Discrete Čerenkov power spectrum

E. B. Manoukian*
Physics Department, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand
(Received 4 September 1992)

A field-theoretical derivation is given for the Čerenkov radiation off a charged particle moving in an isotropic permeable medium trapped in a hollow cylinder with a perfectly conducting neutral surface. An exact expression is derived for the power spectrum and the explicit allowed discrete frequencies are obtained. A Čerenkov counter and the observation of the resulting spectrum would not only provide information on the presence of a charged particle, in the conventional sense, but would also indicate the presence of an enveloping surface to the medium and provide information on the "geometry" of the surface.

PACS number(s): 41.60.Bq, 12.20.Ds, 11.10.−z, 03.50.−z

I. INTRODUCTION

Much theoretical work has been done on the classic Čerenkov radiation [1] off a charged particle (see, e.g., Refs. [2–5]) in a medium. This radiation is emitted by the charged particle when moving in the medium with a spread greater than that of the speed of light in the medium. In this paper, we give a field-theoretical derivation of the Čerenkov radiation off a charged particle moving in an isotropic permeable medium trapped in a hollow (infinite) cylinder with a perfectly conducting neutral surface. We obtain an exact expression for the power spectrum and the explicit allowed discrete radiation frequencies. For simplicity, the charged particle is made to move along the axis of the cylinder and, as in the classic computations, any recoil of the particle is neglected. In Sec. II, the explicit boundary conditions on the causal Green's function are studied and the explicit expression for the Green's function is derived in the absence of the filling medium. Section III deals with the so-called vacuum-to-vacuum transition amplitude (see, e.g., Refs. [3,5–7]) of field theory of the problem in the presence of the medium, the conducting cylinder, and the charged particle. The explicit expression for the power spectrum is derived (Sec. III). The motivation of the work is that to go beyond the classic spectrum with a so-called Čerenkov counter, and the observation of the resulting spectrum of the problem, would not only indicate the presence of a charged particle, in the conventional sense, but would also indicate the presence of an enveloping surface to the medium and provide some information on its "geometry." Throughout, the index of refraction is assumed to be independent of the radiation frequency, i.e., the dispersion is ignored. The generalization of the results to include indices of refraction that are frequency dependent is beyond the scope of the present paper and will be dealt with elsewhere. Section IV deals with our main conclusions and additional comments.

II. GREEN'S FUNCTION AND BOUNDARY CONDITIONS

Maxwell's equations in an isotropic permeable medium (in rationalized cgs units) may be written as (see, e.g., Ref. [5])

$$\partial_\mu \bar{F}^{\mu\nu} = -\frac{\bar{J}^\nu}{\bar{c}} , \tag{1}$$

and in a standard notation

$$D^i = \epsilon E^i, \quad B^i = \mu H^i , \tag{2}$$

where

$$\begin{aligned} &\bar{F}^{ij} = \sqrt{\mu}\epsilon^{ijk} H_k , \\ &\bar{F}^{0i} = \sqrt{\epsilon} E^i , \\ &X^0 = \bar{c}t, \quad \bar{c} = \frac{c}{\sqrt{\mu\epsilon}}, \quad X^i = x^i , \\ &\partial_\mu = \left[\frac{\partial}{\partial X^0}, \frac{\partial}{\partial X^i} \right] , \\ &\bar{J}^\nu = \left[\frac{\bar{c}\rho}{\sqrt{\epsilon}}, \frac{J^i}{\sqrt{\epsilon}} \right], \quad \partial_\mu \bar{J}^\nu = 0 . \end{aligned} \tag{3}$$

