PHYSICAL REVIEW E VOLUME 49, NUMBER 1 JANUARY 1994

Self-organized criticality in computer models of settling powders

R. E. Snyder* and R. C. Ball
Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE, United Kingdom
(Received 28 June 1993)

We present numerical simulations of powder flow in the regime of vacancy hopping, under gravity, in two dimensions, for simplicity. Bulk properties such as density and angle of rest are measured and correlated with the microscopic parameters of the model. Avalanches are identified as the damage spreading from a single new vacancy introduced. They are found to exhibit universal power-law distributions of both total size S and maximum height reached H, with $P_H(H) \sim H^{-1.47\pm0.02}$ and $P_S(S) \sim S^{-1.34\pm0.01}$. At height h, the average width of avalanches (reaching $H \geq h$) scales as $\langle w \rangle \sim h^{0.46\pm0.09}$, consistent with the assumption that $S \sim Hw(H)$. We also show that the distribution of w at fixed h can be scaled as a universal function of $w/\langle w \rangle$. The average lateral deviation of the core of the avalanche from the avalanche origin, $x(h)$, scales as $\langle |x| \rangle \sim h^{0.33\pm0.09}$. We have investigated the correlation between successive avalanches precipitated from the same site. Both their survival to any given height and their horizontal displacements at fixed height are strongly correlated—implying that the critical behavior of the avalanches is dictated by organized structure in the powder.

PACS number(s): 05.40.+j, 05.70.Jk, 46.10.+z

INTRODUCTION

In recent years it has become clear that a wide range of irreversible systems can exhibit stable critical behavior in that they are naturally driven to a regime of scale-invariant and universal behavior without fine tuning of external parameters. Early examples were in growth models such as diffusion-limited aggregation [1] and colloidal (cluster-cluster) aggregation [2,3] and have been confirmed by experiment [4–6]. In the context of stick-slip systems such as sandpile surfaces [7] and earthquake models [8], this was interpreted in terms of self-organized criticality (SOC), highlighting the contrast with equilibrium critical phenomena which depend unstably on external parameter values.

Bak, Tang, and Wiesenfeld (BTW) have shown in a toy model that the surface of a powder can exhibit SOC when driven towards the angle of rest by incrementing the local slope. In particular, BTW observed universal and scale-invariant (i.e., power law) behavior for the size and duration of surface avalanches resulting from a single local addition to the pile [7].

Several groups have looked for these effects in real powders. Held *et al.* observed a power-law distribution for surface avalanche sizes in small, circular sandpiles [9]. Larger sandpiles, however, were characterized by periodic large avalanches caused by slope hysteresis. In an unpublished study, Rosendahl and Rutledge [10] also used a circular pile and observed a power-law distribution of smaller avalanches between the larger avalanches. On the other hand, Liu, Jaeger, and Nagel [11] and Evesque and Rajchenbach [12] used planar setups and did not observe any evidence of SOC. Liu, Jaeger, and Nagel have noted that in sufficiently small sandpiles, the motion of a single grain can bring the system from a slope greater than the maximum angle of stability to a slope less than the angle of rest. They have suggested that apparent SOC in the data of Held *et al.* data stems from this lack of a metastable state [11].

In this paper we consider the related question of whether the interior of a powder organized into a critical structure as it settles, in our case under gravity, and without any grain adhesion, plasticity, or inertial effects. Our model is kinematic, with grains falling one step down when they are insufficiently supported, thereby propagating a corresponding vacancy upwards. Experiments suggest that kinematic models are more successful than plasticity models at describing the velocity distributions of coarse powders, although kinematic models break down for fine powders [13]. Kinematic models have also had triumphs in predicting flow patterns around obstacles [14], and films of particle motion reveal the unsteady motion predicted by kinematic models [15].

Our results mainly concern the flow due to vacancies introduced from below, corresponding to the "hour glass" problem. However, they are also relevant to the question of how a powder selects its resting density, which various authors have previously investigated in terms of models of mechanical shaking [16,17].

