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Stochastic electronic motion and high-efficiency free-electron lasers

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We consider the asymptotic behavior of high-power Compton free-electron laser oscillators. We observe that the electronic motion along the wiggler presents intrinsic stochasticity. We show that the high efficiency in the broad spectrum regime is due to chaotic diffusion of the electrons toward lower energies, rather than to standard synchrotron motion. We obtain simple estimates of the properties of the system at saturation, which agree with complete numerical simulations. Namely, both efficiency and spectral width are predicted to behave like the square roots of the electronic density, of the cavity quality factor, and of the wiggler length, in such a way that their ratio is a universal brightness of the order of $\sqrt{3}/2$.

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High-power free-electron lasers (FEL's) are typical of a wide class of dynamical systems where particles are coupled to waves, comparisons and cross validations of which should be profitable. For instance, they are formally described in the same way as a "cold-beam"–plasma instability. Because of the simplicity of its ingredients, the FEL model offers a favorable context for the specific study of such phenomena. The linear regime is by now well understood [1], but an analytical description in the nonperturbative domain up to saturation is still to be built. Our aim is to obtain the most simple global description of the dynamics in the system at saturation, in order to obtain a qualitative understanding and quantitative estimates of the efficiency and spectral width as simple functions of the physical and operating parameters.

In an FEL oscillator, an electron beam of linear energy $E_e=\rho_e\gamma_0 mc^2$ travels through a wiggler (ρ_e is the electron linear density), where it experiences a periodic static magnetic field of vector potential:

$$A_w(z)=\frac{mc}{e}a_w\cos(k_w z) .$$

Since radial effects are of little importance at saturation for FEL oscillators, we will consider that the radial profiles T and S of the electron and photon beams are frozen, so that the only sequel of the radial dimensions is a filling factor $\langle ST\rangle=\int 2\pi r\,dr\,S(r)T(r)$. However, complete simulations, including dynamical variations of the radial profiles, have been performed, without changing the conclusions. Under the action of A_w, the electrons radiate a field A_L. We introduce a reference wave number k_L, which is fixed later by Eq. (2), and discretize $k_n=(1+nh)k_L$, so that the radiated field and the linear laser energy are

$$A_L(z,t)=\frac{mc}{e}\,\mathrm{Re}\left[\sum_n \frac{1}{k_n}\mathcal{E}_n(z)e^{ik_n(z-ct)}\right] ,$$

$$E_L=\sum_n hE_n ,$$

where $E_n\equiv(mc/e)^2|\mathcal{E}_n|^2/2\mu_0 h$ is the energy spectral density. Note that h is small and will eventually go to zero: it does not describe the structure of the spectrum, which is determined by the field components $\mathcal{E}_n$ only. The averaged dynamical equations for the electrons are

$$d_z\psi=k_w-\frac{k_L}{2\gamma^2}(1+a_w^2/2) , \tag{1a}$$

$$2\gamma d_z\gamma=\sum_n a_w\mathcal{K}_1|\mathcal{E}_n|\langle ST\rangle\sin[(1+nh)\psi-nhk_w z-\phi_n] , \tag{1b}$$

where

$$\mathcal{E}_n=|\mathcal{E}_n|e^{-i\phi_n},\quad \psi=(k_w+k_L)z-ck_Lt ,$$

$$\mathcal{K}_1=J_0(\xi)-J_1(\xi),\quad \xi=a_w^2/(4+2a_w^2) .$$

The retroaction of the electrons on the electromagnetic field is given by the Maxwell equations [2]. This model is basically equivalent to the Colson-Freedman model with periodic boundary conditions [3]. Although the complete self-consistent dynamics is very intricate, we will show that a very crude description of the laser field is sufficient to reach a first analytical description of the average characteristics of the system at saturation.

The linear analysis shows that an instability develops near the resonant wave number k_L:

$$k_L=2\gamma_0^2k_w(1+a_w^2/2)^{-1} . \tag{2}$$

When this mode reaches saturation, and if the coupling is strong enough, a sideband instability with a wave number slightly lower than k_L develops [4]. Nonlinear-mode couplings may then generate new laser modes [2,5], until complete saturation, where both the laser energy and spectral width remain essentially constant from one round trip to the other. During each pass, a fraction $-\Delta E_e=\Delta E_L$ of the electron energy is transferred to the electromagnetic field, and a fraction $1/Q$ of the laser energy is extracted as the output of the system. The extracted efficiency is then $\rho=-\Delta E_e/E_e=E_L/QE_e$.

