and } n \text{ is odd}\\ +1 & \text{if both } m \text{ and } n \text{ are even .}\end{cases}$$

Assume that there exists a number $\varepsilon>0$ such that (a) E_μ is an isolated border-crossing orbit with E_0 on the border of the two regions R_A and R_B, (b) for $-\varepsilon<\mu<0$ the fixed point E_μ is in the region R_A and its index is I_A, (c) for $0<\mu<\varepsilon$ the fixed point E_μ is in the region R_B and its index is I_B, and (d) $I_A \neq I_B$. Since $I_A \neq I_B$, as μ approaches 0^+ or 0^-, the orbit E_μ cannot be the only orbit that approaches the point E_0, since that would violate the invariance of the orbit index [12]. Thus it is necessary that something else happens. What can it be? The following two possibilities, which we shall demonstrate, actually occur: (1) There are additional periodic orbits which shrink to E_0 as $\mu\to0^+$ and/or $\mu\to0^-$ (these bifurcating orbits need not be stable), and (2) a chaotic set or sets shrinks on to E_0 as $\mu\to0^+$ and/or $\mu\to0^-$.

We emphasize that E_μ may be nonattracting in both R_A and R_B but other orbits which shrink to E_0 may be attractors [7]. Using arguments similar to those in [7], we can show the following: If the index of an isolated-border-crossing fixed point E_μ of F_μ changes as μ crosses 0, then at $\mu=0$, a bifurcation occurs at this point E_0, a bifurcation involving at least one additional periodic orbit. Since this bifurcation occurs while the fixed point (or periodic point) crosses the border of the regions R_A and R_B, we call it a *border-collision bifurcation*. In other words, a border-collision bifurcation is a bifurcation at a fixed point (or periodic point) on the border of two regions when the orbit index of the fixed point before the crossing of the border is different from the orbit index of the fixed point after the collision. We find that border-collision bifurcations constitute an extremely rich class of bifurcation phenomena. As an example of possibility (1) above, there may be a periodic attractor of period P_1 existing for $\mu<0$ and a periodic attractor of period P_2 existing for $\mu>0$, and these may shrink to the fixed point E_0 as $\mu\to0$. In this case we say there is a *period-P_1 to period-P_2 border-collision bifurcation.*

We can now summarize our results for the impact oscillator in Eq. (1) as follows. In the majority of the cases, the fixed-point attractor for $\mu<0$ (index $=+1$) converts to a flip saddle (index $=0$) for $\mu>0$. Hence, at $\mu=0$, the orbit index of the fixed point changes from $+1$ to 0. Therefore, a border-collision bifurcation occurs at $\mu=0$. Applying the orbit-index result described above implies that there must be other bifurcating orbits involved. Our numerical examples include a variety of border-collision bifurcations from a fixed-point attractor to a chaotic attractor. (The numerical simulation in [5] with $a=1.25$ and $\gamma=0.1$ is a border-collision bifurcation from a fixed-point attractor to a chaotic attractor for the time-2π map.)

Now consider one-dimensional maps that occur in the studies of laser systems in [13,14]. Define the one-parameter family of maps F_μ by

$$F_\mu(x)=\begin{cases}\alpha x+\mu & \text{if } x\le 0\\ \beta x^z+\mu & \text{if } x\ge 0,\quad 0<\alpha<1,\ \beta<-1\ ,\end{cases}\tag{2}$$

where $0<z\le 1$ and the parameter μ belongs to some interval I surrounding 0.

F_μ is continuous, and is smooth in the regions $R_A=\{x: x<0\}$ and $R_B=\{x: x>0\}$. If $\mu<0$, then the map F_μ has a fixed-point attractor, and the index of this fixed point is $+1$. For small positive μ, F_μ has a fixed point which is a flip saddle, and its index is 0. Therefore, at $\mu=0$, a border-collision bifurcation will occur at $x=0$.

