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Transmission of signals by synchronization in a chaotic Van der Pol–Duffing oscillator

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We investigate the phenomena of chaos synchronization and efficient signal transmission in a physically interesting model, namely, the Van der Pol–Duffing oscillator. A criterion for synchronization based on asymptotic stability is discussed. By considering a cascaded synchronization system, we investigate the possibility of the secure communication of analog signals.

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The concept of synchronized chaos, introduced recently by Pecora and Carroll [1–3], allows for the possibility of building a set of chaotic dynamical systems such that their common signals are synchronized. This idea has in fact been successfully tested in a variety of nonlinear dynamical systems, including Lorenz equations, the Rösseler system, phase-locked loops, hysteresis circuits, Chua's circuit, and so on [1–7]. The robustness of chaotic synchronization suggests that it can be effectively used in *spread-spectrum communications* in which the chaotic signals can be ideally utilized to mask the information-bearing signals. A method of transmitting signals in a secure way through chaos synchronization has recently been reported [8]. This has recently been experimentally demonstrated in the case of Chua's circuit [9]. Here, in this report, we wish to discuss the method of transmitting signals using chaos synchronization in a physically interesting model, namely the Van der Pol–Duffing oscillator.

The physical realization of the Van der Pol–Duffing oscillator circuit is shown in Fig. 1 [10]. This circuit bears a close resemblance to that of Chua's circuit [6,11] in that the piecewise linear element of the latter is replaced by a cubic nonlinear element of the form $I(V)=aV+bV^3$ ($a<0$, $b>0$). Such a nonlinear element can be physically constructed by using a set of diodes and an operational amplifier [10,12].

By applying Kirchoff's laws to the various branches of the circuit of Fig. 1 and appropriately rescaling [10], the following set of dynamical equations can be easily obtained:

$$\dot{x}=-\nu[x^3-\alpha x-y]\ , \tag{1a}$$

$$\dot{y}=x-y-z\ , \tag{1b}$$

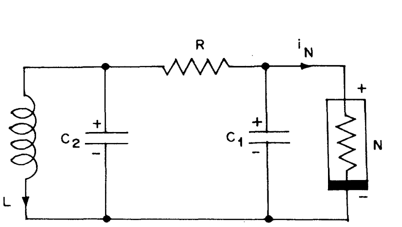

FIG. 1. Circuit realization of the chaotic Van der Pol–Duffing oscillator.

$$\dot{z}=\beta y\ , \tag{1c}$$

where an overdot denotes the operation d/dt. Here x, y, and z correspond to the rescaled form of the voltage across capacitor C_1, the voltage across capacitor C_2, and the current through L, respectively. α, ν, and β are the rescaled circuit parameters [10]. A numerical simulation of Eq. (1) with fixed values of ν and α exhibits period-doubling bifurcations leading to chaos as the parameter β is decreased from a large value [10]. If we choose the parameters as $\nu=100$, $\alpha=0.35$, and $\beta=300$, one observes a double-band chaotic attractor as shown in Fig. 2(a).

In order to observe the synchronization behavior in system (1), by following the approach of Pecora and Carroll [1,2], the system (1) is considered as a *master,* or *drive system.* The *slave,* or *response system,* is chosen to have an identical set of equations for y and z represented with primed variables. However, the initial conditions on y and z are not the same in general as those of y' and z' (response-system variables). Further, the response system is chosen to have exactly the same x signal as that of the drive system by feeding the latter to the former. Thus the response system is taken to be of the form

$$x'=x,\quad \dot{y}'=x-y'-z',\quad \dot{z}'=\beta y'\ . \tag{2}$$

Now considering the dynamics of the combined *drive-response system* (1) and (2) with the same parametric values for ν, α, and β, we find that in spite of the differences in initial conditions of (y',z') and (y,z) variables, the primed and unprimed systems do synchronize,

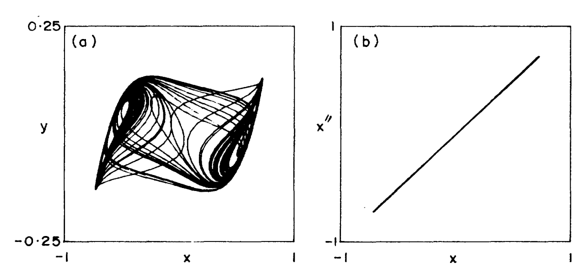

FIG. 2. (a) Chaotic attractor projected on the x-y plane for $\alpha=0.35$, $\nu=100$, and $\beta=300$. (b) Synchronization of chaos between $x-x''$ of Eq. (6).

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such that for $t \to \infty$, $(y-y') \to 0$ and $(z-z') \to 0$. Similar synchronization was observed with a "y" feedback from the drive system. It was noticed, however, that the "z" feedback failed to show synchronization.