SIMULATION MODEL

Our system consists of a vertical, two-dimensional "box" of unit square grains with periodic boundary conditions on the side walls. The vertical coordinate of the grains is discrete, so that they are confined to horizontal layers. However, grains can lie anywhere along the horizontal axis, so long as they do not overlap. Initial configurations were generated by laying down grains

*Address for correspondence: Physics Dept., University of California, Santa Barbara, CA 93106-7160.

along each layer with spacings drawn from a uniform distribution, whose range determines the starting density.

The dynamics are that grains fall if they are insufficiently supported and there is room for them to move. Specifically, at each time step each grain will fall into the row below if the following conditions are satisfied:

(1) Instability condition: the grain overhangs a void in the row below it by more than the critical overhang ϵ;

(2) Dynamical accessibility: There is space in its present row for the grain to slide to overhang the void by 100%;

(3) Volume constraint: The void is wide enough to accommodate the grain falling from there; and

(4) Competition: if the void is only wide enough to accommodate one grain, the grain must overhang that void by more than any competing grain.

The model is thus specified and simulated in terms of the grains, but in practice the effect is that sufficiently large vacancies are mobile upwards. A typical configuration is shown in Fig. 1, together with an indication of the "avalanche" of grains which would move following the removal of a particular grain to create a new vacancy in the bottom layer.

CONSTITUTIVE PROPERTIES

The critical overhang parameter determines a natural density of the powder $\rho^*(\epsilon)$, towards which initially underdense samples spontaneously settle [see Fig. 2(a) and inset]. The flow of vacancies upwards during this process effectively makes a dynamical selection of powder configurations, except in the (less disturbed) bottom few layers. In order to minimize the effect of these less disturbed layers on avalanche statistics, the bottom two layers of the sample are removed once the sample has consolidated. We also investigated further consolidating our model powders by successively removing grains from random positions along the bottom row when no grain could otherwise move, thus driving a vacancy flux throughout the sample. However, this did not affect the height or size distributions. While we always used initially underdense samples, it is interesting to note that if vacancies are driven through the sample in this fashion, even initially overdense samples eventually approach the natural density $\rho^*(\epsilon)$ [see Fig. 2(b)].

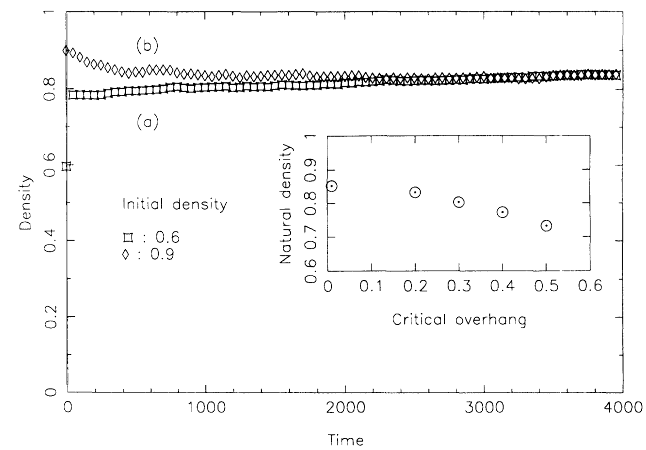

FIG. 2. Sample density vs time, starting from an initial density of (a) 0.6 (underdense) and (b) 0.9 (overdense) for $\epsilon=0.01$. Note the spontaneous settling of the underdense sample during the first 30–50 time steps. Subsequent "equilibration" of both samples is driven by removal of grains from the bottom row. Inset: "natural density" ρ^* attained, as a function of the critical overhang ϵ.

Figure 3 shows a consolidated sample (without periodic boundary conditions) driven to exhibit the angle of rest of its surface by first removing the right-hand wall to expose an unstable "cliff" and subsequently removing the

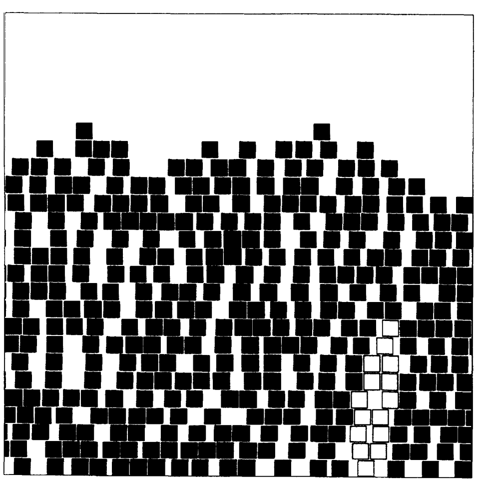

FIG. 1. Typical (but small) configuration of the semilattice powder model, in which no grain can overhang a larger void by more than $\epsilon=0.01$ grain lengths. The avalanche of grains that would move in response to the grain in the bottom row being removed is shown in outline.