Extensive simulations have been used to investigate the FEL nonlinear behavior. This is performed by solving self-consistently Eqs. (1), together with a multifrequency laser field, from noise up to saturation. To get relevant results, a large number of electrons ($\geq 10\,000$) and laser modes are required. Compared to the sideband frequency shift (say, for example, 3%), the computed spectral domain is large ($\simeq 20\%$) and the discretization step small enough (0.3%). The numerical results provided in the following are consistent with and complementary to those given in [2].

It has been observed in these systematic numerical simulations that the extracted efficiency and the relative spectral width $\Sigma=(\langle k^2\rangle-\langle k\rangle^2)^{1/2}/k_L$ at saturation both increase with the coupling in such a way that their ratio, the spectral brightness $\mathcal{B}$, is largely independent of all the physical and operating parameters [2]. This is a remarkable result, since, before entering the broad spectrum regime, the spectral brightness may vary by orders of magnitude from one case to the other. One finds that $\mathcal{B}$ is always slightly smaller than a maximum value $\mathcal{B}_M$, with typically

$$0.6 \leq \mathcal{B} \leq \mathcal{B}_M \sim 0.8 \ . \tag{3}$$

The existence of such a "universal" brightness therefore means that high currents can indeed lead to high efficiencies, but only at the expense of a proportional spectrum broadening. Although, in some cases, tapering allows sideband inhibition with an improved efficiency [6], this is in general not the case for FEL oscillators where one has to use short wigglers [7]. This constraint on the spectral brightness is an important issue concerning the possible outputs of high current FEL oscillators. From a more fundamental point of view, this universality suggests that some important features of the dynamics do not depend on the precise values of the operating parameters, but only on the fact that the couplings and the interaction time are large enough to bring the system in the broad spectrum regime.

Equation (1) can be derived from a Hamiltonian H for the variables ψ and $p=(\gamma^2-\gamma_0^2)/\gamma_0^2$:

$$H(\psi,p)=T(p)+\sum_n V_n(\psi) \ , \tag{4a}$$

where

$$T(p)=k_w[p-\ln(1+p)] \ , \tag{4b}$$

and

$$V_n(\psi)=\frac{a_w\mathcal{K}_1|\mathcal{E}_n|\langle ST\rangle}{(1+nh)\gamma_0^2}\cos[(1+nh)\psi-nhk_wz-\phi_n] \ . \tag{4c}$$

Now we are interested in the saturation, where the gain is equal to the losses $1/Q$, that is, typically a few percent. It is therefore legitimate to consider, as a first step, that the $\mathcal{E}_n$ are constant during one given round trip of the electrons. Each term V_n generates a resonance at $p_n=nh$ corresponding to $d_z\psi=nh/(1+nh)$, with a synchrotron pulsation Ω_n:

$$\Omega_n=(2\mu_0h)^{1/4}\left[k_wa_w\mathcal{K}_1\langle ST\rangle\frac{e}{mc}\right]^{1/2}\gamma_0^{-1}E_n^{1/4} \ . \tag{5}$$

The motion of the electrons in this column of resonances depends on the E_n's. If they were very small, each resonance would keep its identity, and electrons with $p\simeq p_n$ would remain trapped around p_n. However, for larger E_n's, resonances overlap and merge into a column coming down from $p=0$ to $p=p_{\min}=hn_{\min}=-2\sqrt{3}\Sigma$. In this regime, the electrons are no longer confined to a given bucket [8]. On the contrary, their motion is chaotic [9] and eventually leads to an equipartition over all the available phase space.

