Stochastic tracking in nonlinear dynamical systems

Ioana Triandaf and Ira B. Schwartz
Special Project for Nonlinear Science, Code 6700.3, U.S. Naval Research Laboratory, Plasma Physics Division, Washington, D.C. 20375-5000
(Received 8 February 1993)

In a previous paper [Phys. Rev. A **46**, 7439 (1992)] we have introduced an alternative continuation method which does not require an analytical model, but only an experimental time series. Using a predictor-corrector technique, the method tracks a given unstable orbit through different bifurcation regimes by varying an accessible system parameter. In this method, the continuation parameter was varied deterministically. That is, the location of the parameter is chosen by the experimenter. In this paper we introduce a similar algorithm, but now the parameter is varied randomly. A correction procedure is introduced so that control of an unstable orbit is not lost as the parameter changes. Moreover, we show that the small-amplitude feedback-control technique used for correction allows large-amplitude bursts in the parameter. These features are useful to experimentalists for canceling drift in experiments, which is inevitable at some level.

PACS number(s): 05.45.+b, 02.70.−c, 02.50.−r

I. INTRODUCTION

When modeling a dynamical system, theoretical tools and corresponding numerical methods have been developed to determine orbits as a function of a given system parameter. These methods generate complete bifurcation diagrams in which both stable and unstable orbits are located. When an analytical model is known, these methods are called continuation or homotopy methods (see [1,2]). A parallel direction has been followed when the dynamical system is not known analytically, but only an experimentally measured time series is available, along with an accessible parameter.

The time delay embedding methods, initially introduced by Ruelle and Takens (see [3]), allow one to reconstruct an attractor in phase space from a time series. However, to generate a complete bifurcation diagram starting from a time series, it is still necessary to be able to trace unstable branches of solutions, as well as attracting branches, as a function of a single parameter.

Recently we have introduced an alternative continuation method used to track unstable periodic orbits when the process is a time series (see [4]). The versatility and utility of the tracking algorithm has been implemented in two different actual experiments (see [5] and [6]). The method is derived for discrete maps which are generated by flows by taking an appropriate Poincaré section. Specifically, we can track orbits of a map:

$$\mathbf{x}_{n+1}=f(\mathbf{x}_n,p)\ , \quad \mathbf{x}_n\mathbf{x}_{n+1}\in\mathbb{R}^n\ , \tag{1.1}$$

as a function of the parameter p, where f is not known analytically. Having x_n and p, we assume the time series for x_{n+1} is generated from a black box. Here the black box is an unknown map, corresponding to the Poincaré section of the time series created from an experiment.

In this method, we assume that we have access to a system parameter, which will be also our continuation parameter. It is a prediction-correction method, in which the prediction step is varied deterministically, by either the experimenter or by some computer-assisted mechanism.

In this paper we introduce a similar algorithm where we allow the same parameter to vary randomly. The reason we introduce such an algorithm is that virtually every experiment has some drift. A well-known example are the problems appearing in the Belousov-Zhabotinsky (BZ) reaction (see [7]). For example, if the flow rate is the bifurcation parameter in a BZ experiment, then it can be affected by the loss of flex in the plastic tubing of the peristaltic pump (see [8]). A more common problem in the BZ reaction is the presence of bubbles in the feed lines, which produce momentary but large perturbations at random times. Similar problems cropped up in the laser Doppler velocity (LDV) measurements of fluid velocity in the Couette-Taylor system (see [9]). The seed particles tended to clump and one needs a fairly uniform distribution of them in order to measure velocity with the laser. Clumped particles meant a momentary signal loss, since there were no particles from time to time in the laser beam (and the photodetector produced dropouts). In [10] experiments on the BZ reaction were done in order to elucidate the subharmonic intermittency route to chaos. This amounted to a very delicate bifurcation problem, the study of which was limited by the small flow rate drift.

We start the paper by reviewing the continuation algorithm in Sec. II below. In Sec. III we introduce the random step algorithm, which is also used for tracking an unstable orbit. However, this time the size as well as the direction of the step is random. We further show that the algorithm is able to recover the control of the orbit when large-amplitude bursts in the parameter are present. These bursts are isolated and exceed the maximum amplitude of the random step for which the method works. In Sec. IV, we present numerical results using the Hénon map and end the paper with a conclusion section.