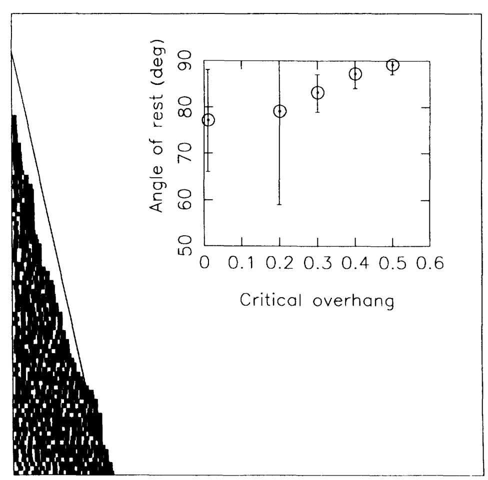

FIG. 3. Sample ($\epsilon=0.01$) driven to exhibit angle of rest by removal of bottom right-hand grain whenever it would otherwise have been stable. The inset shows the measured angle vs ϵ, for samples approximately 100 grains high, with error bars representing the full spread of measured values rather than the uncertainty in the mean.

rightmost grain in the bottom layer when no other could move. Measured over a height of 100 grains, the angle of rest varies systematically with the critical overhang but also shows significant scatter at each value.

SINGLE-AVALANCHE STATISTICS

An avalanche is defined as the totality of grain movements in an otherwise stationary sample "nucleated" by the removal of one grain from the bottom layer. We investigated the possibility of universal scaling properties of avalanche statistics, generally regarded as the hallmark of SOC. Figure 4 indicates how the total height H, width profile $w(h)$, and lateral wandering $x(h)$ are defined for individual avalanches, the size S being defined as the total number of grain movements. In collecting statistics over a large number, successive avalanches were nucleated at random grains along the bottom layer; after each avalanche the loss of one grain from the sample was compensated by creating one new grain at a random point along the top row of the box and allowing it to fall to a stable position.

Figure 5 shows the cumulative distribution of avalanche heights,

$$C_h(h)=\sum_{H=h}^{\infty} P_h(H) ,$$

interpreted as the probability of an avalanche reaching at least height h. The same power-law behavior

$$C_h(h)\sim h^{1-\alpha} , \quad \alpha=1.47\pm0.02 ,$$

is consistent with the data for three quite different values of the critical overhang, suggesting universality. The corresponding distributions by size are shown in Fig. 6, consistent with a universal power law

$$C_s(s)\sim s^{1-\beta} , \quad \beta=1.34\pm0.01 .$$

Figure 7 shows the average avalanche width $\langle w \rangle$ vs height h, averaged over only those avalanches reaching at least height h, and supports a universal power law,

$$\langle w(h) \rangle \sim h^{\gamma} , \quad \gamma=0.46\pm0.09 ,$$

suggesting that the avalanches are statistically self-affine in shape. In this case we have further evidence that a true scaling behavior applies from the superposition of the distributions of w for different heights h and overhang

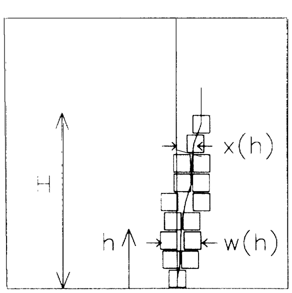

FIG. 4. Definitions of avalanche height H, width $w(h)$, and wander $x(h)$.

