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# Safe, explosive, and dangerous bifurcations in dissipative dynamical systems

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A comprehensive listing of the generic codimension-1 attractor bifurcations of dissipative dynamical systems is presented. It includes local and global bifurcations of regular and chaotic attractors. The bifurcations are classified according to the continuity or discontinuity of the attractor path, which governs the physical outcome that would be observed under a slow control sweep. Related issues of determinacy, hysteresis, basin structure, and intermittency are addressed. Recently discovered chaotic bifurcations are discussed in some detail, with particular reference to the regular or chaotic saddle-type destroyer with which an attractor may collide.

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## I. INTRODUCTION

Following Shilnikov [1], Zeeman [2], and others [3,4] it is useful to classify the generic codimension-1 attractor bifurcations of dissipative dynamics according to the response that would be observed as a control parameter is swept slowly through its critical value. Physical interest often centers on a loss of stability or increase in complexity, and for this reason we focus on a forward sweep which generates these.

Such a classification naturally hinges on the most fundamental property of an attractor path in control-phase space, namely, its continuous or discontinuous dependence on the control, and three categories are immediately perceived [5]. In a *safe* bifurcation, typified by the supercritical local bifurcation of Fig. 1, there is no discontinuous change in the size of the attractor, merely the continuous growth of a new stable form. In an *explosive* bifurcation, there is a discontinuous increase in the size and form of the attractor, the new enlarged attractor after the bifurcation including within itself the phase-space regime of the old attractor. In a *dangerous* bifurcation, such as a fold, or the subcritical local bifurcation of Fig. 1, the current attractor simply disappears, forcing the system to jump in a fast dynamic transient to a remote and entirely new attractor.

Bifurcations lying within one or another of these three categories tend to have other properties in common, relating to questions of determinacy, hysteresis, basin behavior, and intermittency, as shown in Table I. The bifurcations listed are generic for dynamical systems described by differential equations (flows) defined by smooth functions and having a global Poincaré section for which the Poincaré map is a diffeomorphism, including, for example, smooth periodically forced systems. One-dimensional noninvertible smooth mappings, such as the logistic map and sine circle map, are also considered; although strictly speaking they cannot occur in differential equations, their behavior is closely related to the behavior of differential systems with large dissipation.

## II. SAFE BIFURCATIONS

The safe bifurcations, with their continuous attractor paths, initiate no fast dynamic jump or instantaneous enlargement of the attractor. They are totally determinant in nature, with a continuous outcome that is insensitive to the rate of the control sweep and to the inevitable

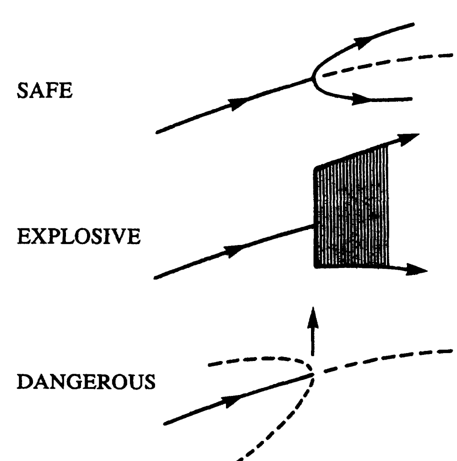


FIG. 1. Classification of bifurcations according to their outcome.

 

TABLE I. Phenomenological classification of attractor bifurcations. In this examination of the generic codimension-bifurcations of dissipative dynamics we speak of a *forward* control sweep of a parameter, $\mu$, as one that generates instability or increased complexity. Attention is focused on flows of continuous systems and their mappings generated by a Poincaré section. The minimum flow dimension necessary to observe a bifurcation in a Euclidean space is denoted by $D$, and descriptions assume that we are in this dimension. So a fold will be described on its center manifold, as a repeller-attractor transition; embedding in a higher-dimensional phase space, with attraction onto this manifold, would give the more familiar terminology of a saddle node. Notice that although a flip can occur on a 2D Möbius strip, our Euclidean restriction necessitates $D=3$. The numbers listed after "Pictures" refer to references (square brackets) and figure numbers (parentheses).

