RAPID COMMUNICATIONS

PHYSICAL REVIEW E VOLUME 48, NUMBER 4 OCTOBER 1993

Cluster effect in initially homogeneous traffic flow

B. S. Kerner and P. Konhäuser
Research Institute, Daimler-Benz AG, 70546 Stuttgart, Germany
(Received 2 March 1993; revised manuscript received 11 June 1993)

It is shown that, in an initially homogeneous traffic flow, a region of high density and low average velocity of cars can spontaneously appear, if the density of cars in the flow exceeds some critical value. This region—a cluster of cars—can move with constant velocity in the opposite direction or in the direction of the flow, depending on the selected parameters and the initial conditions of the traffic flow. Based on numerical simulations, the kinetics of cluster formation and the shape of stationary moving clusters are found. The results presented can explain the appearance of a "phantom traffic jam," which is observed in real traffic flow.

PACS number(s): 05.40.+j, 47.54.+r, 89.40.+k

I. INTRODUCTION

Since the works by Prigogine (e.g., [1]) on kinetic theory of traffic flow, cars have often been considered as interacting "particles." The more the number of cars moving on a road is increased, the more driving according to one's individual intention becomes impossible, but every driver's aim is to keep moving. If the number of cars is sufficiently large and only the average characteristics of the motion are of interest, traffic flow can be considered as a one-dimensional compressible flow of these "particles." Therefore, if, as usual, it is assumed that the hypothesis of continuity holds, a hydrodynamic description for the distribution of the density $\rho(x,t)$ of cars in a lane and their average velocity $v(x,t)$ is possible [1–3]. In this kinetic approach, traffic flow is described by both the continuity equation and the equation of motion [2–5]. The equation of continuity for a one-dimensional compressible flow reads [2–5]

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0\ . \tag{1}$$

The values of ρ should be positive and should not exceed the maximum density $\hat{\rho}$ (for an n-lane road $\hat{\rho}=n/a$, where a is an average length of cars).

The equation of motion, which states that the product of particle density (car density) and acceleration equals the sum of all acting forces, is given by the Navier-Stokes equations (e.g., [6]). Again for a one-dimensional compressible flow these equations reduce to

$$\rho[\partial v/\partial t+v(\partial v/\partial x)]=\partial/\partial x(\mu\partial v/\partial x)-\partial p/\partial x+X\ , \tag{2}$$

where p is the local pressure, μ is the viscosity, and X represents the sum of all inner forces which appear due to interactions between individual particles (cars).

The force X in (2) takes into account the relaxation process of the velocity v to a safe ("maximal and out of danger") velocity $V=V(\rho)$, which is achieved in a both time-independent and homogeneous traffic flow, i.e., [2,3],

$$X=\rho(V(\rho)-v)/\tau\ , \tag{3}$$

where τ is the average relaxation time of this process. The effect of X in (2) can best be understood, if its isolated action on the acceleration of an average driver in homogeneous traffic flow is studied. Under these artificial conditions, (2) reduces to $dv/dt=[V(\rho)-v]/\tau$, and it is apparent from this equation that, for a given ρ, the average driver is decreasing his velocity ($dv/dt<0$) if his instant velocity v is higher than $V(\rho)$ and vice versa. The function $V(\rho)$ is determined by the average balance between safety requirements and the risk readiness of the drivers as well as legal traffic regulations and road conditions, i.e., $V(\rho)$ is a phenomenological function. Despite the variety of quantities that determines the function $V(\rho)$, it strongly depends on ρ, and some important statements with respect to the general shape of the function $V(\rho)$ can be made:

(i) At not too small values ρ the value V should decrease as ρ increases, because drivers decrease their average velocity if the headway to the car in front of them is reduced. At the limit where $\rho\rightarrow\hat{\rho}$, cars cannot move at all, and for this reason $V(\rho)|_{\rho\rightarrow\hat{\rho}}\rightarrow 0$.