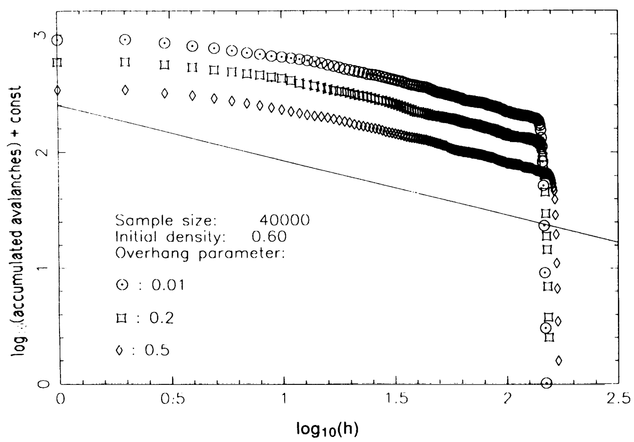

FIG. 5. Number of avalanches $C_h(h)$ reaching at least height h, both on logarithmic scales (base 10). The data are for three different values of critical overhang (as shown), from samples at initial density 0.6 over a 200×200 size grid, collected over approximately 3600 avalanches each. The straight line corresponds to the universal power law

$$C_h(h)\sim h^{1-\alpha} , \quad \alpha=1.47\pm0.02 ,$$

consistent with all three sets of data.

parameters ϵ, as shown in Fig. 8. The scaling function appears consistent with the simple exponential form

$$P_w(w|h,\epsilon)=\langle w \rangle_{h\epsilon}^{-1}\exp(-w/\langle w \rangle_{h\epsilon}) .$$

The power laws observed above are consistent with a simple scaling law based on the assumption that avalanches are compact, so that typically

$$S\sim H\langle w \rangle_h \sim H^{1+\gamma} ,$$

whereupon it follows by comparing the cumulative distributions that

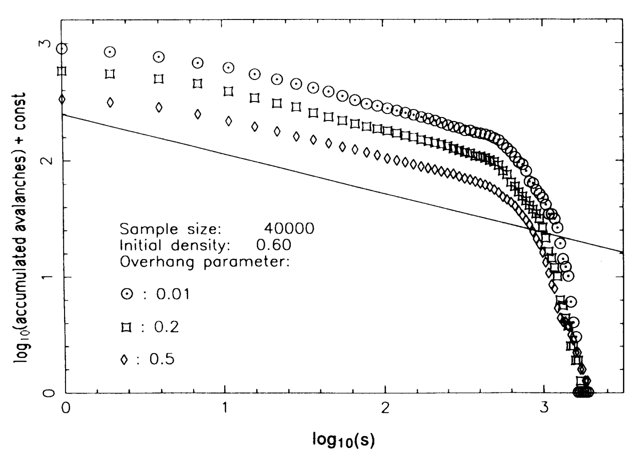

FIG. 6. Number of avalanches $C_s(s)$ reaching at least size s, samples as per Fig. 5. All are consistent with the power law

$$C_s(s)\sim s^{1-\beta} , \quad \beta=1.34\pm0.01 ,$$

as shown.

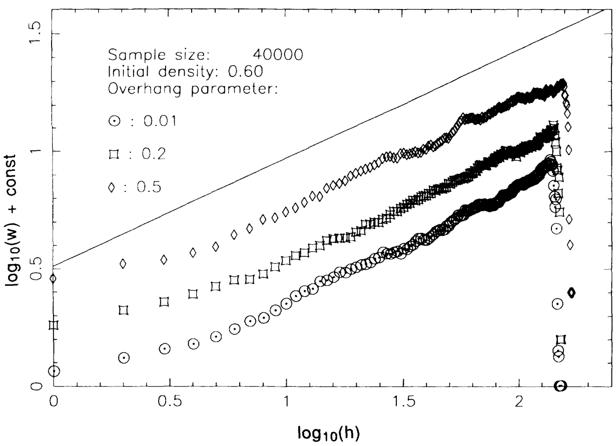

FIG. 7. The width $w(h)$ at height h averaged over the first 200 avalanches, samples as per Fig. 5. All averages are restricted to avalanches reaching h. The straight line indicates the universal power law

$$\langle w(h)\rangle \sim h^{\gamma}\ , \quad \gamma=0.46\pm0.09\ ,$$

which is consistent with all of the data.

$$1+\gamma=\frac{\alpha-1}{\beta-1}\ .$$

Our independent estimates for the two sides of this equation are 1.46 ± 0.09 and 1.38 ± 0.07, respectively.