(a) SAFE BIFURCATIONS

*Subtle* (i.e., continuous) bifurcations with the continuous supercritical growth of a new attractor path
*Safe* with no fast dynamic jump or instantaneous enlargement of the attracting set
*Determinate* with a single outcome even under small noise excitation
*No hysteresis* with attractor paths retraced on reversal of the control sweep
*No basin change*, with basin boundary remote from the bifurcating attractors
*No intermittency* in the steady-state responses of the attractors

(a1) Local supercritical bifurcations

Supercritical Hopf ($D=2$): point to cycle
A spiral point attractor becomes a spiral repeller as a complex conjugate pair of flow eigenvalues, $\lambda=\alpha\pm i\beta$, leaves the left-hand stable half-space. A stable supercritical periodic attractor expands parabolically around the primary path of point repellers. Examples are $\dot{x}=-y+x[\mu-(x^2+y^2)]$, $\dot{y}=x+y[\mu-(x^2+y^2)]$; $\dot{r}=r(\mu-r^2)$, $\dot{\theta}=1$.
*Precursor*: Local transients have the form $e^{\alpha t}\sin(\beta t)$ with the negative $\alpha$ increasing linearly through zero at $C$.
*Other names*: In aeroelasticity, galloping or flutter; the mapping equivalent is the Neimark bifurcation.
*Applications:* The galloping and flutter of elastic solids in a fluid flow, chemical oscillations.
*Pictures:* [7] (7.7.2), [5] (first excitation, 17.1.8), [8] (3.4.4), [25] (7.5,7.6).

Supercritical Neimark ($D=3$): cycle to torus
A spirally attracting cycle becomes repelling as a complex conjugate pair of mapping eigenvalues, $\Lambda=\alpha\pm i\beta=\rho e^{\pm i\phi}$, leaves the stable unit disk. A stable supercritical toroidal attractor grows parabolically around the primary path of repellers. Special resonances occur when the eigenvalues satisfy $\Lambda^3=1$ or $\Lambda^4=1$.
*Precursor:* Local transients in a suitable polar map are $r_i=\rho^i r_0$, $\theta_i=\theta_0+i\phi$ with $\rho$ increasing linearly through $+1$.
*Other names:* Supercritical secondary Hopf bifurcation; the flow equivalent is the Hopf bifurcation.
*Applications*: Taylor-Couette flow, internal autoparametric resonance in coupled driven oscillators.
*Pictures*: [7] (7.7.6), [5] (second excitation, 17.2.6), [25] (8.18).

Supercritical flip ($D=3$): cycle to cycle
An inversely attracting nodal cycle becomes an inverting saddle as a real mapping eigenvalue $\Lambda$ leaves the stable unit disk at $-1$. A stable supercritical periodic attractor, with twice the period of the fundamental cycle, grows parabolically away from the saddles of the primary path. Example in a 1D map is $x_{i+1}=-(1+\mu)x_i+x_i^3$.
*Precursor:* Local mapping transients separate as $\Lambda^i$ with $\Lambda$ decreasing linearly through $-1$ at the bifurcation.
*Other names*: Supercritical period-doubling bifurcation, subharmonic resonance; the flip has no flow equivalent.
*Applications:* Subharmonic resonances in driven oscillators, and is a building brick of the Feigenbaum cascade.
*Pictures:* [7] (7.7.4), [5] (octave jump, 17.3.7, 17.4.5), [8] (3.5.1), [25] (7.10, 8.6, 8.7, 9.4).

(a2) Global bifurcations

Band merging ($D=3$): chaos to chaos
A chaotic attractor with noisy $2^n$ periodicity becomes a chaotic attractor with noisy $2^{n-1}$ periodicity on absorbing a period $2^{n-1}$ inverting saddle. These bifurcations, discussed by Lorenz, form the noisy reverse cascade in the logistic map which follows the more familiar Feigenbaum cascade.
*Precursor:* The separation between adjacent chaotic bands decreases in a locally linear fashion.
*Other names:* Lorenz reverse cascade.
*Applications:* An ingredient of the universal period-doubling route to chaos, encountered in many fields.
*Pictures*: [5] (chaotic octave jump, 22.2.2), [25] (logistic map, 9.8).

(b) EXPLOSIVE BIFURCATIONS

*Catastrophic* (i.e., discontinuous) global bifurcations with an abrupt enlargement of the attracting set
*Explosive* enlargement, but no jump to remote disconnected attractor
*Determinate* with a single outcome even under small noise excitation
*No hysteresis* with attractor paths retraced on reversal of control sweep
*No basin change*, with basin boundary remote from the bifurcating attractors
*Intermittency:* supercritical lingering in old domain, flashes through the new domain

Flow explosion ($D=2$): point to cycle
A path of equilibrium fixed points exhibits locally a regular saddle-node fold; meanwhile the global dynamics

TABLE I. (*Continued.*)

are such that the saddle outset flows around a closed loop to the node; when the saddle and node on this loop annihilate one another at the fold, a stable limit cycle is created; initial period is infinite, due to critical slowing.
*Precursor:* The local subcritical behavior is identical to that of the static fold.
*Other names*: Omega explosion; the mapping equivalent is the map explosion.
*Applications:* Chemical oscillations, low-temperature semiconductors.
*Pictures*: [5] (blue loop in 2D, 21.1.6), [8] (3.4.5), [25] (13.3).

