RAPID COMMUNICATIONS

PHYSICAL REVIEW E VOLUME 48, NUMBER 3 SEPTEMBER 1993

Band-random-matrix model for quantum localization in conservative systems

G. Casati
Università di Milano, Sede di Como, Via Castelnuovo 7, 22100 Como, Italy
and Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Milano, Italy

B. V. Chirikov
Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russia

I. Guarneri
Università di Milano, Sede di Como, Via Castelnuovo 7, 22100 Como, Italy
and Istituto Nazionale di Fisica Nucleare, Sezione di Pavia, 27100 Pavia, Italy

F. M. Izrailev
Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russia
(Received 18 January 1993)

We study the band-random-matrix model for conservative Hamiltonian systems, originally proposed by Wigner in 1955. On the basis of numerical data we show that both the global structure of eigenstates and the level statistics obey a simple scaling law based on a single scaling parameter.

PACS number(s): 05.45.+b, 72.15.Rn

The first attempt to describe statistical properties of complex quantum systems by means of a random matrix model goes back to Wigner [1]. He introduced a band-random-matrix (BRM) model to describe conservative systems like atomic nuclei [2]. Specifically, he considered an ensemble of real, infinite, Hamiltonian matrices of the type

$$H_{mn}=\frac{m}{\rho}\delta_{mn}+v_{mn},\quad v_{mn}=v_{nm}\ . \tag{1}$$

Here, ρ is the mean level density. The off-diagonal matrix elements are random and statistically independent, with $\langle v_{mn}\rangle=0$ and $\langle v_{mn}^2\rangle=v^2$ for $|m-n|<b$, while $v_{mn}=0$ otherwise; b is the bandwidth. In particular, the simplest case of matrix elements with random signs $v_{mn}=\pm v$ was chosen in [1].

Wigner introduced the weighted level density

$$\rho_W(E;m)=\sum_l a_{ml}^2\delta(E-E_l)\ , \tag{2}$$

where a_{ml} are components of the eigenfunctions ψ_l of the Hamiltonian (1) in the physically significant unperturbed basis $\{\varphi_m\}$:

$$\psi_l=\sum_m a_{lm}\varphi_m\ , \tag{3}$$

and E_l are eigenvalues corresponding to ψ_l.

The weighted level density $\rho_W(E;m)$, termed *strength function* by Wigner, proved very useful in studies of quantum statistics, and is now called *local spectral density* (see, e.g., [3]). It characterizes the level density of the so-called *operative eigenfunctions* [4], which actually control the dynamics of the initial state φ_m.

The analytical evaluation of the density (2) turned out to be extremely difficult. Only in some limit cases was Wigner able to derive an explicit expression for it. In particular, for $\rho=\infty$ and $b\gg 1$, he obtained the semicircle law

$$\rho_W(E;m)=\frac{1}{4\pi v^2 b}\sqrt{8bv^2-E^2} \tag{4}$$

(rigorous proofs of the semicircle law are given in recent papers [5]).

After Wigner's pioneering work, BRM's were almost forgotten (curiously enough by Wigner himself [6]), apparently because of their mathematical inconvenience, namely, noninvariance with respect to basis rotation. Due to this, attention was paid mainly to full random matrices for which a fairly complete mathematical analysis has been developed [7]. However, full random Hamiltonian matrices can be used to describe only local statistical properties of spectra and were criticized by Dyson because of the "unphysical" semicircle law.

A physically meaningful approach to the analysis of global properties of Hamiltonian systems can be obtained by just going back to the original Wigner model with increasing diagonal elements (1). In this model the semicircle law holds for the weighted level density (2) only, while the total level density is approximately uniform in the semiclassical region. Moreover, in physical applications, the interaction of unperturbed states always has a finite range which determines the band structure of Hamiltonian matrices. For this reason, there has been a revival of interest in BRM's [8]. Particularly, in Refs. [9,10] spectrum statistics and the structure of eigenstates have been studied in the original Wigner model (1). Another source of interest in BRM's is related to solid-state physics where band matrices are widely used to describe dynamics of electrons in disordered solids. Here, the localization properties of eigenstates are important as well as

their relation to the spectrum properties.