II. REVIEW OF THE EXPERIMENTAL CONTINUATION METHOD

In what follows we are going to consider the deterministic method as it applies to the map (1.1) where the equations of the map are not explicitly known. Our continuation method is able to track an unstable periodic orbit over a wide parameter range using a prediction-correction technique. See [1] for details. Initially, we need a good approximation for the orbit for a few values of the parameter. Subsequent values of the orbit as we increase or decrease the parameter p will be determined by a prediction-correction technique.

We initialize the process by controlling a periodic orbit at two different, but close, values of the parameter p. The prediction step consists of taking the new value of the orbit, for the new parameter value, along the line (considered as a function of the parameter p) through the previous two orbits. (Using the parabola through the previous three points sometimes improves the parameter range over which tracking is possible.) We assume that the step in a parameter is made such that the orbit is still controllable. Simply increasing the parameter and taking the previous value of the orbit to be the predicted value may also work well (see [5]). The main working hypothesis is that the predicted orbit must always lie in the controllability region of the previously controlled orbit. If it does not, then the technique fails since control is lost at the new parameter value. We are going to address this issue in a separate theoretical paper.

Once a successful prediction step is made, it is followed by a correction. This correction reduces the error made in making a prediction. To correct this, we make use of any small-amplitude feedback linear control technique. This is used in conjunction with an estimate, which relates the error in prediction to the observed mean fluctuations in the parameter (see [1]) for details.

For the control technique, we used the Ott-Grebogi-Yorke (OGY) algorithm (see [11]) which was initially designed to stabilize unstable periodic orbits embedded in a chaotic attractor. Using our tracking algorithm with a deterministically adjusted parameter we were able to track through chaotic as well as nonchaotic regimes. In the OGY algorithm the linearized dynamics about the orbit we are tracking is considered. The control also involves approximating eigenvalues and eigenvectors of the orbit, all of which can be obtained by prediction so that a model is not required.

III. THE RANDOM-WALK CONTROL

In this section we will describe in detail the algorithm with random step. We consider also the issue of the robustness of the scheme under large bursts in the parameter.

In what follows we introduce an algorithm which tracks a given orbit of (1.1). The map f is used to generate a time series and, as in the continuation method, we have access to a system parameter. However, this time we allow the parameter to vary randomly.

Suppose for simplicity that the unstable orbit is a fixed point of the map (1.1) and let $\mathbf{x}_F(p_m)$ denote the current fixed point, where m is the number of random steps previously taken. Then we stochastically increase the parameter to $p=p_m+\epsilon$, where ϵ is a random number chosen uniformly between $-\alpha$ and α, where α stands for the amplitude of the step. We assume α is such that control will not be lost (for the Hénon map $\alpha=0.04$; see the example below). Then we approximate the value of the orbit at this new parameter value $\mathbf{x}_{\rm fix}(p)$, by $\mathbf{x}_F(p_m)$; i.e., by the preceding value of the orbit before the parameter drifted.

Next we are going to correct this approximation by using the correction procedure as in the deterministic continuation method. That means we start by applying the OGY algorithm. We slightly change the parameter to some value $p+\delta p$, where δp is to be determined. The idea is to ensure that the next iterate $\mathbf{x}_{n+1}$ in (1.1) will fall on the stable manifold of the predicted orbit, $\mathbf{x}_{\rm fix}(p)$.

We now take the total linear approximation for the map:

$$\mathbf{x}_{n+1}-\mathbf{x}_{\rm fix}(p)\cong\delta p\,\mathbf{g} +[\lambda_u\mathbf{e}_u\mathbf{f}_u+\lambda_s\mathbf{e}_s\mathbf{f}_s]\cdot[\mathbf{x}_n-\mathbf{x}_{\rm fix}(p)-\delta p\,\mathbf{g}]\;. \tag{3.1}$$