Map explosion ($D=3$): cycle to torus
This mapping equivalent of the flow explosion has a saddle-node annihilation of two cycles; these lie on a drift ring in the Poincaré section, so that a toroidal attractor is created after the collision; critical slowing is observed in the toroidal flow as the mapping point passes through the region of the recent cyclic fold.
*Precursor:* The local subcritical behavior is identical to that of the cyclic fold.
*Other names*: Omega explosion, mode locking, mode unlocking; the flow equivalent is the flow explosion.
*Applications:* Arises in the mode locking and unlocking of oscillators with two intrinsic frequencies.
*Pictures:* [5] (blue loop in 3D, 21.2.5), [25] (13.2, 13.3).

Intermittency explosion: flow ($D=4$): point to chaos
According to a geometrical model that has been proposed in 4D, a subcritical Hopf bifurcation can give, not a *dangerous* jump to a remote attractor, but an *explosive* enlargement of the attracting set from a point to a chaotic attractor containing the unstable focus. It is not clear whether this is possible in 3D.
*Precursor:* The subcritical transient behavior is identical to that of the Hopf bifurcation.
*Other names:* None.
*Applications*: None.
*Pictures*: [5] (no illustration).

Intermittency explosion: map ($D=3$): cycle to chaos
As in the map explosion, a cyclic fold can give, not a *dangerous* jump, but an *explosive* enlargement of the attracting set; when this is to a chaotic attractor (as in the opening of a periodic window), critical slowing gives an irregular intermittency in the chaotic motion. Subcritical Neimark and flip bifurcations can replace the fold.
*Precursor:* The subcritical transient behavior is identical to that of the constituent local bifurcation.
*Other names*: Temporal intermittency, Pomeau-Manneville (types I, II, III) intermittency.
*Applications:* Arises as the opening of a periodic window in the ubiquitous period-doubling route to chaos.
*Pictures:* [5] (Zeeman's blue tangle, 21.3.4), [25] (13.7, 13.8; opening of window in logistic map, 9.7, 9.9).

Regular-saddle explosion ($D=3$): chaos to chaos
A chaotic attractor has an abrupt enlargement to a larger chaotic attractor, which includes the smaller as a subset, on colliding with a regular saddle cycle. The larger attractor has a global annular structure, as in the bagel or folded torus. The regular-saddle and chaotic-saddle explosion pair has an analogy in the catastrophes.
*Precursor:* Subcritical lingering near the impinging saddle, significant when the saddle is only weakly repelling.
*Other names:* (regular) interior crisis [3].
*Applications:* Explosion from band attractors to bagel (folded torus) in the velocity-forced Van der Pol equation.
*Pictures*: [5] (no illustration), [25] (13.9).

Chaotic-saddle explosion ($D=3$): chaos to chaos
A chaotic attractor undergoes an abrupt instantaneous enlargement to a larger chaotic attractor, which includes the original attractor as a subset, on colliding with a chaotic saddle [21,3,22]. A simple example of this is seen in the closing of a periodic window of the logistic map.
*Precursor:* Subcritical lingering near the impinging saddle, significant when the saddle is only weakly repelling.
*Other names*: (chaotic) interior crisis [3].
*Applications:* Escape from chaotic one-well to chaotic cross-well motions in the twin-well Duffing oscillator.
*Pictures:* [5] (Ueda's chaotic explosion, 21.4.9), [25] (closing of periodic window in logistic map, 9.9).

(c) DANGEROUS BIFURCATIONS

*Catastrophic* (i.e., discontinuous) bifurcations with the blue-sky disappearance of the attractor
*Dangerous* with a sudden fast dynamic jump to a distant unrelated attractor of any type
*Determinate* or *Indeterminate* in outcome, depending on the global topology
*Hysteresis* with original attractor not reinstated on reversal of the control sweep
*Basin* shrinks to zero (c2) or attractor hits boundary of a residual basin (c1) and (c3)
*No intermittency*, but note the critical slowing in the global bifurcations

(c1) Local saddle-node bifurcations

Static fold ($D=1$): from point
A path of point attractors folds back parabolically as a path of repellers on reaching an extreme value of the control. Moving around the path a real flow eigenvalue $\lambda$ leaves the left-hand stable half-space at the node-repeller transition. Example: $\dot{x}=\mu+x^2$. Related pitchfork (cusp) and transcritical bifurcations are not directly generic.

TABLE I. (*Continued.*)

*Precursor:* Local transients have the form $e^{\lambda t}$, with the real negative $\lambda$ increasing linearly through zero at $C$.
*Other names*: Saddle-node bifurcation; in elasticity, limit point; mapping equivalent is below.
*Applications:* Snap-buckling of shallow elastic structures, including arches and shells; runaway of reactors.
*Indeterminacy:* In $D=2$ the 1D flow outset needs a nongeneric coincident saddle connection for indeterminacy.
*Pictures:* [7] (7.7.1), [5] (static fold, 18.1.10, 18.2.6), [8] (3.4.1), [25] (7.4).