We first consider the case of a vacuum, i.e., for which $\mu=1$, $\epsilon=1$. Throughout we work in the radiation gauge $A^0=0$. The Green's function $D^{ij}(x,x')$ $=\langle 0_+|[A^i(x)A^j(x')]_+|0_-\rangle_0/\langle 0_+|0_-\rangle_0$ in this case satisfies the well-known differential equation (see, e.g., Refs. [8,5])

$$(-\Box\delta^{ij} + \partial^i\partial^j) D^{jk}(x,x') = \delta^{ik}\delta^4(x,x') , \tag{4}$$

where $i,j=1,2,3$, $x=(x^0,x^i)$, and $\mathbf{x}=(r\cos\theta, r\sin\theta, z)$. In particular, the electric-field components are then given by

$$\langle E^i(x)\rangle = \partial^0 \int (dx') D^{ij}(x,x') \frac{J^j(x')}{c} . \tag{5}$$

Since

$$\nabla\cdot\mathbf{E} = \rho , \tag{6}$$

we have

$$\partial^0\partial^i \int (dx') D^{ij}(x,x') J^i(x') = J^0(x) . \tag{7}$$

From the current conservation

$$\partial_\mu J^\mu = 0 \tag{8}$$

1063-651X/93/48(1)/543(5)/$06.00

we may then infer that

$$\partial^i D^{ij}(x,x') = \frac{\partial^j}{(\partial^0)^2}\delta^4(x,x') . \quad (9)$$

Hence (4) may be rewritten as

$$-\Box D^{ij}(x,x') = \left[\delta^{ij} - \frac{\partial^i\partial^j}{(\partial^0)^2}\right]\delta^4(x,x') . \quad (10)$$

The support of the current J^μ is well confined within the cylinder far away from its surface. In particular, the charged particle, described by J^μ, is restricted to move along the axis of the cylinder (the z axis) and hence $J^1=0$, $J^2=0$. Only the $D^{33}(x,x')$ is relevant to the analysis dealing with the vacuum-to-vacuum transition amplitude (see Sec. III). From (10) the latter satisfies the differential equation

$$-\Box D^{33}(x,x') = \left[1 - \frac{(\partial^3)^2}{(\partial^0)^2}\right]\delta^4(x,x') . \quad (11)$$

To solve (11) we need the boundary condition (BC) to be imposed on $D^{33}(x,x')$. This may be extracted from (5). Since $\langle E^3(x)\rangle = 0$ at $r=a$, where a is the radius of the cylinder, the BC to be imposed in (11) is

$$D^{33}(x,x')|_{r=a} = 0 . \quad (12)$$

We may expand $D^{33}(x,x')$ in terms of Bessel functions $J_L(\gamma_i^L r)$ of arbitrary integer order L:

$$D^{33}(x,x') = \sum_{i=1}^{\infty} C_{iL} J_L(\gamma_i^L r), \quad J_L(\gamma_i^L a) = 0 , \quad (13)$$

where C_{iL} depends also on r',θ,θ',z,z'. To solve (11) we make the following expansions:

$$\delta(\theta-\theta') = \frac{1}{2\pi}\sum_{N=-\infty}^{\infty} e^{iN(\theta-\theta')} , \quad (14)$$

$$\frac{\delta(r,r')}{r} = \frac{2}{a^2}\sum_{i=1}^{\infty}\frac{J_L(\gamma_i^L r)J_L(\gamma_i^L r')}{[J_{L+1}(\gamma_i^L a)]^2} , \quad (15)$$

$$\delta(z-z') = \int_{-\infty}^{\infty}\frac{dq}{2\pi}e^{iq(z-z')} , \quad (16)$$

$$\delta(x^0-x'^0) = \int_{-\infty}^{\infty}\frac{dQ^0}{2\pi'}e^{-iQ^0(x^0-x'^0)} , \quad (17)$$

where

$$\delta^4(x,x') = \frac{\delta(r,r')}{r}\delta(\theta-\theta')\delta(z-z')\delta(x^0-x'^0) . \quad (18)$$

Also in cylindrical coordinates

$$\Box = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^{02}} , \quad (19)$$

and we have for the Bessel differential equation

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + (\gamma_i^L)^2\right]J_L(\gamma_i^L r) = \frac{L^2}{r^2}J_L(\gamma_i^L r) . \quad (20)$$