Let us now consider the electrons entering the wiggler, when the large spectrum regime has been reached. The detuning and the initial energy spread of the electrons have little effect in this case, so that we can consider that the electrons start with $p=0$. The chaotic diffusion leads to an equipartition between $p=0$ and $p=p_{\min}=-2\sqrt{3}\Sigma$. Therefore the average p variation for the electrons across the wiggler will be given by $-\Delta p=\sqrt{3}\Sigma$ for an infinite wiggler, so that $-\Delta p\leq\sqrt{3}\Sigma$ for a finite wiggler. The corresponding laser energy variation is then $\Delta E_L=-\Delta E_e\leq\frac{1}{2}\sqrt{3}\Sigma E_e$. Since we consider the asymptotic equilibrium, this is also the energy $\Delta E_L=E_L/Q$ extracted from the cavity so that

$$\mathcal{B}=\frac{E_L}{QE_e\Sigma}\leq\mathcal{B}_M=\frac{\sqrt{3}}{2}\simeq 0.86 \ , \tag{6}$$

which is in agreement with the values issued from extensive numerical simulations [Eq. (3)].

The diffusion of electrons along the p axis may be described by a Fokker-Plank equation [9]. The width of the p distribution then evolves like $z^{1/2}$, so that the electrons have $p\sim -L_w^{1/2}$ at the end of the wiggler, and couple only to laser modes with $\Delta k/k_L\leq(\Delta k/k_L)_{\max}\sim L_w^{1/2}$. We see then that the spectral width is controlled by the diffusion coefficient and the wiggler length, so that the spectral width and the extracted efficiency should be both proportional to $L_w^{1/2}$. This has been tested by complete self-consistent numerical simulations, as shown in Fig. 1. Note that this behavior is radically different from the monochromatic regime, where the efficiency is essentially determined by the detuning and therefore behaves like L_w^{-1}.

As a first step we may not consider the corrections due to the fluctuations of the laser field, and suppose that the average spectrum is smooth enough to avoid trapping. Then the diffusion coefficient D_n around p_n is locally determined in p, and can be evaluated by replacing in Eq. (4) our stochastic column by an infinite and uniform column, with the local value of Ω_n. The Hamiltonian is z periodic, with period $\Delta z=2\pi/hk_w$. Within the quasilinear (or random-phase) approximation, which considers that the phase ψ is rapidly decorrelated by the dynamics, the diffusion coefficient along p is $D=\langle(\Delta p)^2\rangle/\Delta z$, where $\langle(\Delta p)^2\rangle$ is the average over ψ of the quadratic momentum variation during one period Δz. Now since ψ is considered as a random variable, the relative given

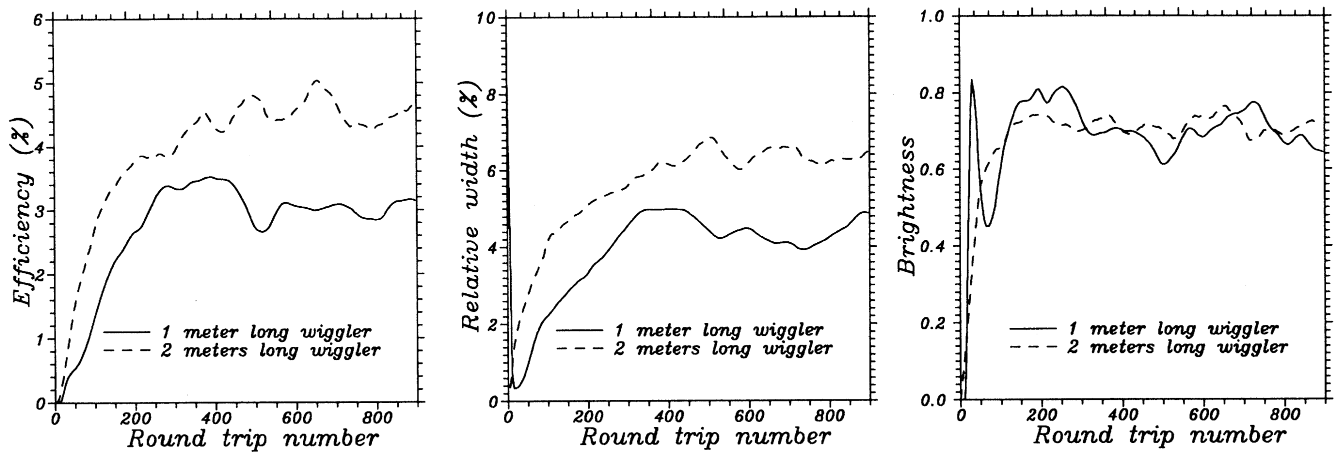

FIG. 1. Efficiency ρ, relative spectral width Σ, and spectral brightness $\mathcal{B}=\rho/\Sigma$, as functions of pass number, for two wiggler lengths. When the wiggler length is multiplied by 2, the efficiency and the spectral width at saturation are both multiplied by $\sqrt{2}$, so that the spectral brightness is invariant.