First consider the case $z=\frac{1}{2}$. In this case there is a square-root singularity in the derivative. This case is particularly interesting because it illustrates generic features of impact oscillators such as (1). The phenomenology and scaling relation given below has been previously reported by Nordmark [5] in the two-dimensional case, and provides a good illustration of a border-collision bifurcation. Indeed, it can be shown that results for (2) with $z=\frac{1}{2}$ correspond to bifurcations of the impact oscillator when the oscillator is overdamped [$a>1$ in (1)]. For example, for the parameters $\alpha=0.5$ and $\beta=-2$, Fig. 1(a) shows that decreasing μ from 0.1 to 0, the bifurcation diagram (and consecutive blowups not shown) exhibit an infinite cascade of reversed "period addings." That is, at the right-hand side of Fig. 1(a) there is a stable period-3 orbit, then a stable period-4 orbit to the left of it, then a stable period-5 orbit to the left of it, then a stable period-6 orbit to the left of it, and so on. As μ continuously de-

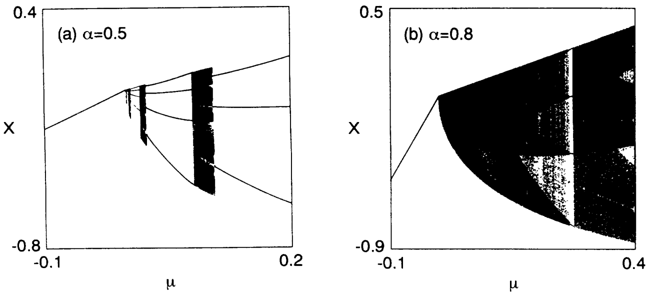

FIG. 1. Border-collision bifurcations for F_μ, where $F_\mu(x)=\alpha x+\mu$ if $x\le 0$ and $F_\mu(x)=-2\sqrt{x}+\mu$ if $x\ge 0$. The chaotic attractor for $\mu>0$ lies in the region $F_\mu^2(0)=\mu-2\sqrt{\mu}\le x\le\mu=F_\mu(0)$.

creases toward 0, there are infinitely many such windows with the period going to infinity. These windows are geometrically accumulating on $\mu=0$ with μ scaling by the factor α^2 at successive period addings. We can show the following.

If $0<\alpha<\frac{2}{3}$, then, as μ decreases through 0, the map F_μ with $z=\frac{1}{2}$ exhibits a border-collision bifurcation from a fixed-point attractor to a reversed infinite cascade of period addings accumulating at $x=0$. If $\frac{2}{3}<\alpha<1$, then, at $\mu=0$, the map F_μ with $z=\frac{1}{2}$ has a border-collision bifurcation from a fixed-point attractor to a chaotic attractor at $x=0$ [see, e.g., Fig. 1(b)].

For $\frac{2}{3}<\alpha<1$, as μ decreases towards zero from large positive values, there will be several period addings but these will cease after a finite number of addings, followed by an interval in μ extending to $\mu=0$, in which the attractor is chaotic, that is, the attractor contains a trajectory consisting of infinitely many points whose Lyapunov exponent is positive [Fig. 1(b)]. The number of period addings that occur before the cascade aborts increases toward ∞ as α approaches $\frac{2}{3}$ from above. Figure 1(b) for $\alpha=0.8$ shows a case where the cascade aborts after the period-4 window.

To understand the geometric accumulation of period addings and the associated scaling factor, we focus our attention on the periodic orbit attractor in a period-adding window. Say that this orbit has a period m and denote the range in the parameter μ over which this orbit exists as an attractor by

$$\mu_m^* > \mu > \mu_m \ . \tag{3}$$

Here, μ_m is the parameter value at which the periodic orbit loses its stability, and μ_m^* is the "tangent" bifurcation parameter value. We wish to find μ_m and μ_m^*. We observe from our numerics that the period-m orbit spends $m-1$ iterates in $x<0$ and one iterate in $x>0$. Let