Recently, a criterion based on the *asymptotic stability* has been developed as a necessary and sufficient condition for the synchronization of periodic and chaotic systems [2,13]. One of the practical ways to establish this asymptotic stability of the subsystem is to find an appropriate Lyapunov function [13]. Its use can be shown, at first by considering this function in connection with the subsystem given by Eqs. (1b) and (1c), namely,

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ \beta & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} x \\ 0 \end{bmatrix} . \tag{3}$$

The subsystem for the slave, or response system, is also represented by the same equation (3) for the primed variables [Eq. (2)]. Now calling $(y-y')=y^*$ and $(z-z')=z^*$, we have

$$\begin{bmatrix} \dot{y}^* \\ \dot{z}^* \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ \beta & 0 \end{bmatrix} \begin{bmatrix} y^* \\ z^* \end{bmatrix} . \tag{4}$$

Now, if we consider the Lyapunov function,

$$E = \tfrac{1}{2}[(\beta y^* + z^*)^2 + \beta y^{*2} + (1+\beta) z^{*2}] , \tag{5}$$

then

$$\dot{E} = (\beta y^* + z^*)(\beta \dot{y}^* + \dot{z}^*) + \beta y^* \dot{y}^* + (1+\beta) z^* \dot{z}^*$$
$$= -\beta (y^{*2} + z^{*2}) \le 0 \quad (\beta > 0) .$$

The equality sign applies only at the origin; therefore the subsystem [(1b) and (1c)] is globally asymptotically stable [13]. Thus the drive [Eqs. (1b) and (1c)] and response [Eq. (2)] systems eventually synchronize.

One may actually have a cascade of response systems [3]. Let us assume that our drive system is represented by Eq. (1) and that the first response system is represented in terms of the variables y' and z' driven by the original $x(t)$ from the drive [Eq. (2)]. In addition, we can have another response system consisting of the variable x'' and driven by the y' variable [3]. The total cascade system of equations is represented in the following manner:

Drive,

$$\dot{x} = -\nu [x^3 - \alpha x - y] , \tag{6a}$$

$$\dot{y} = x - y - z , \tag{6b}$$

$$\dot{z} = \beta y ; \tag{6c}$$

response 1,

$$\dot{y}' = x - y' - z' , \tag{6d}$$

$$\dot{z}' = \beta y' ; \tag{6e}$$

response 2,

$$\dot{x}'' = -\nu [(x'')^3 - \alpha (x'') - y'] . \tag{6f}$$

If all the response systems are synchronized, then $x''(t)$ is identical to the drive signal $x(t)$, even if the drive system (6) exhibits chaotic behavior. If one varies a parameter in the drive or response systems, then $x''(t)$ will not be the same as $x(t)$. A synchronized chaotic behavior which exists between $x(t)$ and $x''(t)$ for $\alpha = 0.35$, $\nu = 100$, and $\beta = 300$ is shown in Fig. 2(b).

In the following, we focus on the use of a synchronizing chaotic signal, in which the above synchronized chaotic Van der Pol–Duffing system can be effectively utilized as a vehicle to transmit analog signals in the context of *secure communications.* By following the scheme adopted by Oppenheim *et al.* [8] and Kocarev *et al.* [9] for our subsequent numerical analysis, we use the $x(t)$ signal of the drive system [Eqs. (6a)–(6c)] as a noiselike "masking signal" and $s(t)$ as an information signal to be transmitted in a secure way. Now let us consider the actual transmitted signal $r(t) = x(t) + s(t)$. The subsystem or, response 1 [Eqs. (6d) and (6e)], is now modified as

$$\dot{y}' = r(t) - y' - z' , \tag{6d'}$$

$$\dot{z}' = \beta y' . \tag{6e'}$$

The second response system (response 2) is the (x'') subsystem driven by the signal y', which is the same as that represented by Eq. (6f). Now from Eqs. (6a)–(6c), (6d'), (6e'), and (6f) we have the following inhomogeneous linear equation:

$$\dot{y}^* = s(t) - y^* - z^*, \quad \dot{z}^* = \beta y^* . \tag{7}$$

By assuming the power level of the information bearing signal $s(t)$ to be significantly lower than that of the $x(t)$ signal and the solution $x^* = (x'' - x)$ to be significantly small with respect to $s(t)$, we see that $s(t)$ can be recovered from response system 2 as [9]

$$\tilde{s}(t) = r(t) - x''(t) = x(t) + s(t) - x''(t)$$
$$\approx s^1(t) . \tag{8}$$