Also associated with an avalanche is its flux $j(h)$ defined as the net flow of grains across the row at height h. However, for the density of the powder to be preserved, the average number of grains crossing each row per grain removed from the bottom must be precisely unity. Therefore, if as for the width, we restrict averages at height h to avalanches reaching at least that height and we have

$$\langle j\rangle_h C_h(h)=1$$

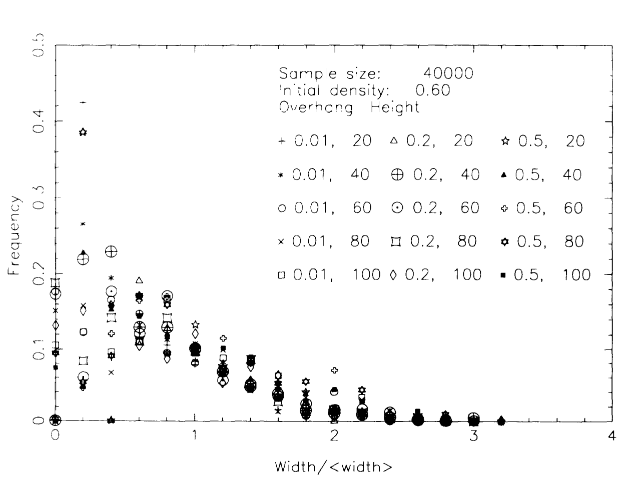

FIG. 8. Overlaid distributions of reduced width $w/\langle w\rangle$ for various values of the height h and critical overhang parameter ϵ. All samples were size 200×200.

and, hence,

$$\langle j\rangle_h \sim h^{\alpha-1}\ .$$

If the avalanche flux were to be interpreted as the width $w(h)$ falling through the order one row, then it would follow that

$$\langle j\rangle_h \sim \langle w\rangle_h\ ,$$

which would give us a second scaling law

$$\alpha-1=\gamma\ .$$

This is consistent with our simulation results.

We also measured the lateral wandering $x(h)$ of the avalanches and found

$$\langle |x(h)|\rangle \sim h^{\eta}\ , \quad \eta=0.33\pm0.09\ ,$$

as shown in Fig. 9. This gives $\eta-\gamma=-0.13\pm0.13$ (taking their errors as independent), compatible with $\eta=\gamma$ but favoring the peculiar result $\eta<\gamma$. While it might be expected from our definitions that typically $|x|<w$, a smaller exponent $\eta<\gamma$ would be more surprising. At face value it suggests that as avalanches grow large and wide they grow increasingly symmetric about the vertical line through their nucleation point. Note, however, that our values of $|x(h)|$ are not particularly large compared to unity and so the scaling of this quantity might be suspect.

Finally in this section, we note that measurements of avalanche lifetime, defined as the total number of time steps for it to be complete, showed this to be almost identical to avalanche height, as might be expected for our dynamics.

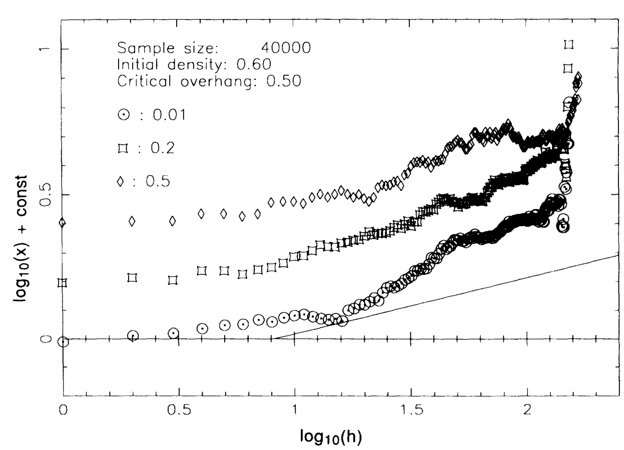

FIG. 9. The lateral wandering $x(h)$ at height h averaged over the first 200 avalanches, samples as per Fig. 5. All averages are restricted to avalanches reaching h. The power law

$$\langle |x(h)|\rangle \sim h^{\eta}\ , \quad \eta=0.33\pm0.09$$

is consistent with all of the samples.