$$x_1<x_2<\cdots<x_{m-1}<0<x_m$$

denote the orbit points for this periodic orbit. We now ask, what is the condition for such an orbit to exist? Since $x_{m-1}<0$, we have, from (2), $x_m<\mu$ and $x_1>\mu+\beta\sqrt{\mu}$. On the other hand, iteration of $x_{n+1}=\alpha x_n+\mu$ from x_1 to x_{m-1} yields

$$x_{m-1}=\alpha^{m-2}x_1+(1-\alpha^{m-2})(1-\alpha)^{-1}\mu \ .$$

Combining these yields the following condition for the existence of the orbit:

$$\mu<\mu_m^*=\alpha^{2m}\left[\frac{(\beta/\alpha^2)(1-\alpha)}{1-\alpha^{m-1}}\right]^2 . \tag{4}$$

We now ask, when is the orbit stable? Differentiating (2), and noting that the orbit has $m-1$ points in $x<0$ and one in $x>0$, we have that $1>\frac{1}{2}\alpha^{m-1}(-\beta)/\sqrt{x_m}$ for stability. Combining this with

$$x_m=\alpha^{m-1}x_1+(1-\alpha^{m-1})(1-\alpha)^{-1}\mu$$

[which follows from $x_m=F_\mu^{m-1}(x_1)$] and $x_1=\mu+\beta\sqrt{x_m}$, we have the stability condition

$$\mu>\mu_m=\frac{3}{4}\frac{\beta^2}{\alpha^2}\frac{1-\alpha}{1-\alpha^m}\alpha^{2m} \ . \tag{5}$$

Thus we see that both μ_m and μ_m^* scale as α^{2m} yields geometric convergence of the window width to zero at the rate α^2. Also, we obtain a condition for the period-adding cascade to occur. In particular, from (3) we must have $\mu_m^*>\mu_m$. Using (4) and (5) with $m\to\infty$, this yields the condition $\alpha<\frac{2}{3}$ for occurrence of an infinite cascade of period addings.

Now consider the case $z=1$ in Eq. (2) for which F_μ is piecewise linear. Let $n\ge 2$ be any integer. Using kneading theory [15] and the results in [16], we can show the following.

If $0<\alpha<1$ and $-\alpha^{1-n}<\beta<\alpha(1-\alpha)^{-1}(1-\alpha^{1-n})$, then, at $\mu=0$, the map F_μ exhibits a "period-1 to period-n" border-collision bifurcation at $x=0$. If $0<\alpha<1$ and $\alpha(1-\alpha)^{-1}(1-\alpha^{-n})<\beta<-\alpha^{1-n}$, then, at $\mu=0$, the map F_μ has a border-collision bifurcation from a fixed-point attractor to a chaotic attractor at $x=0$.

In these bifurcations an interesting bifurcation phenomenon occurs. Let $n\ge 3$ and $0<\alpha<1$ be given, and consider border-collision bifurcations while decreasing β continuously in the interval

$$\frac{\alpha}{1-\alpha}(1-\alpha^{-n})<\beta<-\alpha^{1-n} \ .$$

Then we can show the following: There exist constants β_n^*,β_n^+ (depending on α) for which

$$\frac{\alpha}{1-\alpha}(1-\alpha^{-n})<\beta_n^*<\beta_n^+<-\alpha^{1-n}$$

such that if $\beta_n^+<\beta<-\alpha^{1-n}$, then the map F_μ exhibits a border-collision bifurcation from a fixed-point attractor to a $2n$-piece chaotic attractor; if $\beta_n^*<\beta\le\beta_n^+$, then the map F_μ exhibits a border-collision bifurcation from a fixed point attractor to an n-piece chaotic attractor, and if

$$\frac{\alpha}{1-\alpha}(1-\alpha^{-n})<\beta<\beta_n^* \ ,$$

then the map F_μ exhibits a border-collision bifurcation from a fixed point attractor to a 1-piece chaotic attractor.

The case $n=2$ has not been resolved yet.