Hence from (11) the solution for $D^{33}(x,x')$ is

$$D^{33}(x,x') = \int_{-\infty}^{\infty}\frac{dq}{2\pi}e^{iq(z-z')}\int_{-\infty}^{\infty}\frac{dQ^0}{2\pi}e^{-iQ^0(x^0-x'^0)}\left[1-\frac{q^2}{Q^{02}}\right] \times \frac{1}{2\pi}\sum_{N=-\infty}^{\infty}e^{iN(\theta-\theta')}\frac{2}{a^2}\sum_{i=1}^{\infty}\frac{J_N(\gamma_i^N r)J_N(\gamma_i^N r')}{[(\gamma_i^N)^2+q^2-Q^{02}-i\delta][J_{N+1}(\gamma_i^N a)]^2}, \quad \delta\to 0+ . \quad (21)$$

$J_N(\gamma_i^N a)=0$, where for a given N, the integer L in (15) was chosen to be equal to N.

Since the charged particle moves along the z axis, that is, we have to consider $r=0, r'=0$ (see Sec. III), and only $J_0(0)\neq 0$ [$J_0(0)=1$] for the Bessel functions in (21), only the $N=0$ term will contribute in the sum in (21) for $r=0$, $r'=0$, i.e.,

$$D^{33}(x,x')|_{r=0,r'=0} = \int_{-\infty}^{\infty}\frac{dq}{2\pi}e^{iq(z-z')}\int_{-\infty}^{\infty}\frac{dQ^0}{2\pi}e^{-iQ^0(x^0-x'^0)} \times \left[1-\frac{q^2}{Q^{02}}\right]\frac{1}{\pi a^2}\sum_{i=1}^{\infty}\frac{1}{[\gamma_i^2+q^2-Q^{02}-i\delta][J_1(\gamma_i a)]^2}, \quad \delta\to 0+ \quad (22)$$

where now

$$J_0(\gamma_i a) = 0 \quad (23)$$

defines the zeros of the Bessel function J_0 of zeroth order.

III. POWER SPECTRUM

The vacuum-to-vacuum transition amplitude (see, e.g., Refs. [3,5–7]) is written down from the elementary dimensional scaling defined in (3) to be simply

$$\langle 0_+|0_-\rangle = \exp\left[\frac{i}{2\hbar c^3}\int (dX)(dX')\bar{J}^3(x)D^{33}(X,X')\bar{J}^3(x')\right] . \tag{24}$$

By using the conservation law $\partial_\mu \bar{J}^\mu = 0$ in (3), the latter may be rewritten as

$$\langle 0_+|0_-\rangle = \exp\left[\frac{i}{2\hbar c^3}\frac{n}{\epsilon}\int (dx)(dx')[J^3(x)J^3(x') - \frac{c^2}{n^2}\rho(x)\rho(x')]D(X,X')\right] \tag{25}$$

where, in particular ($n = \sqrt{\mu\epsilon}$),

$$D(X,X')|_{r=0,r'=0} = \frac{1}{\pi a^2}\int_{-\infty}^{\infty}\frac{dq}{2\pi}e^{iq(z-z')}\int_{-\infty}^{\infty}\frac{dQ^0}{2\pi}e^{-i(Q^0/n)c(t-t')}\sum_{i=1}^{\infty}\frac{1}{[\gamma_i^2+q^2-Q^{02}-i\delta][J_1(\gamma_i a)]^2} , \quad \delta\to 0+ . \tag{26}$$

The charged particle will be described in ($J^1=0$, $J^2=0$)

$$J^3(x) = evF(z-vt)\frac{\delta(r)}{r}\delta(\theta) , \tag{27}$$

$$\rho(x) = eF(z-vt)\frac{\delta(r)}{r}\delta(\theta) , \tag{28}$$

where v is its speed, and the explicit form of the function $F(z-vt)$ will be specified later. The vacuum-to-vacuum transition amplitude may be then rewritten as

$$\langle 0_+|0_0\rangle = e^{iW} , \tag{29}$$

$$W = \frac{ne^2v^2}{2\pi\hbar ca^2\epsilon}\left[1-\frac{1}{\beta^2n^2}\right]\int dz\, dz'dt\, dt'F(z-vt)F(z'-vt') \times\int_{-\infty}^{\infty}\frac{dQ^0}{2\pi}e^{-i(cQ^0/n)(t-t')}\int_{-\infty}^{\infty}\frac{dq}{2\pi}e^{iq(z-z')}\sum_{i=1}^{\infty}\frac{1}{[\gamma_i^2+q^2-Q^{02}-i\delta][J_1(\gamma_i a)]^2} , \quad \delta\to +0 \tag{30}$$

where $\beta = v/c$.