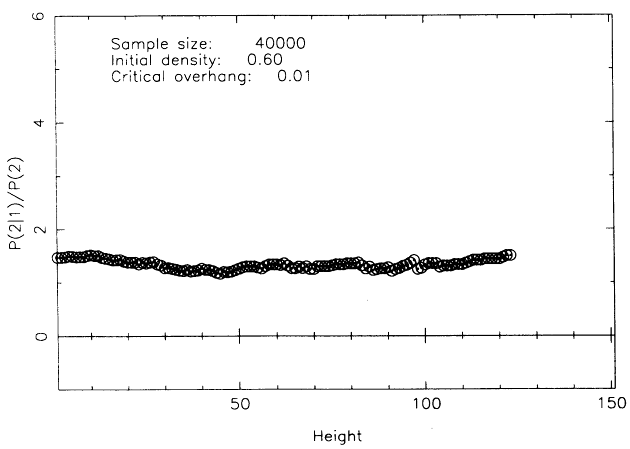

FIG. 10. Height correlations between pairs of successive avalanches started from (nearly) the same place. Plotted is the probability for the second of an avalanche pair to reach height h *conditional on the first having done so*, $P(2|1)$, compared to the unconditional probability for it to do so, $P(2)$. In the absence of correlations, this ratio would be unity. Data are taken from a sample at initial density 0.6 over a 200×200 size grid, collected over 400 avalanche pairs.

CORRELATION MEASUREMENTS

The power-law "survival" probabilities of avalanches and also their width scaling suggest that nontrivial structure is organized in the powder under vacancy flow. To test this interpretation we investigated the correlation between pairs of successive avalanches nucleated at (almost) the same location along the bottom layer.

Figure 10 shows the probability for the second of an avalanche pair to reach height h conditional on the first having done so, $P(2|1)$, compared to the unconditional probability for it to do so, $P(2)$. There is clear enhancement by a factor of 1.4. For cases where both avalanches did reach height h, Fig. 11 shows the normalized correlation of their lateral wandering at height h,

$$c(x_1,x_2)=\frac{\langle (x_1-x_1)(x_2-x_2)\rangle}{(x_1-x_1)_{\rm rms}(x_2-x_2)_{\rm rms}}$$

and shows correlation of order 50%.

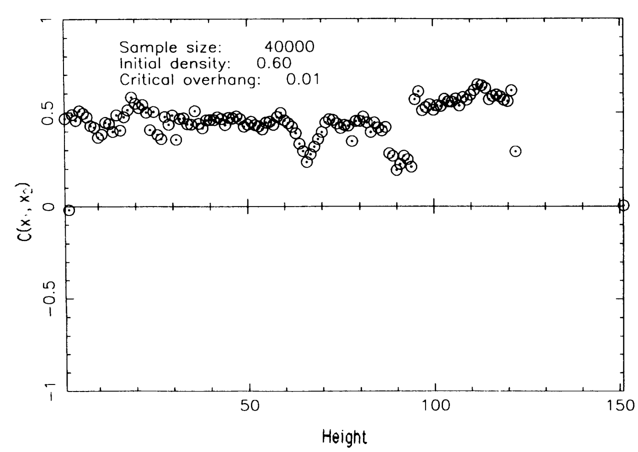

FIG. 11. The normalized correlation of the lateral wanderings of avalanche pairs at height h,

$$c(x_1,x_2) = \langle (x_1-\bar{x}_1)(x_2-\bar{x}_2)\rangle / [(x_1-\bar{x}_1)_{\rm rms}(x_2-\bar{x}_2)_{\rm rms}] \;.$$

Data as per Fig. 10.

Taken together, these results clearly show that the scaling behavior of the avalanches is dictated by the structure selected in the powder, in a manner which is to some extent persistent. The powder does not behave as a simple, uncorrelated random medium. Equivalently, one cannot take the vacancies as performing a biased random walk upward [18–20].

DISCUSSION

Our two-dimensional model shows all the hallmarks of self-organized criticality for the bulk structure of a powder consolidated by vacancy flow. In particular, the avalanches show universal, self-affine scaling of their shape and power-law distribution by size.