Cyclic fold ($D=2$): from cycle
A path of periodic attractors folds back parabolically as a path of periodic repellers on reaching an extreme value of the control. Moving around the path a real mapping eigenvalue, $\Lambda$, leaves the stable unit disk at $+1$ as the stable cycle becomes unstable at the extremum. Example in a one-dimensional map is $x_{i+1}=x_i+\mu+x_i^2$.
*Precursor:* Local mapping transients separate as $\Lambda^i$ with $\Lambda$ increasing linearly through $+1$ at the extremum.
*Other names*: Saddle-node bifurcation, dynamic fold, periodic fold; flow equivalent is previous entry.
*Applications:* Jumps to and from resonance in driven oscillators under direct and parametric excitation.
*Indeterminacy:* Smooth mosquito-coil [31,32], or fractal basin [33] can accumulate onto saddle remote outset.
*Pictures*: [7] (7.7.3), [5] (periodic fold, 18.3.10, 18.4.9), [25] (7.9, 8.3).

(c2) Local subcritical bifurcations

Subcritical Hopf ($D=2$): from point
A spiral point attractor becomes a spiral repeller as complex conjugate flow eigenvalues, $\lambda=\alpha\pm i\beta$, leave the left-hand stable half-space. An unstable subcritical periodic repeller shrinks parabolically around the attractor, pinching its basin of attraction to zero. Example: $\dot{x}=-y+x[\mu+(x^2+y^2)]$, $\dot{y}=x+y[\mu+(x^2+y^2)]$; $\dot{r}=r(\mu+r^2)$, $\dot{\theta}=1$.
*Precursor:* Local transients have the form $e^{\alpha t}\sin(\beta t)$ with the negative $\alpha$ increasing linearly through zero at $C$.
*Other names*: In aeroelasticity, galloping, or flutter; the mapping equivalent is the Neimark bifurcation.
*Applications:* The galloping and flutter of elastic solids in a fluid flow, onset of turbulence.
*Indeterminacy:* An example is the aeroelastic galloping in an asymmetric well [32].
*Pictures:* [5] (spiral pinch, 19.1.4), [25] (reversed time; 7.5, 7.6).

Subcritical Neimark ($D=3$): from cycle
A spirally attracting cycle becomes repelling as a complex conjugate pair of mapping eigenvalues, $\Lambda=\alpha+i\beta=\rho e^{\pm i\phi}$, leaves the stable unit disk. An unstable subcritical toroidal repeller shrinks parabolically around the attractor, pinching its basin to zero. Special resonances occur when $\Lambda^3=1$ or $\Lambda^4=1$.
*Precursor:* Local transients in a suitable polar map are $r_i=\rho^i r_0$, $\theta_i=\theta_0+i\phi$ with $\rho$ increasing linearly through $+1$.
*Other names*: Subcritical secondary Hopf bifurcation; the flow equivalent is the Hopf bifurcation.
*Applications:* Electrical circuits.
*Indeterminacy:* Generically possible, but no specific example known to the authors.
*Pictures:* [5] (vortical pinch, 19.2.6), [25] (reversed time; 8.18).

Subcritical flip ($D=3$): from cycle
An inversely attracting nodal cycle becomes an inverting saddle as a real mapping eigenvalue $\Lambda$ leaves the stable unit disk at $-1$. An unstable subcritical periodic saddle, with twice the period of the fundamental cycle, shrinks parabolically onto the attractor, pinching its basin of attraction to zero. Example is $x_{i+1}=-(1+\mu)x_i-x_i^3$.
*Precursor*: Local mapping transients separate as $\Lambda^i$ with $\Lambda$ decreasing linearly through $-1$ at the bifurcation.
*Other names*: Subcritical period-doubling bifurcation; the flip has no flow equivalent.
*Applications*: Rayleigh-Bénard convection, subharmonic resonance.
*Indeterminacy*: Generically possible, but no specific example known to the authors.
*Pictures*: [7] (7.7.5), [5] (octave pinch, 19.3.6, 19.4.6), [25] (8.8).

(c3) Global bifurcations

Saddle connection ($D=2$): from cycle
A stable cycle expands towards a saddle fixed point whose inset forms the boundary of its basin of attraction; the period of the cycle goes to infinity as it touches the saddle in a homoclinic saddle connection; at the connection the cycle is its own basin boundary, and beyond it the cycle and basin vanish into the blue.
*Precursor:* The period of the cyclic attractor tends to infinity due to slow dynamics near the approaching saddle.
*Other names*: Homoclinic connection, separatrix loop; no mapping equivalent because connection becomes a tangle.
*Applications:* Chemical oscillations.
*Indeterminacy:* In $D=2$ the 1D flow outset needs a nongeneric coincident saddle connection for indeterminacy.
*Pictures:* [5] (periodic blue sky, 20.2.7), [8](4.4.2), [25] (13.4, 13.5, cover).