In this paper we consider the model of infinite BRM's of type (1) with Gaussian distribution for off-diagonal elements. On the basis of numerical data [9,10] we show that the global structure of eigenfunctions can be described by a simple scaling theory based on a single scaling parameter which has a simple physical meaning. A similar scaling approach accounts very well also for energy-level statistics and gives a new insight into the connection between the statistical properties of eigenstates and those of the eigenvalues.

Our starting point is the semicircle law (4), which holds for sufficiently large level density [11–13] $\rho \gg 1$. The finite energy width of the semicircle distribution, $\Delta E = 4v\sqrt{2b}$, encompasses the whole spectrum and therefore allows for an estimate of the maximum number $l_\perp$ of unperturbed states that can be coupled by the perturbation

$$l_\perp \equiv c\rho\Delta E = 4c\rho v\sqrt{2b} \ , \tag{5}$$

where the numerical factor c (of the order one) depends on how $l_\perp$ is practically measured (see below). The physical meaning of $l_\perp$ is that of a maximal localization length of eigenstates in a number of unperturbed levels. We emphasize that $l_\perp$ is determined by the "energy shell" ΔE and makes no reference to any finite matrix size; for this reason $l_\perp$ will be called *transverse localization length* (across the energy shell), see [14]. The actual localization length l, however, is, in general, different. It depends on the various parameters in a complicated way [9,10], and in the limit case $\rho = \infty$ we have $l = l_\infty \sim b^2$ [15,16].

The key point in our approach is that all global properties of eigenfunctions are described by one *localization parameter:*

$$\beta_{\rm loc} = \frac{l}{l_\perp} \ , \tag{6}$$

which is expected to obey a scaling law, i.e., to depend only on the ratio of the two characteristic lengths l_∞ and $l_\perp$:

$$\beta_{\rm loc} = \beta_{\rm loc}(\lambda) \ , \quad \lambda = \frac{l_\infty}{l_\perp} = a\frac{b^{3/2}}{\rho v} \tag{7}$$

with some numerical factor $a \sim 1$. The scaling parameter λ may also be called *ergodicity parameter,* because when it is large, the localization length approaches its maximal value $l_\perp$, which means that the eigenfunctions become ergodic, i.e., completely delocalized within the energy shell. In the opposite case, when $l < l_\perp$, we speak of *longitudinal localization* (along the layer) [14].

The case of *finite* matrices of size N has already been investigated in some detail. In that case, N is another characteristic length and therefore, besides λ, one more scaling parameter appears:

$$\lambda_N = \gamma\frac{b^2}{N} \ . \tag{8}$$

In this case, the statistical properties have been conjectured to depend on both parameters λ and λ_N [$\beta_{\rm loc} = \beta_{\rm loc}(\lambda, \lambda_N)$] [9,10,15]. This was confirmed by numerical experiments [9,10,17], both for spectral statistics and for localization properties. In particular, for homogeneous ($\rho = \infty$) BRM's the parameter λ_N alone is sufficient; moreover, the scaling law was numerically found [16,18] to be (with $\gamma \approx 1.4$) [19]

$$\beta_{\rm loc}(\lambda_N) \approx \frac{\lambda_N}{1+\lambda_N} \ . \tag{9}$$

Recently, an analytical proof of this scaling law was given in [20].

Finite matrices are just an approximation for real conservative Hamiltonians; being interested in the latter, only those statistical properties that do not depend on finite truncation are physically relevant. Therefore, we are addressing here the case of *infinite* matrices. In numerical studies the matrices are obviously of finite size N, but some data from [9,10] pertain to large matrices, for which $\lambda \gg \lambda_N$ and the finite (but large) size of the matrix is not relevant. We have therefore used these very data in order to investigate our scaling assumption (7), which we claim to hold for infinite Hamiltonian matrices. Specifically, we have taken from [9,10] data for the dependence of the localization length l on the model parameters.