As a special case, illustrating the above, consider $\alpha=0.5$, which applies for Fig. 2. If

$$-2^{n-1}<\beta<-2^{n-1}+1\ ,$$

then the family F_μ has a "period-1 to period-n" border-collision bifurcation at $x=0$. For $n=3$ this gives $-4<\beta<-3$ which is the case for Fig. 2(a). If $-2^n+1<\beta<-2^{n-1}$, then at $\mu=0$, there is a border-collision bifurcation at $x=0$ from a fixed point attractor to a chaotic attractor. For $n=3$ this gives $-7<\beta<-4$, which is the case for Figs. 2(b), 2(c), and 2(d). Here Figs. 2(b), 2(c), and 2(d) have chaotic attractors consisting, respectively, of six pieces ($\beta_3^+<\beta<-4$), three pieces ($\beta_3^*<\beta\le\beta_3^+$) and one piece ($-7<\beta\le\beta_3^*$). For Fig. 2(a) note that the fixed point in $\mu<0$ and the period-3 attractor in $\mu>0$ both have index $+1$, while the fixed-point attractor in $\mu<0$ converts to a fixed-point flip saddle (index 0) in $\mu>0$.

We now show why the chaotic attractor has six pieces, three pieces, and one piece in the ranges of β corresponding to Figs. 2(b)–2(d). Let $\mu>0$; then for $-7<\beta\le-4$ there are three intervals (depending on μ,α), say, J_1, J_2, and J_3, which are invariant under the third iterate of F_μ. For $\beta=-4$, every point in J_k $(1\le k\le3)$ which is not a period-3 point, is a period-6 point. Therefore, there is no chaotic attractor for $\beta=-4$. The absolute value of the derivative of F_μ^6 at points in J_k $(1\le k\le3)$ where it exists, exceeds 1 for $-7<\beta<-4$. Decreasing the constant β, initially there are two invariant subintervals for F_μ^6 in each J_k $(1\le k\le3)$. When the constant β is decreased, there exists a value β_3^+, such that in each J_k $(1\le k\le3)$ the two invariant subintervals will merge. It can be shown that for $\beta_3^+<\beta<-4$, the map F_μ has a dense trajectory on the union of these six intervals. Similarly, there exists a value β_3^*, $-7<\beta_3^*<\beta_3^+$, such that if $\beta_3^*<\beta<\beta_3^+$, then F_μ has a dense trajectory on the union of three intervals J_k $(1\le k\le3)$. If $-7<\beta<\beta_3^*$, then F_μ has a dense trajectory on the interval $[F_\mu^2(0),F_\mu(0)]$. The conclusion is that if $\beta_3^+<\beta<-4$, then F_μ exhibits a border-collision bifurcation from a fixed-point attractor to a six-piece chaotic attractor; if $\beta_3^*<\beta\le\beta_3^+$, then F_μ exhibits a border-collision bifurcation from a fixed-point attractor to a three-piece chaotic attractor; and if $-7<\beta\le\beta_3^*$, then F_μ exhibits a border-collision bifurcation from a fixed point attractor to a one-piece chaotic attractor. The main conclusion is that border-collision bifurcations occur in a wide variety of physical circumstances.