(ii) On the other hand, at small enough values of ρ there is almost no interaction between cars and they can move with some average velocity v_f, which depends on road conditions and average car technology. Therefore, $V(\rho)$ is a monotonous decreasing function of ρ, i.e., their derivative $\xi(\rho)=dV/d\rho<0$ [2–5].

The pressure p in (2) is the product of the density of particles ρ and their "temperature," or more precisely, an average square of the difference between velocities of cars and their average velocity (the variance of the velocity distribution), which will be denoted as c_0^2, i.e., $p=\rho c_0^2$. If c_0 is constant [2–5] and (3) is used, Eq. (2) can be written in the form

$$\partial v/\partial t+v(\partial v/\partial x)=-(c_0^2/\rho)(\partial\rho/\partial x)+[V(\rho)-v]/\tau +\rho^{-1}\partial/\partial x(\mu\partial v/\partial x)\ . \tag{4}$$

It is well known from theory and experiment that homogeneous traffic flow can be unstable [3–5] and one can expect a lot of nonlinear intricate phenomena to appear in traffic flow. In this Rapid Communication the nonlinear cluster effect, which can explain the spontaneous appearance of a traffic congestion without obvious reasons [7], a "phantom traffic jam," will be presented.

II. CLUSTERS IN TRAFFIC FLOW

A. Integral condition of problem

Integrating Eq. (1) over x from $x=0$ to $x=L$, where L is the length of the road, one can find the equation of the integral balance of the quantity of cars on the road:

$$dN/dt=q(0,t)-q(L,t)\ , \tag{5}$$

where

$$N=\int_0^L \rho(x,t)dx \tag{6}$$

is the total number of cars on the road (it is naturally supposed that $N \gg 1$); $q(x,t)=\rho(x,t)v(x,t)$—the flux of cars. The integral condition (5) physically expresses the fact that if one seeks some new distribution for $\rho(x,t)$ and $v(x,t)$, the initial quantity of the total number of cars N and its possible change in time have to be taken into account.

For the sake of clarity a homogeneous traffic flow on a circular road of circumference L will be considered in this article, i.e., the case when

$$q(0,t)=q(L,t)\ , \tag{7}$$

$$v(0,t)=v(L,t),\quad \partial v/\partial x|_0=\partial v/\partial x|_L\ . \tag{8}$$

In this case (5) takes the form

$$N=\int_0^L \rho(x,t)dx=\rho_h L\ , \tag{9}$$

where ρ_h is the car density in the homogeneous flow. The corresponding value of v_h follows from the evident relationship

$$v_h=V(\rho_h)\ . \tag{10}$$

If N and L are given, there is only one homogeneous state $\rho=\rho_h$, $v=v_h$ for this flow.

B. Critical fluctuation

By linearizing Eqs. (1), (4), and (9) with boundary conditions (8) in the neighborhood of this homogeneous state $\rho=\rho_h$, $v=v_h$ with respect to the fluctuations

$$\begin{aligned}\delta\rho(x,t)&=\delta\rho(x)\exp(-\gamma t),\quad \delta\rho(x)=\delta\rho_0\exp(ikx)\ ,\\ \delta v(x,t)&=\delta v(x)\exp(-\gamma t),\quad \delta v(x)=\delta v_0\exp(ikx)\ ,\end{aligned} \tag{11}$$

one can find the conditions and the dispersion equation, which determine the behavior of the perturbations (11):

$$\int_0^L \delta\rho(x,t)dx=0\ , \tag{12}$$

$$\exp(ikL)=1\ , \tag{13}$$

$$\gamma^2-\gamma(2ikv_h+k^2\rho_h^{-1}+1) +ik(k^2v_h\rho_h^{-1}+v_h+\xi(\rho_h)\rho_h)+k^2(c_0^2-v_h^2)=0\ . \tag{14}$$

Here, and as of now, $\xi(\rho)=dV/d\rho$, $\tau=$const, and $\mu=$const; v, V, and c_0 are measured in units of l/τ, the length in units of l, the time in units of τ, the density of cars in units of $\hat{\rho}$, where $l=\sqrt{\mu\hat{\rho}^{-1}\tau}$.