slowly varying laser phases ϕ_n have no effect, and can be set to zero without changing the diffusion coefficient (the corrections to the quasilinear value will, however, be sensitive to the ϕ_n's [10]). Then the dynamical equations can be integrated on Δz intervals, giving $\langle(\Delta p)^2\rangle=\frac{1}{2}(2\pi\Omega_n^2/hk_w^2)^2$, so that, taking Eq. (5) into account, the diffusion coefficient is found to be proportional to the spectral density of laser energy (it does not depend on the discretization parameter h):

$$\partial_z g(z,p)=\tfrac{1}{2}\partial_p D(p)\partial_p g(z,p)\ , \tag{7}$$

with

$$D(p_n)=2\mu_0\pi\left[a_w\mathcal{K}_1\langle ST\rangle\frac{e}{mc}\right]^2\gamma_0^{-4}k_w^{-1}E_n\ .$$

The precise z dependence of the density $g(z,p)$ will depend on the detailed shape of the spectrum. However, the asymptotic distribution for large z is uniform between 0 and $p_{\min}$, and does not depend on this spectrum shape. Furthermore, as long as the spectrum is smooth, and because of the regularizing properties of the diffusion equation, the speed at which the asymptotic distribution is reached depends very little on the details of the variations of $D(p)$ with p. It is mainly controlled by a mean value D of the diffusion coefficient over the available p interval: the lowest-magnitude eigenvalue of the Sturm-Liouville problem corresponding to the diffusion equation [Eq. (7)] is $\lambda_0=0$, which corresponds to equilibrium. The next eigenvalue can be estimated [11] by $\lambda_1\simeq-\pi^2[\int dp/\sqrt{D(p)}]^{-2}$, which is the same as if the diffusion coefficient were uniform with $1/\sqrt{D}\equiv\langle 1/\sqrt{D(p)}\rangle$, or $D=\langle D(p)\rangle$ up to the second order in the relative variations $1-D(p)/\langle D(p)\rangle$. This eigenmode dominates the route to equilibrium, since the next eigenvalue can be similarly estimated to be four times larger than λ_1. Then, everything goes on just as if the diffusion were uniform, with

$$D\equiv\frac{1}{|n_{\min}|}\sum_n D(p_n)$$

$$=\mu_0\pi\left[a_w\mathcal{K}_1\langle ST\rangle\frac{e}{mc}\right]^2\gamma_0^{-4}k_w^{-1}\mathcal{B}QE_e\ . \tag{8}$$

Neglecting for the moment the effects of the lower bound at $p=p_{\min}$, the solution of Eq. (7) is

$$g(z,p)=\rho_e\Theta(-p)\left[\frac{2}{\pi Dz}\right]^{1/2}\exp\left[\frac{-p^2}{2Dz}\right]\ ,$$

where Θ is the Heaviside function, so that the average energy is $\langle p(z)\rangle=-(2Dz/\pi)^{1/2}$. This leads to the efficiency $\rho\simeq-\frac{1}{2}\langle p(z=L_w)\rangle$:

$$\rho=\alpha(Q\rho_e L_w)^{1/2}\ , \tag{9a}$$

with

$$\alpha^2=\frac{\mu_0 e^2}{m}(a_w\mathcal{K}_1\langle ST\rangle)^2\gamma_0^{-3}k_w^{-1}\frac{\mathcal{B}}{2\sqrt{3}}\ . \tag{9b}$$

A first estimation α_M of the coefficient α is available since we know that the spectral brightness is smaller than, and of the order of magnitude of, $\mathcal{B}_M=\sqrt{3}/2$. A complete *ab initio* evaluation of the brightness would necessitate refining the argument stating that laser modes do not appear where electrons do not have time to diffuse within one wiggler length. This means taking into consideration the whole self-consistent dynamics, which cannot be done analytically at that time.