We have numerically solved the cascade system of equations (6a)–(6c), (6d'), (6e'), and (6f) simultaneously with parameters $\alpha = 0.35$, $\nu = 100$, and $\beta = 300$. The information-bearing signal $s(t)$ is assumed to be any one of the following type: (i) $s(t) = F \sin(\omega t)$ (single tone, $F = 0.02$, $\omega = 1.0$); (ii) $s(t) = F \sin(\omega t)[1 + f \sin(\Omega t)]$ (amplitude-modulated wave, $F = 0.02$, $\omega = 1.0$, $f = 1.0$, and $\Omega = 0.2$); and (iii) $s(t) = F \sin[\omega t + f \sin(\Omega t)]$ (phase-modulated wave, $F = 0.02$, $\omega = 1.0$, $f = 0.2$, and $\Omega = 0.2$). From the numerical simulation results, the information signal $s^1(t)$ is recovered at the response system by adopting Eq. (8). Figures 3(a)–3(c) depict the power spectrum of the information signal $s(t)$, the actual transmitted signal $r(t)$ [$= s(t) + x(t)$], and the recovered signal $s^1(t)$ for the above three different cases, respectively. As the power level of $s(t)$ is significantly lower than that of the $x(t)$ signal, the component of signal frequency of $s(t)$ is not discernible or detectable in Figs. 3(a)(ii)–3(c)(ii) because of the chaotic (broadband) nature of the actual transmitted signal $r(t)$. Also, the quality of the recovered signal $s^1(t)$ is significantly comparable to that of the original signal $s(t)$. In view of the typical broadband spectra, the chaotic signal $x(t)$ becomes an ideal candidate for *spread-spectrum* communication applications [8,9].

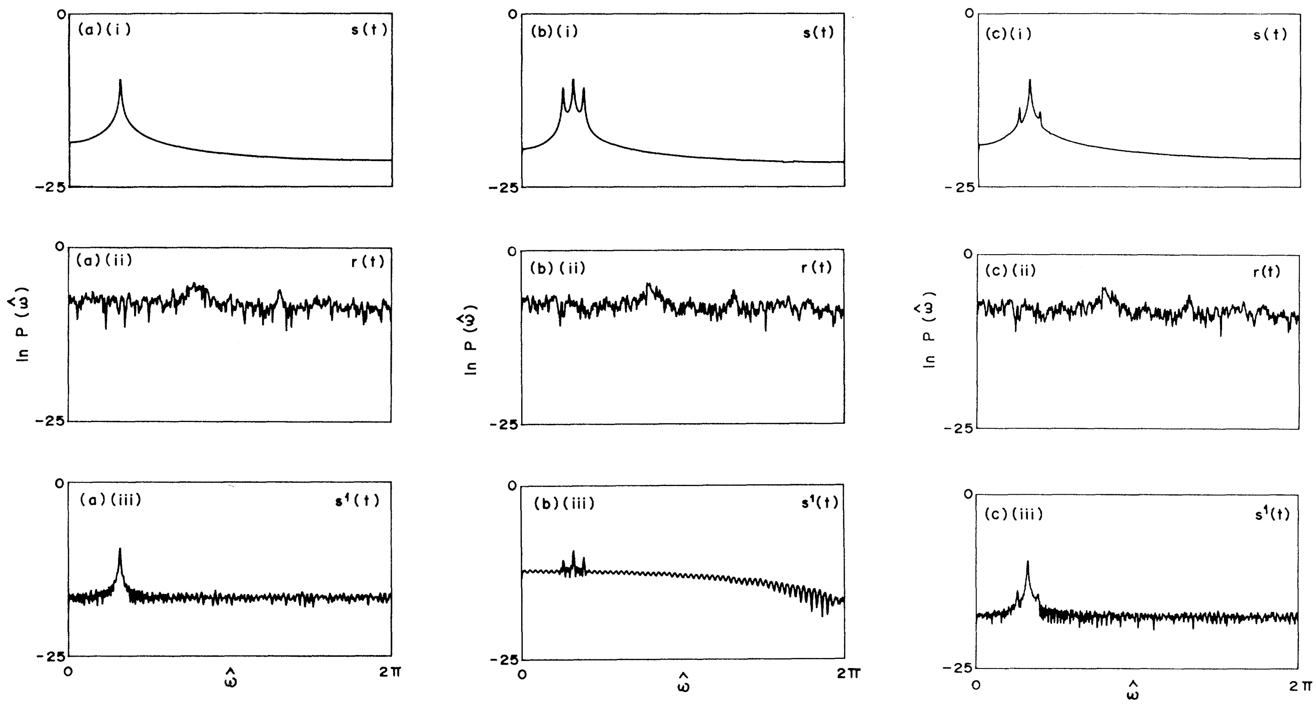

FIG. 3. (a) Power spectra of signals (i) $s(t)=F\sin(\omega t)$ (single tone, $F=0.02$, $\omega=1.0$), (ii) $r(t)$, and (iii) $s^1(t)$. (b) Power spectra of signals (i) $s(t)=F\sin(\omega t)[1+f\sin(\Omega t)]$ (amplitude-modulated wave, $F=0.02$, $\omega=1.0$, $f=1.0$, $\Omega=0.2$), (ii) $r(t)$, and (iii) $s^1(t)$. (c) Power spectra of signals (i) $s(t)=F\sin[\omega t+f\sin(\Omega t)]$ (phase-modulated wave, $F=0.02$, $\omega=1.0$, $f=0.2$, $\Omega=0.2$), (ii) $r(t)$, and (iii) $s^1(t)$.

In summary, we have numerically investigated the synchronization aspects of the Van der Pol–Duffing oscillator. A criterion for synchronization of chaos, based on the asymptotic stability has been discussed. By having a cascade of response systems, we have shown that this model can be effectively used to transmit a variety of analog signals for secure or spread-spectrum communications.

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