Regular-saddle catastrophe ($D=3$): from chaos
A chaotic attractor expands to hit a saddle cycle whose smooth, untangled inset forms its basin boundary; at the collision the saddle simultaneously becomes homoclinic and the chaotic attractor and its residual basin vanish into the blue. A simple example is seen at the end of the logistic map.
*Precursor*: subcritical lingering near the impinging saddle, significant when the saddle is only weakly repelling.
*Other names*: (regular) boundary crisis [3], chaotic blue sky catastrophe.

TABLE I. (*Continued.*)

*Applications:* Blue sky instability of Birkhoff-Shaw folded torus in asymmetrically forced Van der Pol equation.
*Indeterminacy:* Generically possible with a smooth mosquito-coil accumulation, but no example known.
*Pictures:* [5] (chaotic blue sky, 20.3.11), [25] (13.11; end of logistic map, 9.8).

Chaotic-saddle catastrophe ($D=3$): from chaos
A chaotic attractor expands to hit the accessible saddle orbit within a fractal basin boundary, remaining however at a distance from the main saddle cycle whose prior homoclinic tangling generated the fractal boundary. At the collision the chaotic attractor and its residual basin vanish into the blue [36].
*Precursor:* Subcritical lingering near the impinging saddle, significant when the saddle is only weakly repelling.
*Other names*: (chaotic) boundary crisis [3], chaotic blue sky catastrophe.
*Applications:* Escape from a cubic potential well where bounded chaotic motions suddenly jump out of the well.
*Indeterminacy:* Determinate and indeterminate (fractal accumulation) examples in the twin-well Duffing equation [31].
*Pictures:* [5] (Roessler's blue sky, 20.4.8), [25] (13.12).

noise present in a computer or laboratory experiment. They exhibit no hysteresis, with the attractor path precisely retraced on a reversal of the control sweep. The bifurcating attractor lies totally within a basin whose remote boundary plays no part in the event, and remains topologically unchanged. There is no observed intermittency or critical slowing in the response of the attractors before or after the bifurcation.

Attractor continuity means that the phase-space regime immediately after bifurcation can be kept arbitrarily close to the old regime by holding the control sufficiently near the critical value. Thus any undesirable consequences of bifurcation can be avoided by sweeping very slowly and reversing the sweep as soon as the bifurcation is observed; it is therefore natural to follow Shilnikov [1] and term such bifurcations safe. In [1], Shilnikov defines safe with reference to a typical orbit just after bifurcation remaining in a neighborhood of the old attractor; Zeeman [2] introduced the notion of continuity of a function, mapping each control value to the corresponding attractor. The two approaches are equivalent for generic bifurcations [6].

The safe category includes the well-known supercritical forms of the local Hopf, Neimark, and flip bifurcations [7,8]. These local bifurcations are often amenable to closed-form analysis of the underlying differential equation models. Powerful analytical techniques, such as center manifold theory (also known as elimination of passive coordinates) and reduction to normal form, can be brought to bear. A useful introduction to these methods, and further references, can be found in a recent review article [9].

A fourth type of safe bifurcation is the band merging most familiar in the bifurcations of the one-dimensional logistic or quadratic map, where the Feigenbaum cascade of period doublings is followed by a reverse cascade in which at each stage a chaotic attractor consisting of $2^n$ bands or intervals grows and consolidates to an attractor of $2^{n-1}$ bands [10]. As merging occurs, an unstable $2^{n-1}$ periodic orbit with negative multiplier is absorbed into the attractor. The fundamental measure of complexity of a chaotic attractor is the ensemble of unstable periodic orbits within an attractor, so it is appropriate to consider a control sweep away from the Feigenbaum point to be a forward sweep. Equivalently, this may be understood in terms of syncope [11]: the strict rhythm with which the $2^n$ bands are visited before merging becomes syncopated after merging, and only a $2^{n-1}$ rhythm is strictly followed. A scaling law for the broadening of $2^{n-1}$ spectral peaks has been proposed for band merging [12–14]. The final merging in the reverse cascade, from the two-band to one-band attractor, is sometimes called Ruelle's point [15].

## III. EXPLOSIVE BIFURCATIONS

The explosive bifurcations violate the continuity of the attractor path by causing the attractor to suddenly enlarge. The new attractor includes the old attractor as a proper subset, so there is no jump to a remote attractor. The new attractor is determinant, with a single outcome independent of the rate of control sweep and insensitive to the presence of small noise. Upon reversal of the control sweep, the attractor implodes to the old attractor at the same critical control value, with no hysteresis under infinitely slow control sweep. (When sweeping at a finite rate, apparent hysteresis may be observed, but over a control range that can be reduced to zero by slowing the rate of sweep.) There is no associated change in basin structure as the newly enlarged attractor is still remote from its basin boundary. Just after an explosion, the new region of the attractor is visited infrequently, and the system spends long periods in or very near the old phase-space regime, resulting in temporal intermittency [16,17].