In the correction step we will iterate the above equality, taking initially $x_1=x_F(p_m)$; i.e., the previously controlled fixed point. In the above we approximated

$$\mathbf{g}=\left.\frac{\partial\xi_F(p)}{\partial p}\right|_{p=0}\cong\frac{1}{p}[\mathbf{x}_n-\mathbf{x}_{\rm fix}(p)]\;,$$

and $\mathbf{e}_u$, $\mathbf{e}_s$, λ_u, and λ_s are eigenvectors and eigenvalues of the Jacobian of the corresponding map. $\mathbf{f}_s$ and $\mathbf{f}_u$ are contravariant basis vectors defined by $\mathbf{f}_s\cdot\mathbf{e}_s=\mathbf{f}_u\cdot\mathbf{e}_u=1$, $\mathbf{f}_s\cdot\mathbf{e}_u=\mathbf{f}_u\cdot\mathbf{e}_s=0$.

These eigenvectors and eigenvalues are obtained by linear interpolation, from the previous values of the orbit. Another way to evaluate these eigenvalues and eigenvectors would be to determine the matrix of the linearization (3.1) through the least-squares solution of a set of linear equations, having the form (3.1). This technique will be presented more accurately in a future paper. In (3.1) we choose δp in such a way that the next iterate $\mathbf{x}_{n+1}$, falls on the stable manifold of the predicted fixed point. That means we must have

$$\mathbf{f}_u\cdot(\mathbf{x}_{n+1}-\mathbf{x}_{\rm fix})=0\;. \tag{3.2}$$

From (3.2) and the approximation (3.1), we get that

$$\delta p\equiv\frac{\lambda_u[\mathbf{x}_n-x_{\rm fix}(p)]\cdot\mathbf{f}_u}{(\lambda_u-1)\mathbf{g}\cdot\mathbf{f}_u}\;. \tag{3.3}$$

As in the OGY method we change p to $p+\delta p$ only if the fluctuation in the parameter is small; otherwise we take $\delta p=0$. However, taking x_{n+1} as the corrected value is not sufficient. The essential observation we made in order to correct the error made by prediction is that the error in the fixed point is proportional to the mean of the fluctuations in the parameter as we apply OGY repeatedly.

Therefore in order to correct the orbit, in an experiment, we would iterate OGY repeatedly and record the corresponding fluctuations in the parameter, δp. Then we would move the fixed point $x_{\rm fix}(p)$ slightly until the mean in the fluctuations of the parameter is brought as close to zero as possible.

To see why this correction works, let $\xi = \mathbf{x}_F(p) - \mathbf{x}_{\rm fix}(p)$ denote the error in the predicted fixed point introduced by prediction. Here $\mathbf{x}_F(p)$ is the real fixed point and $\mathbf{x}_{\rm fix}(p)$ is the predicted fixed point. We rewrite (3.3) as follows:

$$\delta p_n = \frac{\lambda_u (\mathbf{x}_n - \mathbf{x}_F)\cdot \mathbf{f}_u}{(\lambda_u - 1)\mathbf{g}\cdot \mathbf{f}_u} + \frac{\lambda_u \xi \cdot \mathbf{f}_u}{(\lambda_u - 1)\mathbf{g}\cdot \mathbf{f}_u} , \tag{3.4}$$

where n stands for the number of times we applied the OGY algorithm.

Taking the temporal average in Eq. (3.4), we see the first term has the mean value zero since $\mathbf{x}_F$ is the real fixed point, by hypothesis. In the second term all quantities are known except the error vector ξ. So taking the mean in both sides of Eq. (3.4) over a large number of iterates we get that

$$|\langle \delta p_n \rangle| = \left| \frac{\lambda_u \xi \cdot \mathbf{f}_u}{(\lambda_u - 1)\mathbf{g}\cdot \mathbf{f}_u} \right| , \tag{3.5}$$

which clearly shows that $|\langle \delta p_n \rangle|$ is proportional to ξ. Thus the control point may be moved in some small ball about the exact fixed point such that $|\langle \delta p_n \rangle|$ is minimized. This ensures that ξ, the error in the fixed point, is minimized.

In practice, the correction procedure requires sampling several control iterates of the fluctuations to get a mean value of the error made in predicting the new fixed point. The assumption that is required, therefore, is that the parameter drifts slowly compared to the sampling rate. That is, if the drift is sufficiently slow, then the parameter appears to be approximately constant during the correction procedure.

