RAPID COMMUNICATIONS

PHYSICAL REVIEW E VOLUME 48, NUMBER 5 NOVEMBER 1993

Relativistic solitary wave in an electron-positron plasma

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(Received 8 July 1993)

Relativistic solitary-wave propagation is studied in a cold electron-positron plasma embedded in an external arbitrary strong magnetic field. The exact, analytical, solitonlike solution corresponding to a localized, purely electromagnetic pulse with an arbitrarily large amplitude is found.

PACS number(s): 52.35.Sb

Recently, the problem of wave propagation and related phenomena in electron-positron (e-e^+) plasmas, or more generally in plasmas with equal-mass components, has attracted considerable attention [1]. Such plasmas are, for example, found in Van Allen belts, in active galactic nuclei, and near the polar caps of pulsars [2]. The e-e^+ pair production and subsequent plasma formation are also possible when electrons are accelerated to relativistic velocities either by intense laser beams, or by large-amplitude wake fields generated by intense short laser pulses [3]. Under certain conditions, even an ultrarelativistic electron-proton plasma can behave very much like an electron-positron plasma [4].

It is also obvious that different types of such plasmas will be the essential constituents of the early universe [5]. It is conjectured that intense relativistic plasmas could exist in the vicinity of cosmic defects like strings [6]. Most of all, the dynamics of an e-e^+ plasma could be of great interest to further our understanding of the MeV epoch in the evolution of the universe; it may, indeed, be possible that a deeper insight into the behavior of an interacting e-e^+ fluid in this era may provide valuable clues to its later evolution. A stable localized solution with density excess may, coupled with gravity, create templates for confining matter and creating inhomogeneities necessary to understand the observed structure of the visible universe.

Although our investigations could have a wider scope, we concentrate in this paper primarily on finding localized pulselike solutions which may be of relevance to the early universe dynamics. We deal with a pure e-e^+ plasma embedded in an arbitrary strong magnetic field $\mathbf{B}_0=B_0\mathbf{z}$. The two-cold-fluid system is exactly solved for arbitrary electric- and magnetic-field perturbations that propagate following the z direction of the ambient field. We find that the equal-mass constraint, coupled with the demands of a pulselike (localized) solution, forces strict charge neutrality leading to a pure electromagnetic pulse.

The cold magnetized relativistic electron-positron plasma can be described using two-fluid hydrodynamic equations together with the set of Maxwell equations:

$$\frac{\partial N_\alpha}{\partial t}+\nabla\cdot(N_\alpha \mathbf{u}_\alpha)=0 , \tag{1}$$

$$\frac{\partial \mathbf{p}_\alpha}{\partial t}+(\mathbf{u}_\alpha\cdot\nabla)\mathbf{p}_\alpha=\mathbf{s}_\alpha[\mathbf{E}+\mathbf{u}_\alpha\times\mathbf{B}+\mathbf{u}_\alpha\times\mathbf{B}_0] , \tag{2}$$

$$\nabla\times\mathbf{B}=\frac{\partial \mathbf{E}}{\partial t}+\sum_{\alpha=e,e^+} s_\alpha N_\alpha \mathbf{u}_\alpha , \tag{3}$$

$$\nabla\times\mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t} , \tag{4}$$

$$\nabla\cdot\mathbf{E}=N_{e^+}-N_e , \tag{5}$$

$$\nabla\cdot\mathbf{B}=0 , \tag{6}$$

where α is the species index (e for the electron and e^+ for the positron) so that s_e and s_{e^+} denote, respectively, the negative sign and the positive sign. In these equations all variables are dimensionless. The time and space variables are respectively normalized to the electron plasma frequency ω_e and the collisionless electron skin depth c/ω_e. The electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields of the wave are given in units of $m_e\omega_e c/e$. In these units the constant external magnetic field is $|\mathbf{B}_0|=\Omega_e/\omega_e$ where Ω_e is gyrofrequency. The relativistic momentum vector $\mathbf{p}_\alpha$ is in units of $m_e c$ and the particle number density N_α is normalized to the equilibrium number density $n_0=n_{e0}=n_{e^+0}$.

For definiteness we choose the constant external magnetic field $\mathbf{B}_0$ to be parallel to the z axis. It is also convenient to write the equations in terms of the scalar (φ) and vector ($\mathbf{A}$) potentials defined by the standard relations

$$\mathbf{E}=-\frac{\partial \mathbf{A}}{\partial t}-\nabla\varphi , \quad \mathbf{B}=\nabla\times\mathbf{A} . \tag{7}$$

Without any loss of generality we work in the gauge $\nabla\cdot\mathbf{A}=0$.