Explosive bifurcations of regular attractors are triggered by local subcritical and fold bifurcations occurring within a global structure that brings an orbit recurrently back to the neighborhood of the old regular attractor. The most familiar examples are mode unlocking, where a cyclic fold occurring within an invariant torus (in Poincaré section, a closed curve), and the analogous static fold occurring within an invariant circle. If an orbit on the new enlarged attractor recurs via a homoclinic tangle or horseshoelike global structure, the bifurcation is a route to chaos by temporal intermittency. The routes from cycle to chaos were classified by Pomeau and Manneville [16] according to the form of the initiating local subcritical bifurcation, and Pomeau [18] proposed a geometric model for explosion from an equilibrium point attractor to chaos via a subcritical Hopf bifurcation. A common example of the cycle to chaos route is the opening of a periodic window in the logistic or quadratic map; this explosion is generated by decreasing the usual parameter, which is a forward sweep considering the complexity of the attractor only. Intermittency explosions have been observed in several diverse experiments as, for example, in [19].

Intermittency explosions are subject to a scaling law for the proportion of time spent near the old regular attractor; the dependence on the control comes in part from the generic form of the initiating local bifurcation [16]. In experiments, this scaling may be governed by subtleties not included in the most naive analysis [19].

Recently, a new form of intermittency has been reported in which hysteresis was observed [20]. The mechanism involves the coincidence of the initiating local bifurcation together with an additional and independent constraint on the reinjection produced by the global recurrent structure. In the perspective of the present classification, the two independent constraints require coordination of two different controls, and hence this phenomenon is not generic when sweeping a single control. The robustness of the phenomenon observed in [20] is due to the fact that, in the particular systems studied there, the two independent codimension-1 bifurcation arcs are not strongly transverse, but happen to meet in a very acute angle, and so remain close together over an extended region in the space of two controls. This well illustrates the caution necessary in interpreting Table I.

Chaotic attractors can also explode in size. Such a bifurcation was first documented in a differential equation in [21] and was independently recognized in [3] and called an *interior crisis.* A common example is the closing of a periodic window in the logistic map.

In all types of explosive bifurcation, the old, smaller attractor comes near and touches an unstable equilibrium or unstable periodic motion. In other words, just as the stable behavior is associated with a well-defined structure in phase space, the attractor, so too is the instability associated with a well-defined geometric structure in phase space. It is useful to have a name for this impinging unstable structure, and we suggest the term *destroyer.* In one-dimensional maps, the destroyer is a repeller, but in higher dimensions it is usually a saddle with one-dimensional outset or unstable manifold. In the flow and map explosions, the outset branch facing away from the old attractor is not tangled, and the destroyer is just a saddle point or periodic orbit, an isolated component of the nonwandering set that remains remote from all other nonwandering points until the bifurcation threshold is reached. That is, the destroyer is a *regular saddle.* In intermittency explosions to chaos, the destroyer outset branch away from the old attractor is tangled, and the destroyer is part of a complicated component of the nonwandering set with horseshoelike structure, that is, a *chaotic saddle.*

A similar distinction can be drawn within the category of chaotic explosions or interior crises. Examples of chaotic explosion by collision with a chaotic saddle include [21], which shows clearly the tangled structure of the outset branch facing away from the old attractor, and [22], in which the nonwandering set itself just prior to explosion is illustrated. The most familiar example of a chaotic-saddle explosion is the closing of a periodic window in the logistic map; the gaps filled in by such an explosion were filled with chaotic transients prior to the explosion. Another simple case is the escape from chaotic one-well to cross-well motions in the twin-well Duffing equation [23,24]; here a simply folded band attractor containing just one inversely unstable harmonic explodes and absorbs the simple band in the adjacent well. On the other hand, chaotic explosion by collision with a regular saddle can also occur given the appropriate global structure. Examples include Fig. 13.9 in [25], and [26], and also the escape from chaotic librational motions of a periodically forced rigid-arm pendulum to rotational chaotic motions.