We see in Eq. (9) that the extracted efficiency and the spectral width are both proportional to the square root of the wiggler length, as previously stated. They are also proportional to the square root of the electron density and to the square root of the cavity quality factor Q: the more we close the cavity, the more we extract energy [12]. Now, there is no difficulty in calculating the corrections to α induced by the lower bound $p_{\rm min} = -2\sqrt{3}\Sigma$ of the phase-space region available for diffusion, and one finds a theoretical value $\alpha_{\rm th}$ slightly lower than α_M. The scaling law [Eq. (9a)] has been tested, and is in good agreement with full numerical simulations (Fig. 2).

The model described in this Rapid Communication has enabled us to obtain in a simple way most of the average characteristics of the high-power saturation regime, otherwise issued from heavy numerical simulations. It shows that the electronic energy is transferred to the laser via chaotic diffusion toward lower energies, rather than via a coherent synchrotron motion as suggested by the standard models [1,4]. This is why the efficiency is no longer simply related to the detuning, which we neglect here, but appears to be more directly related to the spectral width. One of the purposes of the ELSA experiment by Joly *et al.* [13] will be to reach high efficiency and test this behavior.

The quantitative consequences of this idea could be made more precise with a detailed analysis of the diffusion behavior, testing the relevance of the quasilinear approximation, and of our crude description of the laser spectrum. They could also be made more complete by taking into account the time evolution of the laser field along the wiggler. In particular, the values and dynamics of the laser phases may affect the diffusive behavior [10]. Furthermore, an analysis of the self-consistent coupling between the electrons and the laser could provide some insight into the transient regime, and into the nature of the fluctuations in the asymptotic regime.

It appears from this discussion that FEL's may be efficient experimental devices to investigate the issues related to wave plus particle systems with self-consistent couplings. Finally, and from a more practical point of view, we have shown how allowing a broad laser spectrum makes possible an important increase of the expected efficiencies for high current FEL oscillators.

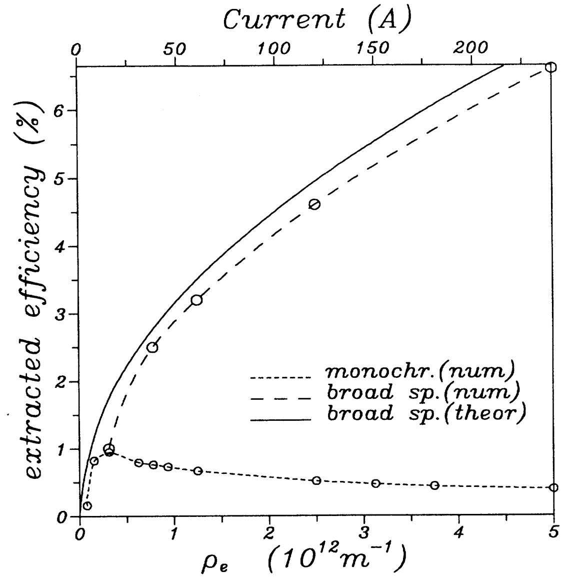

FIG. 2. Efficiency vs linear electron density ρ_e. For a monochromatic laser (dotted curve), the efficiency (related to the detuning $\Delta k/k_L \leq 0.41/N_w$, $N_w \simeq 31$) remains below 1%. If the spectral dynamics is not constrained [dashed curve from full self-consistent numerical simulation, solid curve from Eq. (9a) with $\alpha = \alpha_{\rm th}$], the spectrum broadens and the efficiency increases like the square root of the charge. The considered numerical values are $a_w = 1.075$, $\mathcal{K}_1 = 0.9$, $\gamma_0 = 33$, $Q = 25$, $L_w = 1$ m, $\langle ST \rangle = 0.22$ mm^{-1}. Equation (9b) gives $\alpha_M = 7.38 \times 10^{-9}$, the corrected theoretical value is $\alpha_{\rm th} = 6.12 \times 10^{-9}$, while the observed value is $\alpha_{\rm num} = 5.72 \times 10^{-9}$ (the relative error is about 7%).

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