We look for the propagating localized solutions (vanishing at infinity) described by Eqs. (1)–(6), in which all fields depend only on the combination $\xi=z-vt$. Since we are interested in localized solutions, we consider only the subluminous case where the normalized wave velocity v (given in c units) is always less than 1. The chosen spatial dependence (on z alone) coupled with the gauge condition implies that $A_z=0$.

Multiplying the vector equation of motion (2) by the velocity $\mathbf{u}_\alpha$ and integrating it together with its longitudinal component, we can derive the relation

$$\gamma_\alpha-vp_{\alpha z}=1-s_\alpha\varphi , \tag{8}$$

where

$$\gamma_\alpha=\sqrt{1+\mathbf{p}_{\alpha\perp}^2+p_{\alpha z}^2}\ , \tag{9}$$

and the transverse momentum $\mathbf{p}_{\alpha\perp}$ is defined as $\mathbf{p}_{\alpha\perp}=p_{\alpha x}\mathbf{x}+p_{\alpha y}\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are unit vectors along the two independent transverse directions.

The longitudinal component of the momentum is now written more explicitly using Eq. (8) and the factor $\gamma_v=(1-v^2)^{-1/2}$,

$$p_{\alpha z}=\gamma_v^2[v(1-s_\alpha\varphi)-D_\alpha]\ , \tag{10}$$

where

$$D_\alpha=[(1-s_\alpha\varphi)^2-(1+\mathbf{p}_{\alpha\perp}^2)\gamma_v^{-1}]^{1/2}\ . \tag{11}$$

Also the relativistic factor γ_α for the particles can be expressed in terms of the transverse momentum

$$\gamma_\alpha=\gamma_v^2[(1-s_\alpha\varphi)-vD_\alpha]\ . \tag{12}$$

Using Eqs. (10) and (12), we integrate the continuity equation (1) for the boundary condition that all variables are vanishing at infinity and we get the positron and the electron density perturbation

$$\delta N_\alpha=N_\alpha-1=\gamma_v^2[(1-s_\alpha\varphi)v/D_\alpha-1]\ . \tag{13}$$

The remaining two components of Eq. (2) for the transverse momenta can be expressed [using Eqs. (10) and (11)] as follows:

$$\frac{dp_{\alpha x}}{d\xi}=-s_\alpha\left[\frac{dA_x}{d\xi}+\frac{p_{\alpha y}B_0}{D_\alpha}\right]\ , \tag{14}$$

$$\frac{dp_{\alpha y}}{d\xi}=-s_\alpha\left[\frac{dA_y}{d\xi}-\frac{p_{\alpha x}B_0}{D_\alpha}\right]\ . \tag{15}$$

To close the system, we need Maxwell's equations,

$$\frac{d^2\mathbf{A}_\perp}{d\xi^2}=-\gamma_v^2\sum_{\alpha=e,e^+}s_\alpha\frac{N_\alpha\mathbf{p}_{\alpha\perp}}{\gamma_\alpha} \tag{16}$$

and

$$\frac{d^2\varphi}{d\xi^2}=-v\gamma_v^2\sum_{\beta=e,e^+}s_\beta\frac{1-s_\beta\varphi}{D_\beta}\ . \tag{17}$$

We shall now demonstrate that this system of equations contains an arbitrary-amplitude, propagating, localized, electromagnetic pulse as a solution. It is convenient to solve our system for the transverse momentum, eliminating the vector potential. Introducing the complex variables

$$A=A_x+iA_y \tag{18}$$

and

$$P_\alpha=p_{\alpha x}+ip_{\alpha y}\ , \tag{19}$$

we derive [from Eqs. (13)–(15)] the basic equation of this paper,

$$-s_\alpha\frac{d^2P_\alpha}{d\xi^2}+iB_0\frac{d}{d\xi}\frac{P_\alpha}{D_\alpha}+v\gamma_v^2\sum_{\mu=e,e^+}s_\mu\frac{P_\mu}{D_\mu}=0\ , \tag{20}$$

where D_α reads

$$D_\alpha=[(1-s_\alpha\varphi)^2-(1+|P_\alpha|^2)\gamma_v^{-2}]^{1/2} \tag{21}$$

with new variables. Equation (20) for positrons and for electrons is integrated once using Eqs. (14) and (15). Combining the obtained first-order differential equations with their complex conjugates leads to the relation