The definition of l that was used in numerical studies was that of "entropy localization length," namely,

$$l = N\exp(\langle H\rangle - H_{\rm GOE}) \ , \tag{10}$$

where H is the "entropy" of an eigenstate u_n:

$$H = -\sum_n |u_n|^2 \ln|u_n|^2 \tag{11}$$

and $\langle H\rangle$ is an average over all eigenvectors from an ensemble of random matrices with the same N, ρ, b, v. The normalization used in the expression (11) is such that in the limit case of full random matrices of size N the localization length l is equal to N [18]. To this end, the factor $\exp(-H_{\rm GOE})$ was introduced, which is related to the average entropy of eigenstates taken from the Gaussian orthogonal ensemble (GOE). The definition [(10) and (11)] has the same meaning as an effective number of unperturbed eigenstates covered, on average, by a single eigenvector.

In order to check the scaling (7) and to find the form of the scaling function $\beta_{\rm loc}(\lambda)$, one has to investigate the dependence of the localization length on the various parameters. Actually, the scaling (7) would imply that this dependence has the form

$$l(x) = b^2 f(x) \ , \tag{12}$$

where $x = \lambda/a$ and $f(x)$ is related to the scaling function $\beta_{\rm loc}$ via $\beta_{\rm loc} = c_0^{-1} x f(x)$ with $c_0 = \lim_{x\to\infty} x f(x)$. We have therefore analyzed the behavior of $xf(x) \equiv xb^{-2}l(x)$ as a function of x.

Data [9], and additional data kindly provided by M. Feingold, show that on increasing x at fixed $b = 3$–9, the product $xf(x)$ first increases towards a maximum value in the range 4.4–5.5, and then decreases approximately as $1/x$. In the latter region, perturbation theory is valid

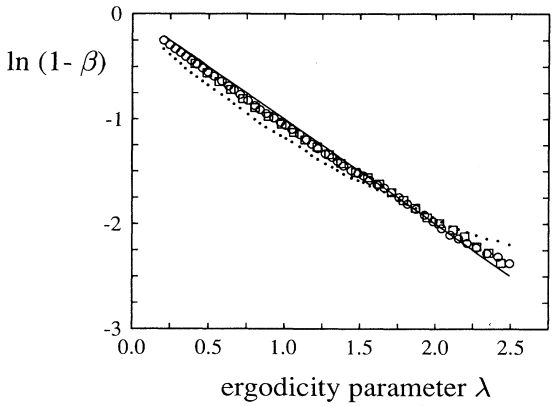

FIG. 1. Dependence of the localization (squares) and repulsion [dots for q and circles for β (16)] parameters on the ergodicity parameter λ as compared to the simple exponential scaling (14) (solid line): the rms deviation for $\beta_{\rm loc}-\beta$ is 0.7%.

[11,13] and there is no scaling of the form (7). Being mainly interested in nonperturbative effects, we have restricted ourselves to values of x below the maximum of $xf(x)$. This range is approximately defined [11] by the condition $x \leq x_b$, where

$$\frac{b}{x_b}=\frac{\rho v}{\sqrt{b}}\sim 0.3 \ . \tag{13}$$

The quantity b/x is a sort of perturbation parameter, giving the ratio of the rms perturbation ($\sim v\sqrt{b}$) to the full detuning ($\sim b/\rho$). The same quantity can also be interpreted as the ratio $l_\perp/b$ of the maximal (transverse) localization length to the number of directly coupled unperturbed states.

As b increases, the border x_b also increases, and the maximum of $xf(x)$ becomes broad and flat; thus the perturbative region shifts to higher and higher values of x, and in the limit it disappears. Therefore, in order to study the asymptotic behavior for large b, one has to consider the region $x \leq x_b$ only.