Condition (12) has a simple physical sense: A perturbation with $k=0$ would change the number of cars N on the road. Because this number is given and cannot be changed, the homogeneous perturbation cannot be realized in the traffic flow under consideration. From (12) and (13) we have the condition for the values k that are suitable for (14):

$$k=2\pi m/L,\quad m=\pm 1,\pm 2,\ldots\ . \tag{15}$$

Substituting γ in (14) for $\gamma=\lambda+i\omega$ (i.e., $\lambda=\mathrm{Re}\gamma$, $\omega=\mathrm{Im}\gamma$) and separating real and imaginary parts, we obtain two equations. From the first of these two equations, near the point where the homogeneous state loses its stability, more precisely for $|\lambda| \ll 1$, $k^2c_0^2$, we find two solutions for ω:

$$\omega_1=k(v_h-c_0)\ , \tag{16a}$$

$$\omega_2=k(v_h+c_0)\ . \tag{16b}$$

Substituting ω in the second of these equations (the imaginary part) for ω_1 (16a) [the solution for ω_2 (16b) does not meet the values k (15)] and taking into account (15), one can find that the stability of the homogeneous traffic flow is lost ($\mathrm{Re}\gamma=\lambda<0$) if

$$[-1-(\rho_h/c_0)\xi(\rho_h)]\rho_h>(2\pi l/L)^2 \tag{17}$$

with respect to the growth of the critical fluctuation where

$$k=k_c=2\pi/L,\quad \mathrm{Im}\gamma=\omega_c=k_c(v_h-c_0)\ . \tag{18}$$

The condition (17) can be fulfilled due to $\xi(\rho_h)<0$. To understand the physical mechanism of this instability, it is important to notice that the change in V caused by a perturbation $\delta\rho(x)$ is $\delta V(x)=\xi(\rho_h)\delta\rho(x)$. In the local region, where ρ is increased, i.e., the perturbation $\delta\rho(x)>0$, the value V decreases. To maintain safe driving conditions, the drivers must reduce their velocities in this region, i.e., $\delta v(x)<0$. On the other hand, the other condition $\delta\rho(x)=-\delta v(x)\rho_h/c_0$, which follows from the linearized equation (1) at the point $\rho_h=\rho_{c1}$, where $\mathrm{Re}\gamma=\lambda=0$ and $\mathrm{Im}\gamma=\omega_1$, indicates that a decrease of the local velocity produces an increase in the local density. An increase in the local perturbation of ρ leads to a local decrease in the values δV and therefore to a decrease in the local perturbation of v, which again amplifies the perturbation of ρ. This avalanchelike process is started when the value $|\xi(\rho_h)|$ is large enough to exceed the influence of the "diffusion (viscosity)" and "relaxation" processes which correspond to the right of, and to the first term on the left of, inequality (17), respectively.

The boundary (17) determines the critical values of the density of cars $\rho_h=\rho_{ci}$, $i=1,2,\ldots$, at which the homogeneous traffic flow becomes unstable. The critical values ρ_{ci} are functions of the length of the road L. The homogeneous traffic flow can lose its stability in one ($\rho_h=\rho_{c1}$ and $\rho_h=\rho_{c2}$, $\rho_{c2}>\rho_{c1}$, Fig. 1) or more intervals on the ρ_h

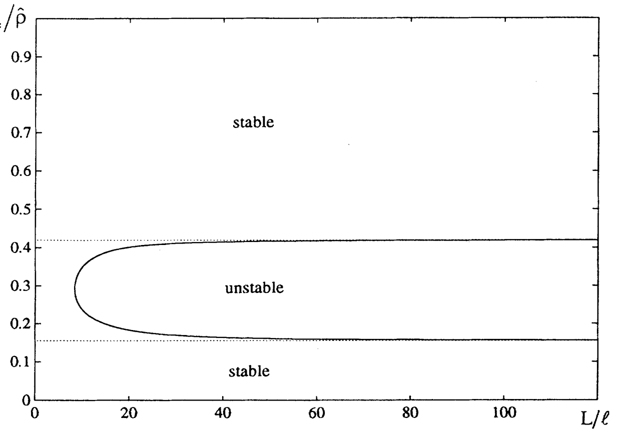

FIG. 1. Dependence of the critical values $\rho_h = \rho_c$ (i.e., $\rho_h = \rho_{c1}, \rho_{c2}$) on L. Results of numerical computations for $V(\rho) = 5.0461(\{1+\exp[(\rho-0.25)/0.06]\}^{-1} - 3.72\times10^{-6})$, $c_0 = 1.8634$.