On the other hand, this technique is also successful in canceling the effects of bursts in the parameter. As we track an orbit by using the above algorithm, a maximum amplitude is set for the random step, which depends on the particular system being studied. Still, the correction step is able to recover the exact orbit even when we allow large random jumps in the parameter, which exceed this amplitude by an order of magnitude (see example in next section).

IV. NUMERICAL EXAMPLES

We demonstrate our scheme using the Hénon map, which is given by the equations

$$x_{n+1} = A - x_n^2 + By_n ,$$

$$y_{n+1} = x_n ,$$

where we take $B = 0.3$ and $A = A_0 + p$, where $A_0 = 1.29$ and p is the control parameter. For this value of A_0, the attractor of the map is chaotic so it contains a dense set of unstable periodic orbits.

In practice we always assume measurement noise is present. This will appear in (3.1) as an additive term $\epsilon\delta_n$, where δ_n is a random variable with zero mean ($\langle \delta_n \rangle = 0$), satisfies ($\langle \delta_n \delta_m \rangle = 1$ for $m \neq n$) and ($\langle \delta_n^2 \rangle = 1$), and has a probability density independent of n.

In a first example we track a period-two orbit, using the deterministic continuation algorithm (see [1]). The orbit as a function of the parameter A is shown in Fig. 1(a) and the corresponding relative error is shown in Fig. 1(b). Notice that for parameter values greater than $A = 1.4$, all finite attractors disappear and there is only an attractor at $-\infty$. Therefore, the tracking method allows control to be continued into nonchaotic regions.

In the application of the control part of the algorithm, both fixed points of the period-two orbit were used, and control was implemented at every iterate. This is in contrast to the original technique, in which control would be implemented at every other iterate by considering the orbit as a fixed point of the second iterate of the map. By complementing control at every iterate, performance is improved because the orbit has less time to wander away from the true fixed point as compared to controlling every other iterate.

Next we demonstrate the random-walk control algorithm for a period-one orbit of the same map as in the previous example. In this example we initialized at $A_0 = 1.39$ and take 300 random steps. Figure 2(a) shows the orbit as a function of the now random parameter A and Fig. 2(b) shows the corresponding relative error. The random walk of A wanders both into the chaotic region (which ends at $A = 1.4$) and outside it. The parameter was randomly adjusted using a uniform probability distribution. Figure 2(c) shows the corresponding random steps as a function of the iterate. For controlling orbits in regions which are very close to boundary crisis [12], this method works very well, since it is very hard to isolate the parameter exactly at the crisis value (see [10]).

In the next example we present the control of a

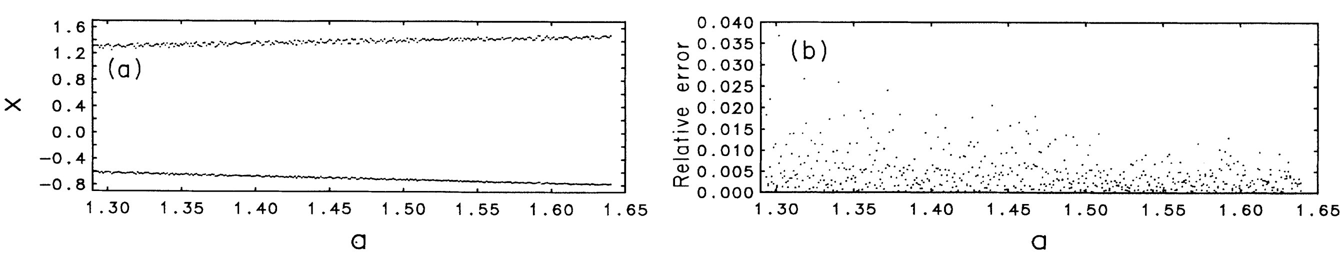

FIG. 1. (a) Deterministic continuation. x_n vs A for a period-two orbit of the Hénon map, $A_0 = 1.29$, $B = 0.3$, noise = 0.01. (b) The relative error vs A.

period-one orbit of the Hénon map, when large bursts in the parameter are present. In Fig. 3(a), we can see the controlled period-one orbit, and in Fig. 3(b) the corresponding relative error is shown. In this example we introduced large-amplitude bursts in the parameter once every 50 iterates. These bursts are random in amplitude. The parameter A discontinuously increases by a random quantity which may be larger than α, then returns to a fixed value, $A=1.29$. As that happens, a large error will appear in the controlled orbit. In order to correct for this error, we apply the correction step of the random-walk control algorithm, described above. That means we apply OGY repeatedly, record the corresponding values of the fluctuations in the parameter, and then use (3.5) to correct. By taking the mean of δp_n over a large number of iterates, the large burst in the parameter is smoothed out. This procedure recovers control of the orbit.