This distinction among chaotic explosions, based on the regular or chaotic structure of the impinging saddle-type destroyer, is rather more subtle than the other categories in Table I and would require some ingenuity to be detected by experimental observations. It is therefore appropriate to ask whether or not this distinction carries with it any predictive power. Figure 2 suggests a possible answer to this question; shown are prototype schematic phase diagrams of four topologically distinct types of attractor explosion. In each case, the phase diagram corresponds to the condition of incipient explosion, with a small attractor prior to explosion, and a nearby saddle-

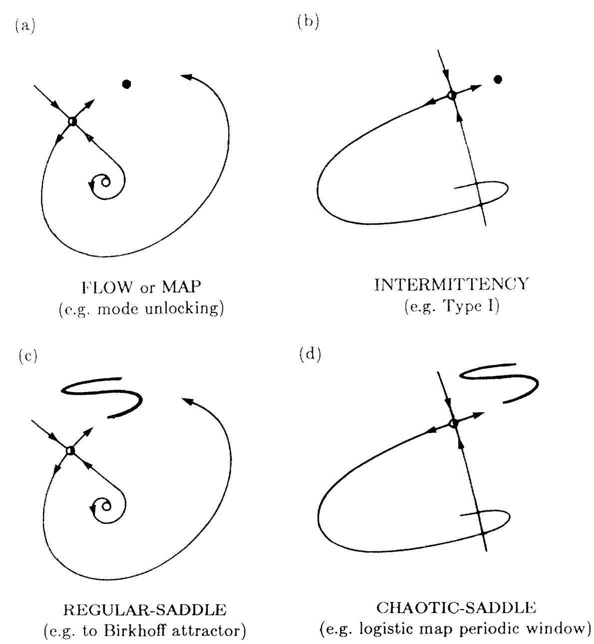


FIG. 2. Four types of attractor explosion, arranged by rows according to the regular or chaotic structure of the small attractor prior to explosion, and by columns according to the type of impinging saddle-type destroyer: (a) flow or map explosion, from point attractor to attracting invariant closed curve; (b) intermittency type I from regular to chaotic attractor; (c) regular-saddle explosion or interior crisis from small chaotic attractor to enlarged chaotic attractor with a folded torus structure; (d) chaotic-saddle explosion or interior crisis from small to enlarged chaotic attractor.

type destroyer, whose collision with the small attractor will trigger explosion. In the upper row of Fig. 2, the small attractor is indicated by a solid (filled) point, representing a stable equilibrium, or a stable periodic orbit in the Poincaré section.

Just as Fig. 2 is arranged so that the two rows contain different types of small attractors (regular or chaotic), so in a similar way has Fig. 2 different types of impinging saddles in the two columns: a regular saddle (drawn as half-hollow dots) in the left column and chaotic saddles in the right column. The chaotic structure of the saddles in the right column is indicated schematically by showing enough of the outset branch facing away from the attractor to identify a transverse homoclinic intersection with a branch of the inset.

The upper left diagram in Fig. 2(a) describes an incipient omega explosion, in either its flow or map form. The hollow dot represents a focal repelling point which will be encircled by the exploded attractor. The upper right diagram, Fig. 2(b), represents a type-I intermittency explosion, from the periodic attractor in the Poincaré section to the chaotic attractor. In the lower left diagram, Fig. 2(c), an explosion from small to enlarged chaotic attractor is triggered by a regular saddle. In the known examples cited above, the resulting enlarged attractor has the global structure of a folded torus, sometimes called a Birkhoff attractor. For this reason, Fig. 2(c) shows the saddle outset branch away from the small attractor, making a circuit around a focus repelling point. This repeller need not be present in systems such as the forced pendulum, where the folded torus attractor structure can arise by virtue of the underlying cylindrical structure of the phase space. Finally, Fig. 2(d) represents a prototype explosion from small to large chaotic attractor involving a chaotic saddle.

It seems likely that a chaotic-saddle explosion could also produce an exploded attractor with folded torus structure, although there are at present no known examples. So Fig. 2(d) is not meant to exclude such a possibility, but Fig. 2(c) is meant to suggest that in the case of regular saddle explosions the folded torus structure of the result is not optional but necessary.

It seems plausible that chaotic explosions involving a regular saddle can only happen if the explosion produces a global annular structure in the Poincaré section of the new attractor. If a chaotic attractor collides with a regular saddle, all orbits will be drained away across the destroyer; without a global annular return circuit, the former attractor could not be part of the new steady-state response. That is, the event would not be an explosion but a dangerous bifurcation. Thus, of the four diagrams in Fig. 2, the two in the left column have in common a necessarily annular structure in the exploded attractor.

A recently published conjecture holds that a folded torus attractor will have among its periodic orbits of least period an equal number of directly and inversely unstable orbits, as a consequence of the global constraint on Poincaré indices in an annular or cylindrical absorbing region having an Euler characteristic equal to zero [27]. This suggests the following.

*Conjecture.* When a generic explosion of a chaotic attractor of a two dimensional smooth diffeomorphism is initiated by collision with a regular saddle, the increase $\Delta D$ in the number of directly unstable periodic orbits of least period within the attractor is at least one greater than the increase $\Delta I$ in the number of inversely unstable periodic orbits of the same period; that is, $\Delta D > \Delta I$, and the large attractor includes a folded annular structure.