axis. For a given L and a fixed function $V(\rho)$, the width of the intervals of instability increases with decreasing c_0.

From the shape of the critical fluctuation $\delta\rho(x)$, which, corresponding to (11) and (18), is a wave with only one maximum moving with the phase velocity

$$v_p = \omega_c / k_c = v_h - c_0 \ , \tag{19}$$

one can expect that, owing to the growth of this fluctuation, the cluster of cars spontaneously appears in the initially homogeneous traffic flow.

C. Formation of stationary clusters and their form

For the investigations of the cluster effect, the problem (1), (4), (8), and (9) has been solved numerically (Figs. 2 and 3). For this purpose, the additional unknown functions $w(x,t) = \partial v / \partial x$ and $\varphi(x,t)$, where $\partial\varphi/\partial x = \rho(x,t)$, were introduced and the problem (1), (4), (8), and (9) was written as a system of four first-order differential equations for the functions $\varphi(x,t)$, $\rho(x,t)$, $v(x,t)$ and $w(x,t)$ with the corresponding boundary conditions: $\varphi(0,t)=0$, $\varphi(L,t)=\rho_h L$, $v(0,t)=v(L,t)$, $w(0,t)=w(L,t)$. This system has been approximated on a grid $x_i = (i-1)dx$, $i=1{:}1{:}I$, $x_I = L$, $t_i = j dt$, $j=0,1,2,\ldots$ by the centered Euler or box scheme [8]. Assuming a known solution at time $t = t_{j-1}$, a system of $4(I-1)$ nonlinear equations for the $4(I-1)$ unknown $\varphi(x_i,t_j)$, $i=2{:}1{:}(I-1)$; $v(x_i,t_j)$ and $w(x_i,t_j)$, $i=2{:}1{:}I$; $\rho(x_i,t_j)$, $i=1{:}1{:}I$ at the new time level is obtained. The nonlinear equations are linearized with the help of Newton's method, which converges quadratically, and are solved iteratively by starting the iteration with a estimated solution, i.e., $\varphi(x_i,t_j)=\varphi(x_i,t_{j-1})$, $i=2;1{:}(I-1)$; $v(x_i,t_j)=v(x_i,t_{j-1})$ and $w(x_i,t_j) = w(x_i,t_{j-1})$, $i=2{:}1{:}I$; $\rho(x_i,t_j)=\rho(x_i,t_{j-1})$, $i=1{:}1{:}I$.

From the numerical simulations made one can conclude the following:

(i) At $\rho_h > \rho_{c1}$ the amplitude of the initial small-amplitude perturbations $\delta\rho$, δv [(11) and (18)] grows in time and the perturbations move with the velocity approximately equal to v_p (19), which, depending on the values v_h and c_0, can be more or less than zero. As a result of the growth of the perturbations a cluster of cars forms in the flow, i.e., a local, as a rule, moving region on the road, where the density of cars is considerably higher and the average velocity of cars is considerably lower than in the initial flow and outside the cluster. In time, this local region transforms into a stationary cluster of cars, which moves with a constant velocity v_g and with an invariable form [Fig. 2(a), $t > 100\tau$, and Fig. 3(a), $t > 95\tau$].

(ii) The velocity of cluster v_g can be positive or negative. It is determined by the shape of the function $V(\rho)$ and the other parameters of the flow (especially by the value c_0). The velocities of the cluster v_g and of the small-amplitude perturbation v_p (19) are often different from each other, not only in their values but also in their sign [see Fig. 2(a), where $v_p > 0$ but $v_g < 0$].