The average number of photons emitted by the charged particle is then [5,7,9–11]

$$\langle N\rangle = \frac{ne^2v^2}{\pi\hbar ca^2\epsilon}\left[1-\frac{1}{\beta^2n^2}\right]\int_{-\infty}^{\infty}\frac{dq}{2\pi}\sum_{i=1}^{\infty}\frac{|F(q,Q_i^0)|^2}{2\sqrt{\gamma_i^2+q^2}[J_1(\gamma_i a)]^2} \tag{31}$$

and hence we must have $\beta n \geq 1$, where $Q_i^0 = +(\gamma_i^2+q^2)^{1/2}$. Equation (31) may be rewritten more conveniently as

$$\langle N\rangle = \frac{ne^2v^2}{2\pi\hbar ca^2\epsilon}\left[1-\frac{1}{\beta^2n^2}\right]\sum_{i=1}^{\infty}\int_{-\infty}^{\infty}dt\left\{\int_0^{\infty}e^{-i\omega t}d\omega\,\delta\left[\omega-\frac{cQ_i^0}{n}\right] \times\int_{-\infty}^{\infty}\frac{dq}{2\pi}\int_{-\infty}^{\infty}dz\,e^{iqz}\frac{F(z-vt)F^*(q,Q_i^0)}{2\sqrt{\gamma_i^2+q^2}[J_1(\gamma_i a)]^2}+\text{c.c.}\right\} , \tag{32}$$

where

$$F(q,Q^0) = \int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dt\,e^{iqz}e^{-i(Q^0c/n)t}F(z-vt) , \tag{33}$$

$$Q_i^0 = +\sqrt{\gamma_i^2+q^2} . \tag{34}$$

The power spectrum may be then easily read from (32), in the spirit of Refs. [3,5,7], by making use of the fact that the energy of a photon is $\hbar\omega$, to be

$$P(\omega) = \frac{ne^2v^2}{2\pi ca^2\epsilon}\left[1-\frac{1}{\beta^2n^2}\right]\sum_{i=1}^{\infty}\frac{\omega\delta\left[\omega-\dfrac{cQ_i^0}{n}\right]}{2\sqrt{\gamma_i^2+q^2}(J_1(\gamma_i a))^2}\left\{e^{-i\omega t}\int_{-\infty}^{\infty}dz\,e^{iqz}F(z-vt)F^*(q,Q_i^0)+\text{c.c.}\right\} . \tag{35}$$

The function $F(z-vt)$ is given simply to be $\delta(z-vt)$ and hence

$$F(q,Q^0)=\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dt\, e^{iqz}e^{-(ic/n)Q^0t}\delta(z-vt)$$

$$=2\pi\delta\left[vq-\frac{cQ^0}{n}\right] \tag{36}$$

and

$$P(\omega)=\frac{2ne^2v^2}{ca^2\epsilon}\left[1-\frac{1}{\beta^2n^2}\right]$$

$$\times\sum_{i=1}^{\infty}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{\omega\delta(\omega-\frac{c}{n}Q_i^0)}{2Q_i^0[J_1(\gamma_i a)]^2}\delta(qv-\omega) \tag{37}$$

or

$$P(\omega)=\frac{e^2v}{2\pi a^2\epsilon}\left[1-\frac{1}{\beta^2n^2}\right]$$

$$\times\sum_{i=1}^{\infty}\frac{\delta\left[\omega-\frac{c}{n}\left[\gamma_i^2+\frac{\omega^2}{v^2}\right]^{1/2}\right]}{[J_1(\gamma_i a)]^2} . \tag{38}$$