Notice that using the error estimate (3.5) to correct is essential. Suppose instead that we correct in a different way. At each new iterate we apply OGY several times. Then we average the values of the orbit, x_n at the new parameter value, obtained in this way. We then take the averaged value of the fixed point as the corrected value. The results of this procedure are shown in Fig. 4(a) along with the error estimates in Fig. 4(b). We see that the tracking procedure fails, because the average value of the iterate does not necessarily lie near the true fixed point.

A final example shows that the correction procedure can regain control even when the burst in the parameter holds for several iterates. We allow the parameter to increase by a random quantity of amplitude $\alpha=0.15$, and stay at the wrong value for several iterates. The controlled orbit in that case is shown in Fig. 5(a), where bursts occur every 50 iterates and are held at the wrong value for 50 more iterates. Figures 5(b) and 5(c) are as in Figs. 3(b) and 3(c). This shows that by using the correction procedure, accurate control is recovered when large parameters are sustained.

We remark that in the last two examples above, correction was applied at each iterate, whether there was a burst in the parameter or not. As a result, the effect of noise is reduced compared to the case when OGY control alone is applied. For comparison see [1].

We exemplified the control of bursts in the parameter when we are controlling an orbit at a fixed parameter

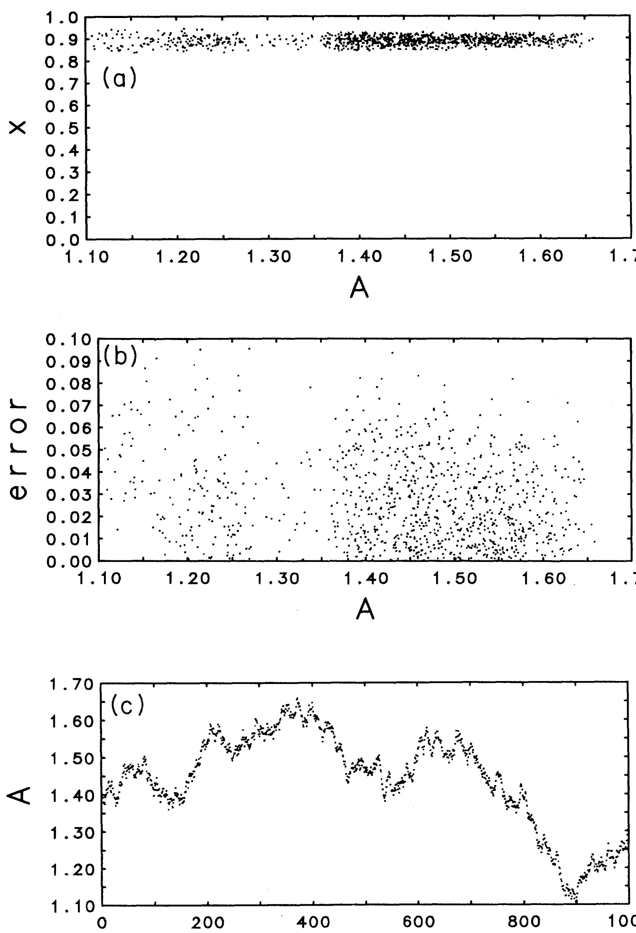

FIG. 2. (a) x_n vs A for a period-one orbit of the Hénon map, using random-walk control $A_0=1.39$, $B=0.3$, noise = 0.01. (b) The relative error vs A. (c) The parameter A vs n.