(iii) One can distinguish two scenarios of kinetics of cluster formation: (a) The first scenario is usually realized at values ρ_h, which are close to the critical values ρ_{ci}. In this case, the monotonous growth in time of the small-amplitude perturbation (11) and (18) leads, as a rule, directly to the appearance of one cluster on the road (Fig. 2). In the example of this scenario, shown on Fig. 2, the value v_p (19) is positive. As the amplitude of the per-

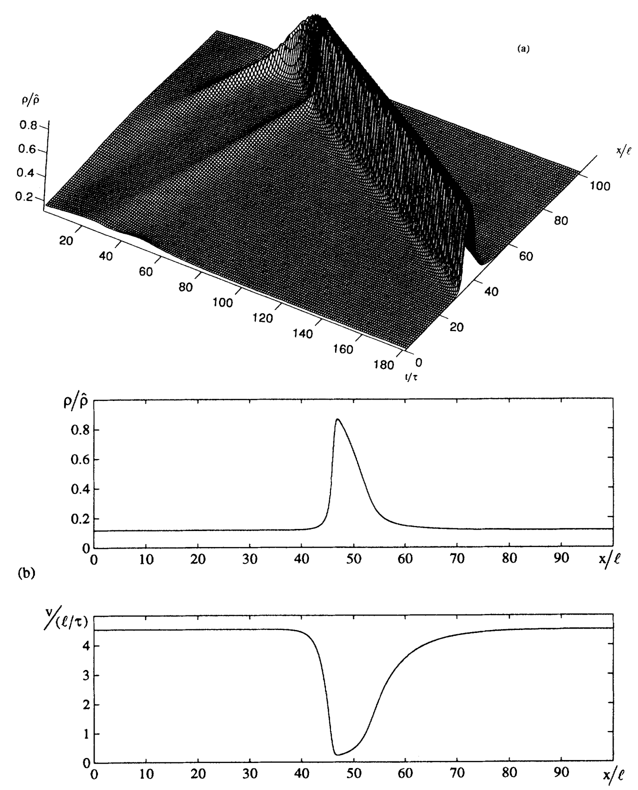

FIG. 2. The kinetics of the cluster formation: (a) the dependence $\rho(x,t)$ for $\rho_h = 0.168$; (b) the distributions $\rho(x)$ and $v(x)$ in the stationary cluster at $t=150$. $L=100$, the initial distribution $\rho(x,0)=\rho_h + \delta\rho(x,0)$ with $\delta\rho$ [(11) and (18)], $\delta\rho_0 = 0.02$. The other parameters are the same as in Fig. 1.

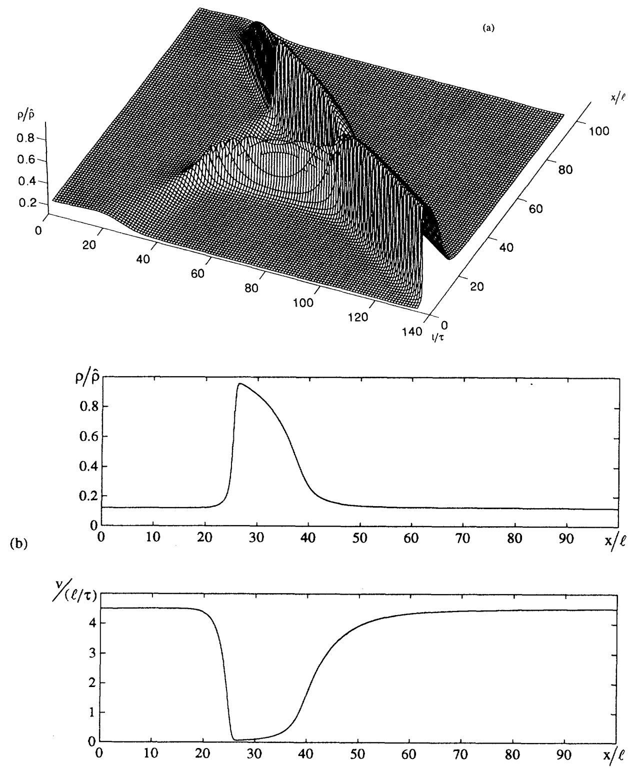

FIG. 3. Kinetics of the cluster formation: (a) the dependence $\rho(x,t)$ for $\rho_h=0.22$; (b) the distributions $\rho(x)$ and $v(x)$ in the stationary cluster at $t=100$. $\delta\rho_0=0.01$. The other parameters are the same as in Figs. 1 and 2.