For example, if a periodically forced oscillator that is known to be uniformly dissipative has a regular saddle explosion, it can be inferred from the above-conjectured attractor topology that the displacement coordinate is an angular variable. For chaotic explosions generally, the mean time between intermittent bursts just after explosion is governed by a scaling law [28], as has been confirmed in experiments; e.g., [29].

## IV. DANGEROUS BIFURCATIONS

The dangerous bifurcations are characterized by the blue sky disappearance of the current attractor, giving rise to a jump to a remote attractor of any type. The term *hard transition* is also used to describe these bifurcations. On reversal of the control sweep, the response will typically remain on the path of the new attractor, giving rise to hysteresis. Since the different attractors are generically remote and separated from each other by a phase-space distance that remains bounded away from zero, the loss of stability of one attractor will not be correlated with the loss of stability of another attractor and will occur at a different value of the control.

Two different basin scenarios can be observed within the dangerous category. In one, typified by the subcritical forms of the Hopf, Neimark, and flip bifurcations, the basin shrinks around the attractor and pinches it off at the critical control value; the size of the basin drops continuously to zero as the bifurcation is approached. In the second scenario, typified by the saddle-node folds, the attractor moves towards the edge of its basin, colliding with a saddle in the basin boundary at the point of bifurcation; there is a residual basin at the critical value of the parameter [30]. In the cyclic fold the saddle will be a cycle, onto which can be accumulated thin fingers of one or more basins of attraction in either a mosquito-coil [31,32] or a fractal [33] structure. There is then an indeterminacy in the outcome of the dynamic jump, because the basin into which the system moves depends sensitively on the precise manner in which the bifurcation is realized and on the inevitable noise in an experimental realization.

Such indeterminism is a generic possibility in all but the simplest of the dangerous bifurcations. It is nongeneric for the static fold and saddle connection in planar flows, where the outset of the saddle fixed point is only one dimensional. In these cases a coincidental additional saddle connection would be required to observe indeterminacy under small noise excitation. In higher-dimensional flows, the one-dimensional outset of the destroyer might lead away to indeterminate outcomes if it were involved in an additional Shilnikov saddle-focus connection. According to Shilnikov [34], even though the saddle-focus connection is structurally unstable and nongeneric, the implied horseshoes are robust, so the phenomenon should be treated as generic. The existence of indeterminate static fold and saddle connection bifurcations is, however, only conjectural, and no examples are known at present.

The dangerous bifurcations break down naturally into three subcategories. The first, (c.1), contains the two well-known saddle nodes, of fixed point and cycles, respectively. The second, (c.2), contains the three subcritical local bifurcations. The third, (c.3), contains three global bifurcations, starting with the saddle connection in which a limit cycle slows down and disappears as it collides with a saddle fixed point in its basin boundary. The final two events, involving the loss of stability of a chaotic attractor, have been elucidated relatively recently and warrant a more detailed discussion.

In both of these bifurcations, a chaotic attractor is eradicated from the phase portrait. This means that the attractor necessarily collides with at least part of its basin boundary; hence the term *boundary crisis* [3]. Following the emphasis by Zeeman [2] and Abraham [4] on the fundamental role of discontinuity of the control-to-phase-portrait function, the terms *catastrophe* or *blue sky catastrophe* have also been applied.

Recently it has been noted that the basin boundary near the points of incipient contact with the chaotic attractor may have a fractal structure [35,36]. The points of contact are an unstable fixed point or periodic orbit lying in the basin boundary called an accessible orbit [35], which is another example of a destroyer. Fractal structure in the basin boundary occurs if the destroyer outset branch facing away from the attractor is homoclinic; thus the regular-fractal basin boundary dichotomy for blue sky catastrophes is completely analogous to the distinction between regular-saddle and chaotic-saddle explosions.

There is another perspective on this dichotomy. Often, as, for example, in the twin-well Duffing oscillator [36] and the Hénon map [35], the basin boundary contains a periodic point that is not a subharmonic, but has the same fundamental period as the chaotic attractor (i.e., the unstable periodic orbit of least period within the attractor). This fundamental periodic point in the basin boundary may remain accessible and touch the chaotic attractor itself, in which case a regular-saddle catastrophe is observed; or it may form a tangle, creating accessible orbits that are subharmonics, and remain remote from the attractor at a chaotic-saddle catastrophe. For example, upon increasing the parameter $a$ in the Hénon map, regular-saddle catastrophes are observed for $b$ between zero and about $-0.08$, with a period-1 accessible orbit acting as destroyer; for more negative values of $b$, chaotic-saddle catastrophes are observed, with destroyers of period 3 or $3\times 2^n$.

## V. CONCLUSION

Table I has presented a comprehensive list of generic codimension-1 attractor bifurcations of dissipative dynamical systems, including local and global bifurcations of regular and chaotic attractors. This list is proposed as a classification scheme based on topological principles that have physical correlates of foremost concern; it is of course far from a complete classification of all possible topological configurations, particularly as regards global structure.

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