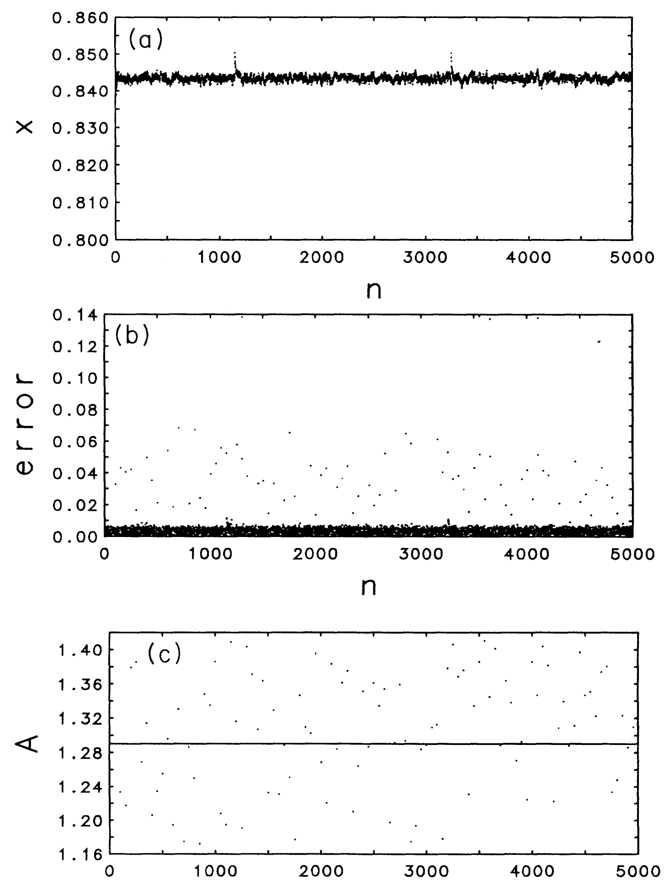

FIG. 3. (a) x_n vs n when controlling a period-one orbit of the Hénon map, $A_0=1.29$, $B=0.3$ bursts in the parameter occur every 50 iterates. (b) The relative error vs n. (c) The parameter A vs n.

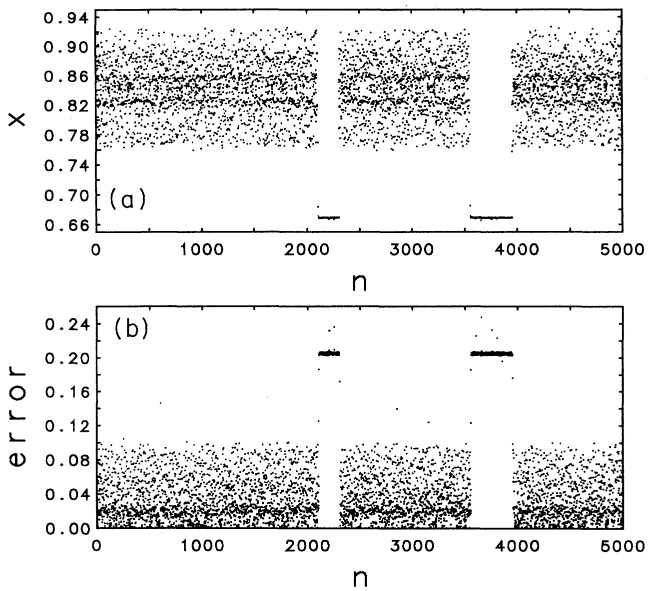

FIG. 4. (a) x_n vs n when controlling a period-one orbit of the Hénon map, the error estimate (4.5) is not applied $A_0=1.29$, $B=0.3$ bursts in the parameter occur every 50 iterates. (b) The relative error vs n.