The power law for avalanche survival vs height means that the avalanches cannot simply be interpreted as independent vacancies diffusing upwards. It is crucial that vacancies can be trapped and also subsequently liberated, so that the avalanche propagation is nontrivially coupled to the structure of the powder built up by previous avalanche events. The strong correlation between avalanche pairs nucleated in the same place further confirms this.

We could find no evidence of significant avalanche branching, and the scaling law relating size to the product of height with width strongly indicates that they are compact rather than fractal. The scaling law relating width to flux further suggests that they can be interpreted as a bodily fall of a whole region. However, it is not the case that these regions of partial collapse spread outwards with anything like a characteristic angle (e.g., of rest), which would have required $\gamma=1$.

Our numerical results suggest that the scaling of avalanche survival, width, and size are all simply related, and it would be particularly interesting to see if this carries over to related models in three space dimensions.

Other avenues for further work must include more realistic particle geometry such as discs or spheres, in fully continuous space. This would raise an important distinction between the propagating vacancies in our present study, which must be at least one grain big and so are not easily subdivided, and those in the continuum, which could, in principle, divide indefinitely. Avalanche branching would seem more likely with divisible vacancies.

The particle-shaking studies of Barker and Mehta [16] and of Jullien, Meakin, and Pavlovitch [17] might be closely related to our work, or at least its continuum generalization discussed above. One has to interpret the shaking as an injection of vacancies, and then their results such as the upwards segregation of larger particles are an obvious consequence of vacancies having to flow around them. In this context it becomes important to establish to what extent the propagating vacancies turn out to be quantized in continuous systems.

ACKNOWLEDGMENTS

One of us (R.E.S.) wishes to acknowledge the support of the Winston Churchill Foundation. We would like to thank J. P. Bouchaud for suggesting that we examine the distribution of $w/\langle w\rangle$.

[1] T. A. Witten and L. M. Sander, Phys. Rev. Lett. **47**, 1400 (1981).
[2] M. Kolb, R. Botet, and R. Jullien, Phys. Rev. Lett. **51**, 1123 (1983).
[3] P. Meakin, Phys. Rev. Lett. **51**, 1119 (1983).
[4] R. M. Brady and R. C. Ball, Nature **309**, 225 (1984).
[5] M. Matsushita, M. Sano, Y. Hayakawa, H. Honjo, and Y. Sawada, Phys. Rev. Lett. **53**, 286 (1984).
[6] D. A. Weitz, J. S. Huang, M. Y. Lin, and J. Sung, Phys. Rev. Lett. **54**, 1416 (1985).
[7] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. Lett. **60**, 2347 (1988); Phys. Rev. A **38**, 364 (1988); J. Stat. Phys. **51**, 797 (1988).
[8] K. Chen, P. Bak, and S. P. Obukhov, Phys. Rev. A **43**, 625 (1991).
[9] G. A. Held, D. H. Solina, P. T. Keane, W. J. Haag, P. M. Horn, and G. Grinstein, Phys. Rev. Lett. **65**, 1120 (1990).
[10] J. Rosendahl and J. E. Rutledge (unpublished).
[11] H. M. Liu, H. M. Jaeger, and S. R. Nagel, Phys. Rev. A **43**, 7091 (1991).
[12] P. Evesque and J. Rajchenbach, C. R. Acad. Sci. Ser. II (Paris) **307**, 223 (1988).
[13] U. Tüzün, G. T. Houlsby, R. M. Nedderman, and S. B. Savage, Chem. Eng. Sci. **37**, 1691 (1982).
[14] R. M. Nedderman, S. T. Davies, and D. J. Horton, Powder Technol. **25**, 215 (1980).
[15] M. Levinson, B. Shmutter, and W. Resnick, Powder Technol. **16**, 29 (1977).
[16] G. C. Barker and A. Mehta, Phys. Rev. A **45**, 3435 (1992).
[17] R. Jullien, P. Meakin, and A. Pavlovitch, Phys. Rev. Lett. **69**, 640 (1992).
[18] H. S. Caram and D. C. Hong, Mod. Phys. Lett. B **6**, 761 (1992).
[19] J. Litwinyszyn, Bull. Acad. Polon. Sci. Ser. Sci. Tech. **11**, 61 (1963).
[20] W. W. Mullins, J. Appl. Phys. **43**, 665 (1972); **43**, 867 (1974).