$$|P_e|^2=|P_{e^+}|^2\ , \tag{22}$$

which is the analog of a relation derived in Ref. [7] for electron-ion plasmas. Due to their equal masses, the radiative pressure is the same for electrons and positrons. In addition to effecting considerable algebraic simplification, Eq. (22) leads to a most interesting feature about the localized pulses in an e-e^+ plasma. To demonstrate this remarkable result, we begin with the Poisson equation [Eq. (17)] with Eq. (22) substituted in it,

$$\frac{d^2\varphi}{d\xi^2}=v\gamma_v^2\left[\frac{1+\varphi}{[(1+\varphi)^2-(1+|P_e|^2)\gamma_v^{-2}]^{1/2}}-\frac{1-\varphi}{[(1-\varphi)^2-(1+|P_e|^2)\gamma_v^{-2}]^{1/2}}\right]\ . \tag{23}$$

Naturally, the structure of φ depends upon the structure of the other field variable $|P_\alpha|^2$, which is the same for the two species. As $\xi\to\infty$, both of the field variables φ and $|P_e|^2$ must become much less than unity for localized pulses. Thus, as $\xi\to\infty$, Eq. (23) must reduce to

$$v^2\frac{d^2\varphi}{d\xi^2}+2\varphi[1+(3v^{-2}-1)|P_e|^2]=0\ , \tag{24}$$

which does not allow a decaying solution; the solution must oscillate (spatially) with a wavelength of order unity. The only way for φ to go to zero as $\xi\to\infty$ is to be zero everywhere. If $|P_e|^2\neq|P_{e^+}|^2$, then Eq. (24) would contain inhomogeneous terms proportional to $|P_\alpha|^2$, i.e., terms independent of φ, and one could seek decaying solutions. Thus the equal mass of the constituent particles forbids the existence of an electrostatic potential and forces a localized pulse to be purely electromagnetic. From Eqs. (14)–(16) one can see that for $\varphi=0$ only linearly polarized electromagnetic stationary localized radiation can propagate.

The system of equations for electrons and positrons (20) is integrated once. Then, two first-order equations are reduced into a single one, using the fact that the potential (φ) is zero in the expression for D_α [see Eq. (21)]:

$$4\frac{d^2P}{d\eta^2}+i\beta P(1-|P|^2)^{-3/2}\frac{d|P|^2}{d\eta}+\beta^2P(1-|P|^2)^{-1}-2P(1-|P|^2)^{-1/2}=0\ , \tag{25}$$

where $\eta=2\gamma_v\xi$,

$$P=P_e(\gamma_v^2-1)^{-1/2}\ , \tag{26}$$

and

$$\beta=B_0(\gamma_v^2-1)^{-1/2}\ . \tag{27}$$

Equation (25) is an exact consequence of Eqs. (1)–(6)

describing a purely electromagnetic propagating pulse in e-e^+ plasma. In order to solve this equation we express the complex momentum as

$$P=|P|\exp(i\theta)\ , \tag{28}$$

and eliminate the phase θ from the set of coupled equations, obtaining

$$\frac{d^2|P|}{d\eta^2}+\beta^2\frac{|P|}{1-|P|^2}\left[1-\frac{|P|^4}{[1+(1-|P|^2)^{1/2}]^4}\right]-2\frac{|P|}{(1-|P|^2)^{1/2}}=0\ . \tag{29}$$

Integrating once Eq. (29) with the boundary conditions $|P|\to 0$ for $\eta\to\infty$, we get

$$\left[\frac{d|P|}{d\eta}\right]^2+2\beta^2\frac{1-(1-|P|^2)^{1/2}}{|P|^2}+(1-|P|^2)^{1/2}-\beta^2-1=0\ . \tag{30}$$

Equation (30) can be simplified making the substitution

$$R=1-(1-|P|^2)^{1/2}\ , \tag{31}$$

where R is constrained to the range $0\le R<1$. In the new variable R Eq. (30) takes the following form:

$$\left[\frac{dR}{d\eta}\right]^2-\frac{R^2(2-\beta^2-R)}{(1-R)^2}=0\ . \tag{32}$$

This equation has an implicit soliton-type solution given by

$$R=R_m\operatorname{sech}^2\left[\frac{R_m^{1/2}}{2}[|\eta|+2(R_m-R)^{1/2}]\right]\ , \tag{33}$$

where the maximum amplitude R_m is

$$R_m=2-\beta^2\ . \tag{34}$$

Since the variable R is always positive and smaller than 1, the parameter β^2 must satisfy the following inequality:

$$1<\beta^2\le 2\ , \tag{35}$$

which, for a given γ_v, imposes a restriction on the strength of the ambient magnetic field. Naturally, this solution ceases to exist when $B_0\to 0$. In spite of the fact that the pulse soliton is expressed in an implicit form, it is not difficult to calculate its physically relevant features. For example, the half width of soliton reads

$$L=1.77(R_m)^{-1/2}-(2R_m)^{1/2}\ . \tag{36}$$

It can be seen from Eqs. (33)–(36) that both the shape and width of the soliton depend on its amplitude. It is also clear that the large-amplitude ($R_m\to 1$) soliton tends to be spiky, while for smaller amplitudes, the wave train is spread out. Such an amplitude dependence of the soliton is displayed in Fig. 1.