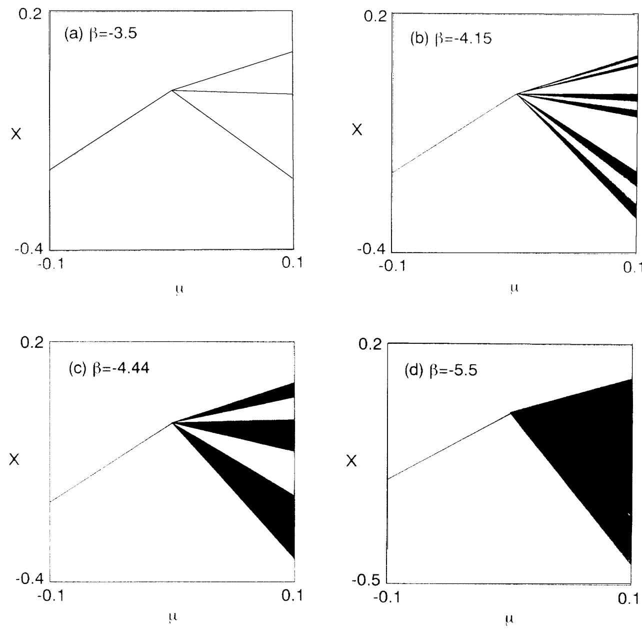

FIG. 2. Border-collision bifurcations for F_μ, where $F_\mu(x)=0.5x+\mu$ if $x\le0$ and $F_\mu(x)=\beta x+\mu$ if $x\ge0$.

The computer-assisted pictures were made by using the program Dynamics [17]. This work was in part supported by the Department of Energy (Scientific Computing Staff Office of Energy Research) and by ONR (Physics).

[1] J. Gullicksen *et al.*, in *Proceedings of the 1st Experimental Chaos Conference,* edited by S. Vohra *et al.* (World Scientific, Singapore, 1992), p.137.

[2] M. de Sousa Vieira *et al.*, Int. J. Bifurcations Chaos **3**, 645 (1992).

[3] R. Vallée and C. Delisle, Phys. Rev. A **31**, 2390 (1985).

[4] C. Marriott and C. Delisle, Physica D **36**, 198 (1989).

[5] A. B. Nordmark, J. Sound Vib. **145**, 279 (1991); Int. J. Bifurcations Chaos **2**, 597 (1992).

[6] C. H. Hommes and H. E. Nusse, J. Econom. **54**, 157 (1991).

[7] H. E. Nusse and J. A. Yorke, Physica D **57**, 39 (1992).

[8] S. W. Shaw and P. J. Holmes, J. Sound Vib. **90**, 129 (1983).

[9] S. W. Shaw, J. Sound Vib. **99**, 199 (1985).

[10] G. S. Whiston, J. Sound Vib. **118**, 395 (1987); **152**, 427 (1992).

[11] F. C. Moon, *Chaotic and Fractal Dynamics* (Wiley, New York, 1992).

[12] J. Mallet-Paret and J. A. Yorke, J. Diff. Eq. **43**, 419 (1982).

[13] A. A. Hnilo, Commun. **53**, 194 (1985).

[14] M. C. de Sousa Vieira, E. Lazo, and C. Tsallis, Phys. Rev. A **35**, 945 (1987).

[15] J. Milnor and W. Thurston, in *Dynamical Systems,* edited by J. C. Alexander, Lecture Notes in Math. Vol. 1342 (Springer-Verlag, Berlin, 1988), p. 465.

[16] S. Ito, S. Tanaka, and H. Nakada, Tokyo J. Math. **2**, 241 (1979).

[17] H. E. Nusse and J. A. Yorke, *Dynamics: Numerical Explorations* (Springer-Verlag, New York, in press).

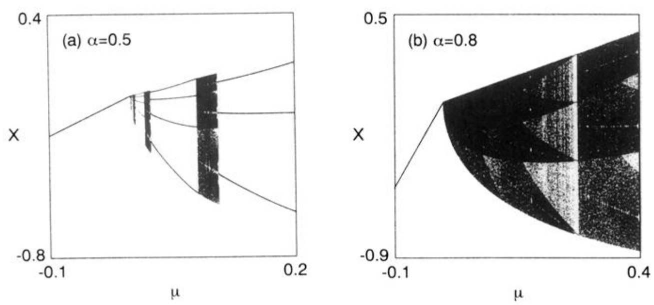

FIG. 1. Border-collision bifurcations for F_μ, where $F_\mu(x)=\alpha x+\mu$ if $x\leq 0$ and $F_\mu(x)=-2\sqrt{x}+\mu$ if $x\geq 0$. The chaotic attractor for $\mu>0$ lies in the region $F_\mu^2(0)=\mu-2\sqrt{\mu}\leq x\leq\mu=F_\mu(0)$.