Because of the δ function we must have

$$\omega^2\left[1-\frac{1}{\beta^2n^2}\right]=\frac{c^2}{n^2}\gamma_i^2>0 , \tag{39}$$

and hence $P(\omega)$ is nonzero *only for* $\beta n>1$. Upon setting

$$\lambda_i=\gamma_i a , \tag{40}$$

the expression in (38) simplifies finally to

$$P(\omega)=\frac{e^2v}{2\pi\epsilon a^2}\sum_{i=1}^{\infty}\frac{\delta(\omega-\omega_i)}{[J_1(\lambda_i)]^2},\quad \beta n>1 \tag{41}$$

with the discrete frequencies ω_i given by

$$\omega_i=\frac{c\lambda_i}{an}\left[1-\frac{1}{\beta^2n^2}\right]^{-1/2} , \tag{42}$$

where

$$J_0(\lambda_i)=0,\quad i=1,2,\ldots , \tag{43}$$

$\beta=v/c$, $n=\sqrt{\epsilon\mu}$, and a is the radius of the cylinder. The first few zeros λ_i of J_0 in (43) are (see, for example, Ref. [12], p. 409) 2.41,5.52,8.65,11.79,14.93,. . . . For a given medium (i.e., a given n) a given cylinder radius a, and a given speed v ($v>c/n$) the allowed frequencies ω_i may be then explicitly calculated from (42).

IV. SUMMARY AND DISCUSSION

For example, for water $n=1.33$, and for a relative speed $\beta=0.8$, the frequencies (written in dimensionless form) are given by

$$\omega_i s\simeq\frac{2.1\times10^9}{(a/m)}\left[\frac{\lambda_i}{\pi}\right] , \tag{44}$$

where s stands for a second and m stands for a meter, and the zeros of the zeroth-order Bessel function $J_0(\lambda_i)$ are given [12] with sufficient accuracy even for small i:

$$\frac{\lambda_i}{\pi}\simeq\left[i-\frac{1}{4}+\frac{1}{20}\frac{1}{(4i-3)}\right] . \tag{45}$$

By a filtering process, an experimentalist may measure the power spectrum in a narrow range about a typical frequency: $\underline{\omega}\le\omega\le\overline{\omega}$. For blue light: $\omega s\simeq3.878\times10^{15}$, which according to (41) would be given by

$$\int_{\underline{\omega}}^{\overline{\omega}}P(\omega)d\omega=\frac{e^2v}{2\pi\epsilon a^2}\sum_i{}'\frac{1}{[J_1(\lambda_i)]^2} , \tag{46}$$

and for a sufficiently large, the sum is over all integers i in the range

$$\frac{\underline{\omega}s}{2.1}\times10^{-9}\frac{a}{m}\le i\le\frac{\overline{\omega}s}{2.1}\times10^{-9}\frac{a}{m} \tag{47}$$

and the sum in (46) is one over a finite but large number of terms. Since $\underline{\omega}s$ and $\overline{\omega}s$ should be of the order 4×10^{15}, there are about 10^6 distinct frequencies possible in the frequency interval for a of the order of 1 m. Since the relation (46) does not take into account the resistive power loss at the surface of the cylinder, the power loss could be enormous for visible light (since it is proportional to the square root of the frequency) and this makes the observation of emitted power for visible frequencies unobservable as such. On the other hand, for microwaves of 10^9 Hz frequency, Eq. (46) should give a reasonable answer to the emitted power in the particular microwave frequency interval of the order of 10^9 Hz for n about 4 and a of the order of 1 m, where now there should be only a few distinct frequencies in the interval of 10^9 Hz. Here, assuming that, because of the smallness of the frequency the resistive power loss can be neglected.