Let us now express all relevant physical quantities in terms of the solution R. The particle relativistic factor γ_α and the density perturbation δN_α [see Eqs. (12) and (13)] are respectively given by

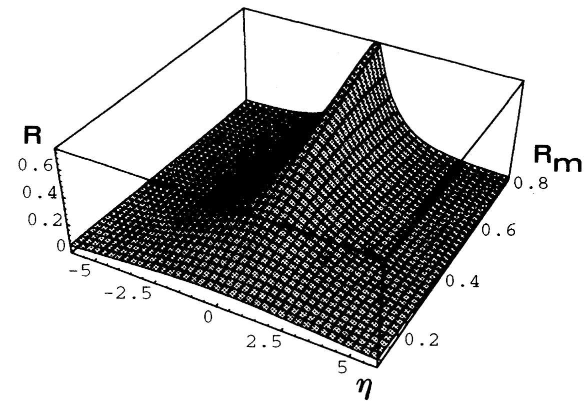

FIG. 1. Solution R as function of space coordinate η and amplitude R_m.

$$\gamma_\alpha=(\gamma_v^2-1)R+1 \tag{37}$$

and

$$\delta N_\alpha=\frac{\gamma_v^2 R}{1-R}\ . \tag{38}$$

The electric field $E=(E_x^2+E_y^2)^{1/2}$ is obtained from Eqs. (7), (14), and (15),

$$E=2(\gamma_v^2-1)R^{1/2}\ . \tag{39}$$

Equations (37)–(39), together with Eq. (33), completely describe the characteristics of a solitary wave in e-e^+ plasma.

From Eqs. (27) and (39), it can be seen that the maximum value of electric field and the factor γ_v [velocity of the soliton $v=\gamma_v^{-1}(\gamma_v^2-1)^{1/2}$] are interrelated,

$$E_m=2(\gamma_v^2-1)[2-B_0^2(\gamma_v^2-1)^{-1}]^{1/2}\ . \tag{40}$$

In this relation the dimensionless magnetic field $B_0=\Omega_e/\omega_e$ is an external parameter which, taking into account the inequality (35), restricts the range of the maximal electric field and hence of γ_v,

$$0\le E_m<2B_0^2\ , \tag{41}$$

$$B_0^2/2+1\le\gamma_v^2<B_0^2+1\ . \tag{42}$$

When the electric field approaches its upper limit ($2B_0^2$) a big density excess (δN) appears [see Eq. (38)]. If the electric field exceeds its upper limit, wave breaking will occur. In this case the electromagnetic waves are overturned and cause multistream motion of the plasma. In order to study such a situation a kinetic approach is necessary. When $B_0^2\ll 1$, it is impossible to have solitons with relativistically big amplitudes, since the particle's kinetic energy is less than the rest energy ($\gamma_\alpha\to 1$). In the opposite case, when $B_0\gg 1$, the velocity of the soliton is close to the velocity of light, and the amplitude can be relativistically big ($1\ll\gamma_{\alpha\max}<B_0^2$). A minimal possible width of the relativistic soliton is $\lambda=L/2\gamma_v\approx B_0^{-1}$.

In conclusion, it is shown that the radiative pressure is the same for electrons and positrons, which means the absence of the charge separation, i.e., the vanishing of the scalar potential. Solving exactly the system of nonlinear equations, we found the one-dimensional propagating localized purely electromagnetic pulse with relativistically large amplitude. The subluminous case of the relativistic solitary-wave propagation is considered in an arbitrary strong magnetic field. The obtained exact unique solution can lead to a density bunching and as a result to the creation of large inhomogeneities in plasma.

The existence of coherent pulselike exact solutions in the e-e^+ fluid is interesting, and in a future paper we shall investigate its stability, as well as what kind of physical effects could be associated with such concentrated, high-amplitude probes. It will be very important to expand the analysis to include kinetic, collisional, and radiation effects to study the interaction and energy exchange between the pulse and the particles.

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