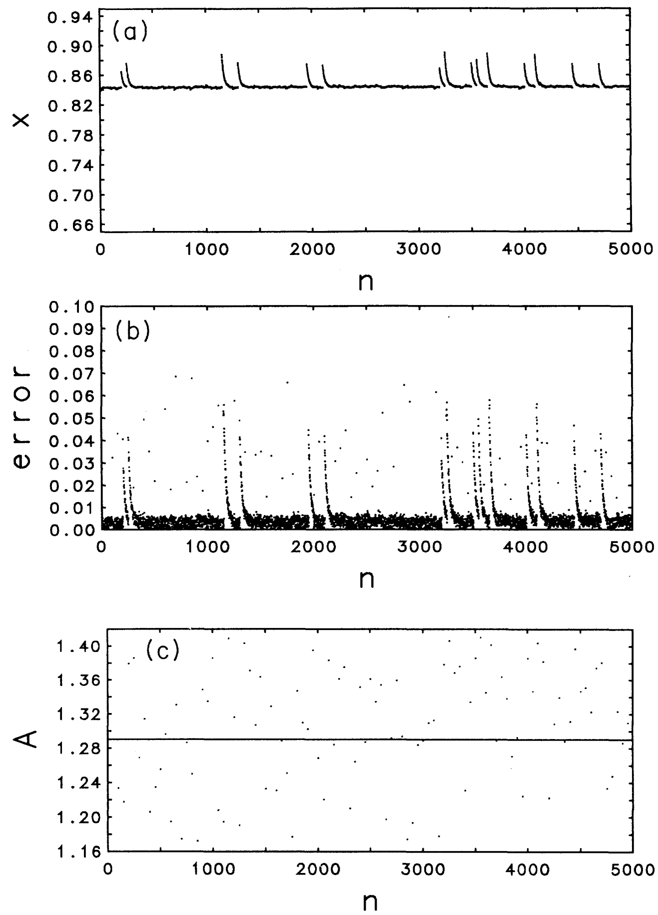

FIG. 5. (a) x_n vs n when controlling a period-one orbit of the Hénon map, the bursts in the parameter occur every 50 iterates and hold for 50 iterates. $A_0=1.29$, $B=0.3$ bursts in the parameter occur every 50 iterates. (b) The relative error vs n. (c) The parameter A vs n.

value. This is in fact only the correction step in the continuation algorithm or in the random-walk control. If we showed the picture for random-walk control, with bursts in the parameter, these bursts would not have been noticeable, since only the corrected value of the orbit would be shown, without the intermediate values appearing in the correction step. So the performance of the scheme would not have been so obvious.

V. CONCLUSIONS

We have introduced a practical algorithm for tracking stable as well as unstable orbits when the process is a time series. This is achieved by a small-amplitude control of an accessible system parameter along with a suitable error estimate. The novelty of this procedure, in addition to the fact that it does not require an analytical model, is that it compensates for a random step in the parameter. This feature cancels the effect of parametric drift, thus improving experimental resolution. It also is a direct measure of the rate at which the drift is occurring, which is usually discovered by postprocessing the data and analyzing it statistically.

ACKNOWLEDGMENT

I.T. gratefully acknowledges the support of the Office of Naval Technology for conducting this research.

[1] Werner C. Rheinboldt, University of Pittsburgh Technical Report No. ICMA-79-04, (1979) (unpublished).

[2] Tien-Yien Li and James A. Yorke (unpublished).

[3] F. Takens, in *Detecting Strange Attractors in Turbulence*, edited by D. Rand and L. S. Young, Lecture Notes in Mathematics Vol. 898 (Springer-Verlag, Berlin, 1981).

[4] Ira Schwartz and Ioana Triandaf, Phys. Rev. A **46**, 7439 (1992).

[5] Thomas L. Carroll, Ioana Triandaf, Ira Schwartz, and Lou Pecora, Phys. Rev. A **46**, 6189 (1992).

[6] Zelda Gills, Christina Iwata, Raiarshi Roy, Ira Schwartz, and Ioana Triandaf, Phys. Rev. Lett. **69**, 3169 (1992).

[7] Eric Kostelich (private communication).

[8] Reuben H. Swinney, Alan Wolf, and Harry L. Swinney, Phys. Rev. Lett. **49**, 245 (1982).

[9] A. Brandstater and Harry L. Swinney, Phys. Rev. A **35**, 2207 (1987).

[10] N. Kreisberg, W. D. Cormick, and Harry L. Swinney, Physica D **50**, 463 (1991).

[11] Edward C. Ott, Celso Grebogi, and James A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).

[12] C. Grebogi, E. Ott, and J. A. Yorke, Physica D **7**, 181 (1983).