The power in (41) may be rewritten equivalently as

$$P_a(\omega)=P_\infty(\omega)\sum_{i=1}^{\infty}\frac{\delta(\omega-\omega_i)}{\left[\frac{\pi\lambda_i}{2}J_1^2(\lambda_i)\right]}\frac{\pi c}{na\left[1-\frac{1}{n^2\beta^2}\right]^{1/2}} \tag{48}$$

where

$$P_\infty(\omega)=\frac{e^2v}{4\pi c^2\epsilon}n^2\omega\left[1-\frac{1}{n^2\beta^2}\right] . \tag{49}$$

For a_0 and a_1 sufficiently large we then have the following power-law behavior:

$$\frac{P_{a_1}(\omega)-P_\infty(\omega)}{P_{a_0}(\omega)-P_\infty(\omega)}=\left[\frac{a_0}{a_1}\right] . \tag{50}$$

Hence if a_0 is some reference point value for which $P_{a_0}(\omega)$ has been measured, the value a_1 for another experiment, for which $P_{a_1}(\omega)$ has been measured, can be formally inferred from (50).

We have obtained an explicit expression for the Čerenkov power spectrum of radiation emission by a charged particle in an isotropic permeable medium trapped within a hollow (infinite) cylinder with a perfectly conducting neutral surface. The charged particle is made to move along the axis of the cylinder and, as in the classic computation, any recoil of the charged particle is neglected. As a result of the boundary conditions on the electromagnetic fields, the spectrum of radiation is discrete. As shown above, the allowed frequencies are easily calculated. The fact that the emission occurs with certain frequencies only may be experimentally interesting from the observational point of view as a selection (filtering) process and goes beyond the classic situation [1]. The observation of such a spectrum would then not only indicate the presence of a charged particle in the medium (with the help of a Čerenkov counter) but would also indirectly signal the presence of an enveloping surface to the medium and provide some information on its geometry by giving an order of magnitude on its location provided by the scale parameter a. Throughout dispersion was ignored. The generalization to include indices of refraction that are frequency dependent is beyond the scope of the present paper and will be dealt with elsewhere with no speculation on it here. Unfortunately, the formal theory of radiation requires the radius of the enveloping surface to be large for its validity in the strict sense and hence puts some limitations on its size and on such analyses. Any departure from the formal theory would make the analysis nontractable and is beyond the scope of the work. Throughout we used the very elegant quantum viewpoint [3,7] within the vacuum-to-vacuum transition amplitude context. The latter not only simplifies the analysis tremendously but also makes the physics of the problem more transparent as formulated in terms of photons.

ACKNOWLEDGMENTS

I would like to thank my colleagues Dr. Edward Jeżak (RMC) and Dr. Ahpisit Ungkitchanukit (Chulalongkorn) for discussions. I would also like to thank my colleagues at Chulalongkorn and, in particular, Dr. Ahpisit Ungkitchanukit and Dr. Somphong Chatraphorn for their very kind hospitality while this work was being completed. This work was supported by the Department of National Defense under Grant No. 3610-637:FUHDT.

---

*Permanent address: Royal Military College of Canada, Kingston, Ontario, Canada K7K 5L0.

[1] P. A. Čerenkov, C. R. Acad. Sci. (USSR) **3**, 413 (1936).
[2] W. Sollfrey and T. H. Yura, Phys. Rev. **139A**, 48 (1965).
[3] J. Schwinger, W.-T. Tsai, and T. Erber, Ann. Phys. (N.Y.) **96**, 303 (1976).
[4] M. Pardy, Phys. Lett. A **134**, 357 (1989).
[5] E. B. Manoukian, Nuovo Cimento **104**, 1459 (1991).
[6] E. B. Manoukian, Phys. Rev. D **34**, 3739 (1986).
[7] E. B. Manoukian, Gen. Rel. Grav. **21**, 501 (1990).
[8] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics* (Pergamon, Oxford, 1984), Pt. 2.
[9] E. B. Manoukian, Int. J. Theor. Phys. **25**, 147 (1986).
[10] E. B. Manoukian, Hadronic J. **7**, 897 (1984).
[11] J. Schwinger, *Particles, Sources, and Fields* (Addison-Wesley, Reading, MA, 1970).
[12] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).