A detailed processing of the data in the region $x < x_b$ yields evidence of a scaling behavior of the form (7), with the scaling function (see Fig. 1),

$$\beta_{\rm loc}(\lambda)=\frac{xf(x)}{c_0}\approx 1-\exp(-\lambda)\equiv\beta_0(\lambda) \ , \tag{14}$$

where β_0 is defined by the last equality. The parameter c_0 is in principle defined as the limit of $xf(x)$ as $x\to\infty$, but we have instead determined it as a fitting parameter. Indeed, in order to directly find c_0 as the limit value of $xf(x)$, a long plateau is needed, and this requires new data for larger b than available. For the largest $b=9$, the mean value of $xf(x)$ on the plateau (which includes only seven points) is 5.42, while the value of c_0 obtained from the fitting (see Fig. 1) is $c_0=5.29$. To suppress fluctuations in numerical data for large λ we used the so-called moving window averaging over six neighboring points. A least-squares fit gives $c_0=5.29$, $a=0.216$ with a rms deviation of 6%. We would like to stress that the scaling (14) is quite different from (9) found for finite homogeneous BRM's. We also remark that our scaling theory leads, in particular, to Eq. (12), which was also suggested in [9,15] on the grounds of different arguments.

Although the agreement between numerical data and the scaling law (14) shown in Fig. 1 is overall quite good, there is also a small systematic deviation. This deviation $\beta_{\rm loc}-\beta_0$ can be substantially reduced by a slight change in the scaling (14), namely,

$$\beta_0(\lambda)\to\beta_1(\lambda)=1-\exp(-\lambda+\mu\lambda^2-\nu) \ , \tag{15}$$

with $\mu=0.039$, $\nu=0.055$ and a close value of $a=0.226$ ($c_0=5.29$ as before). Using the latter parameter values, the numerical factor $\gamma\approx ac_0\approx 1.2$ is also close to that previously found in different models [16]. The factor c in Eq. (5) is equal to 0.94, which corresponds to the definition [(10) and (11)] of localization length used in all the above-mentioned numerical experiments.

We now turn to the relation of the above-discussed global properties of the eigenfunctions to the statistical properties of the energy spectrum. A most widely used quantity for the description of the latter properties is the repulsion parameter for neighboring levels [7]. To determine this parameter one needs to compare numerical data with some analytical expression for the distribution of level spacings. In [10] this was done for the model (1) by using the so-called Brody distribution, which depends on one fitting parameter q. We used instead a different theoretical distribution, which appears more physically meaningful in the analysis of intermediate statistics caused by localization effects [18]. This distribution also depends on one *spectral parameter* β, which is approximately related to the Brody parameter q [21] in the following way:

$$\beta=0.654q+0.411q^2 \ . \tag{16}$$

This relation was obtained by a least-squares fitting of one distribution to the other.

In previous studies [18,22] of dynamical models it was conjectured and numerically supported that the localization parameter $\beta_{\rm loc}$ was close to the repulsion parameter β. In the present case, this conjecture is fully supported by numerical data, see Fig. 1. We emphasize that, in spite of some statistically significant deviations of both β and $\beta_{\rm loc}$ from the simple exponential scaling (14), the difference $\beta_{\rm loc}-\beta$ remains well within the range of statistical fluctuations in the whole range of available data for $q(\lambda)$. The rms fluctuations of $(\beta_{\rm loc}-\beta)/(1-\beta)$ are about 2% only. This in our opinion clearly indicates that the parameter β is much more suitable than the Brody parameter for the description of statistical properties of quantum chaos.

It would be very interesting to extend numerical experiments [9,10] to the region where perturbation theory begins to work, in order to follow the transition to the unperturbed system as represented by the diagonal matrix elements. Also, it is not completely clear whether the present results remain unchanged, with a different statistics of the diagonal matrix elements in the model (1), in particular, for a Poissonian statistics. Most likely, they

do, but a direct check would certainly be desirable.

In conclusion, we have provided evidence for a scaling law that holds for infinite BRM's of the form (1); besides that, we have shown that for this model the spectral parameter β and the localization parameter β_{loc} are surprisingly close. The latter striking result is still waiting a theoretical explanation.

We most gratefully thank M. Feingold, who supplied us with a great deal of published data and unpublished data of his own.

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