turbations (11) and (18) increases and they gradually transform into a cluster of cars [Fig. 2(a), $t \cong 40\tau$], the velocity of this nonstationary cluster monotonously decreases. At some amplitude of the cluster the velocity of the cluster changes sign [Fig. 2(a), $t \cong 47\tau$]. As a result, in time the stationary cluster of cars forms [Fig. 2(a), $t > 100\tau$; Fig. 2(b)], which moves in the opposite direction to the traffic flow (with the velocity $v_g \cong -0.4467 l/\tau$). (b) The second scenario is usually realized, if $\rho_{c1} < \rho_h < \rho_{c2}$ and ρ_h is noticeably distinguished from ρ_{ci}. First, owing to the growth of the small-amplitude perburbation, two (Fig. 3) or more clusters appear on the road, which, as a rule, have different velocities, widths, and amplitudes and are situated at different locations. Due to the collisions between clusters and their successive confluences the number of cluster decreases in time and finally only one cluster can remain on the circular road. The example of this scenario is shown on Fig. 3. First, with the growth of the small-amplitude perturbations, a nonstationary cluster forms, which moves, with negative velocity [Fig. 3(a), $t \cong 22\tau - 32\tau$]. After some time, behind this cluster, where the values $\rho > \rho_{c1}$ ($\rho_{c1} \cong 0.1561\hat{\rho}$, Fig. 1), a new cluster forms, which moves with positive velocity [Fig. 3(a), $t \cong 35\tau - 45\tau$]. At $t \approx 75\tau$ these two clusters meet and within approximately 10τ combine [Fig. 3(a)]. Finally, the only one stationary cluster (moving with the velocity $v_g \cong -0.59 l/\tau$) forms [Fig. 3(a), $t > 95\tau$; Fig. 3(b)].

III. CONCLUSIONS

The growth of the small-amplitude nonhomogeneous perturbation of the density of cars and their average velocity in an initially homogeneous traffic flow leads to a formation of one or several clusters of cars in a traffic flow. A cluster of cars represents a locally moving (along or against the flow) region, where the density of cars is considerably higher and the average velocity of cars is considerably lower than in an initial flow and outside the cluster. A possible collision between two clusters causes their mergence. On a circular road, in time, finally only one cluster can remain. This stationary cluster moves with a constant velocity and has an invariable form. The direction of the cluster motion cannot be correlated with the velocity of the small-amplitude perturbations in a homogeneous flow.

ACKNOWLEDGMENTS

We would like to thank Mr. M. Herrmann and Mr. M. Schilke for their help of carrying out the numerical investigations.

[1] I. Prigogine and R. Herman, *Kinetic Theory of Vehicular Traffic* (Elsevier, New York, 1971).

[2] H. J. Payne, in *Mathematical Models of Public Systems*, Simulation Council Proceedings, 1971 (Simulation Councils, Inc., La Jolla, CA, 1971), Vol. 1, p. 51.

[3] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).

[4] R. Kühne, in *Highway Capacity and Level of Service*, edited by U. Brannolte (Balkema, Rotterdam, 1991), p. 211.

[5] W. Leutzbach, *Introduction to the Theory of Traffic Flow* (Springer-Verlag, Berlin, 1988).

[6] H. Schlichting, *Boundary-Layer Theory* (McGraw-Hill, New York, 1979).

[7] J. Treiterer, Ohio State University Technical Report No. PB 246 094, Columbus, OH (unpublished) (1975).

[8] H. B. Keller, SIAM J. Numer. Anal. **11**, 305 (1974).